P-BIHARMONIC PSEUDO-PARABOLIC EQUATION WITH LOGARITHMIC NON LINEARITY

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Reception: 20/09/2022 **Acceptance:** 05/10/2022 **Publication:** 29/12/2022

Suggested citation:

Sushmitha Jayachandran and Gnanavel Soundararajan (2022). p-Biharmonic Pseudo-Parabolic Equation with Logarithmic Non linearity. *3C TIC. Cuadernos de desarrollo aplicados a las TIC*, *11*(2), 108-122. https://doi.org/10.17993/3ctic.2022.112.108-122

ABSTRACT

This paper deals with the existence of solutions of a p-biharmonic pseudo parabolic partial differential equation with logarithmic nonlinearity in a bounded domain. We prove the global existence of the weak solutions using the Faedo-Galerkin method and applying the concavity approach, that the solutions blow up at a finite time. Further, we provide a maximal limit for the blow-up time.

KEYWORDS

p-Biharmonic, pseudo-parabolic, global existence, blow up

1 INTRODUCTION

Here we examine the following problem for a p-biharmonic pseudo-parabolic equation with logarithmic nonlinearity.

$$\begin{cases} u_t - \Delta u_t + \Delta(|\Delta u|^{p-2}\Delta u) - \mathrm{div}(|\nabla u|^{q-2}\nabla u) = -\mathrm{div}(|\nabla u|^{q-2}\nabla u \log|\nabla u|) & \text{if } (x,t) \in \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{if } (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{if } x \in \Omega. \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ represents a bounded domain whose boundary $\partial\Omega$ is smooth enough, $T \in (0,\infty)$, ν indicates the normal vector on $\partial\Omega$ pointing outward, $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}$ and the condition $2 < p < q < p(1 + \frac{2}{N+2})$ holds for p and q.

Pseudo-parabolic equations address several significant physical processes, like the evolution of two components of intergalactic material, the leakage of homogeneous fluids through a rock surface, the biomathematical modeling of a bacterial film, some thin film problems, the straight transmission of nonlinear, dispersive, long waves, the heat transfer containing two temperatures, a grouping of populations, etc. Shawalter and Ting [18], [22] first examined the pseudo parabolic equations in 1969. After their precursory results, there are many papers studied the nonlinear pseudo-parabolic equations, like semilinear pseudo-parabolic equations, quasilinear pseudo-parabolic equations, and even singular and degenerate pseudo-parabolic equations (see [1], [28], [4], [6], [15], [24], [25]). A pseudo-parabolic equation with p-Laplacian $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ and logarithmic nonlinearity were studied by Nahn, and Truong [13] in 2017. Considering the equation,

$$u_t - \Delta u_t - \Delta_p u = |u|^{p-2} u \log|u|$$

and by using the potential well method proposed by Sattinger [17] and a logarithmic Sobolev inequality, they proved the existence or nonexistence of global weak solutions. Additionally, they provided requirements for both the large time decay of weak global solutions and the finite time blow-up of weak solutions. Later, many authors [26], [27], [23] considered pseudo-parabolic equations with logarithmic nonlinearity and established results for local and global existence, uniqueness, decay estimate and asymptotic behaviour of solutions, blow-up results. Logarithmic nonlinearities in parabolic and pseudo-parabolic equations were studied by Lakshmipriya et.al [11], [10] and other researchers [29], [9], [5] and they proved the existence of weak solutions and their blow up in finite time. Lower bound of Blow-up time to a fourth order parabolic equation modelling epitaxial thin film growth

Recently, higher-order equations have gained much importance in studies. Lower bound of Blow-up time to a fourth order parabolic equation modelling epitaxial thin film growth studied by Liu et.al [3]. The p-biharmonic equation

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) + \lambda|u|^{p-2}u = 0$$

were studied by Liu and Guo [14], and by using the discrete-time method and uniform estimates, they established the existence and uniqueness of weak solutions. Hao and Zhou [7] obtained results for blow up, extinction and non-extinction of solutions for the equation

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) = |u|^q - \frac{1}{|\Omega|}\int_\Omega |u| dx.$$

Wang and Liu [8] studied the p-biharmonic parabolic equation with logarithmic nonlinearity,

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) = |u|^{q-1} u \log|u|$$

for $2 < p < q < p(1 + \frac{4}{n})$ and proved the global existence, blow up, extinction and no extinction of solutions. Then Liu and Li [2] studied,

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{q-1} u \log|u|.$$

Based on the difference and variation methods, they showed the existence of weak solutions and observed large-time behaviour and the transmission of solution perturbations for $\lambda > 0, p > q > \frac{p}{2} + 1, p > \frac{n}{2}$.

https://doi.org/10.17993/3ctic.2022.112.108-122

Comert and Piskin [?] studied a p-biharmonic pseudo-parabolic equation with logarithmic nonlinearity and used the potential well method and logarithmic Sobolev inequality obtained the existence of the unique global weak solution. In addition, they also exhibited polynomial decay of solutions. Motivated by these works, we have formulated our problem (1) for a p-biharmonic pseudo-parabolic equation with logarithmic nonlinearity and studied their existence and non-existence. The problem (1) for the case $p = 2$ is already investigated and proved the existence, uniqueness and blow up of solutions (see [19], [20], [21]).

The rest of this paper is arranged to the two sections below. The preliminary notations, definitions, and results we need to support our main findings are described in Section 2. Section 3 contains the major findings of this paper explained in five theorems.

2 PRELIMINARIES

In this section, we provide some fundamental ideas and facts that are necessary for us to explain our findings. In this article, we follow the notations listed below throughout. $\|.\|_r$ denotes the $L^r(\Omega)$ norm for $1 \leq r \leq \infty$, $\|.\|_{H_0^1}$ denotes the norm in $H_0^1(\Omega)$, $(.,.)_1$ denotes the $H_0^1(\Omega)$-inner product.=, r' denotes the Holder conjugate exponent of $r > 1$ (that is, $r' = \frac{r}{r-1}$).
We define the energy functional J and the Nehari functional I as follows:
$I, J : W_0^{2,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{p}\|\Delta u\|_p^p + \frac{q+1}{q^2}\|\nabla u\|_q^q - \frac{1}{q}\int_\Omega |\nabla u|^q \log|\nabla u| dx \tag{2}$$

$$I(u) = \|\Delta u\|_p^p + \|\nabla u\|_q^q - \int_\Omega |\nabla u|^q \log|\nabla u| dx \tag{3}$$

Then we have,

$$J(u) = \frac{1}{q}I(u) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\Delta u\|_p^p + \frac{1}{q^2}\|\nabla u\|_q^q \tag{4}$$

We introduce the Nehari manifold as

$$\mathcal{N} = \{u \in W_0^{2,p}(\Omega)\backslash\{0\} : I(u) = 0\}$$

also define the potential well as

$$\mathcal{W} = \{u \in W_0^{2,p}(\Omega)\backslash\{0\} : J(u) < d, I(u) > 0\}$$

where $d = \inf_{u\in\mathcal{N}} J(u)$ is referred to as the depth of the potential well.

Definition 1. *A function $u = u(x,t)$ is considered to be a weak solution of problem (1) if $u \in L^\infty(0,T; W_0^{2,p}(\Omega))$, $u_t \in L^2(0,T; H_0^1(\Omega))$ and validates*

$$(u_t, \phi) + (\nabla u_t, \nabla \phi) + (|\Delta u|^{p-2}\Delta u, \Delta \phi) + (|\nabla u|^{q-2}\nabla u, \nabla \phi) = (|\nabla u|^{q-2}\nabla u \log|\nabla u|, \nabla \phi) \tag{5}$$

for all $\phi \in W_0^{2,p}(\Omega)$ and a.e $0 \leq t \leq T$ along with $u(x,0) = u_0(x)$ in $W_0^{2,p}(\Omega)\backslash\{0\}$. Furthermore, it also agrees the energy inequality

$$\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + J(u) \leq J(u_0) \quad ,0 < t \leq T. \tag{6}$$

Lemma 1. *[12] Let ρ be a positive number. Then we have the following inequalities:*

$$x^p \log x \leq (e\rho)^{-1} \text{ for all } x \geq 1$$

and

$$|x^p \log x| \leq (ep)^{-1} \text{ for all } 0 < x < 1.$$

https://doi.org/10.17993/3ctic.2022.112.108-122

The following lemma is similar to one in [8], [16]. However, we explain the proof with some changes due to the occurrence of the non-linear logarithmic term $-\text{div}(|\nabla u|^{q-2}\nabla u \log|\nabla u|)$ and the q-Laplacian $\text{div}(|\nabla u|^{q-2}\nabla u)$.

Lemma 2. *For any $u \in W_0^{2,p}(\Omega)\backslash\{0\}$, we have the following:*

(i) $\lim_{\gamma\to 0^+} J(\gamma u) = 0$ and $\lim_{\gamma\to\infty} J(\gamma u) = -\infty$;

(ii) $\frac{d}{d\gamma}J(\gamma u) = \frac{1}{\gamma}I(\gamma u)$ for $\gamma > 0$;

(iii) there exists a unique $\gamma^* = \gamma^*(u) > 0$ such that $\frac{d}{d\gamma}J(\gamma u)|_{\gamma=\gamma^*} = 0$. Also $J(\gamma u)$ is increasing on $0 < \gamma \leq \gamma^*$, decreasing on $\gamma^* \leq \gamma < \infty$ and takes the maximum at $\gamma = \gamma^*$;

(iv) $I(\gamma u) > 0$ for $0 < \gamma < \gamma^*$, $I(\gamma u) < 0$ for $\gamma^* < \gamma < \infty$ and $I(\gamma^* u) = 0$.

Proof.

(i) Applying the definition of J we have

$$J(\gamma u) = \frac{\gamma^p}{p}\|\Delta u\|_p^p + \frac{\gamma^q(q+1)}{q^2}\|\nabla u\|_q^q - \frac{\gamma^q \log\gamma}{q}\|\nabla u\|_q^q - \frac{\gamma^q}{q}\int_\Omega |\nabla u|^q \log|\nabla u| dx$$

so it is evident that $\lim_{\gamma\to 0^+} J(\gamma u) = 0$ and $\lim_{\gamma\to\infty} J(\gamma u) = -\infty$ since $2 < p < q$.

(ii) Direct computation yields,

$$\frac{d}{d\gamma}J(\gamma u) = \gamma^{p-1}\|\Delta u\|_p^p + \gamma^{q-1}\|\nabla u\|_q^q - \gamma^{q-1}\int_\Omega |\nabla u|^q \log|\gamma\nabla u| dx = \frac{1}{\gamma}I(\gamma u)$$

(iii) We have,

$$\frac{d}{d\gamma}J(\gamma u) = \gamma^{q-1}\left(\gamma^{p-q}\|\Delta u\|_p^p + \|\nabla u\|_q^q - \log\gamma\|\nabla u\|_q^q - \int_\Omega |\nabla u|^q \log|\nabla u| dx\right)$$

Now define,

$$g(\gamma) = \gamma^{p-q}\|\Delta u\|_p^p + \|\nabla u\|_q^q - \log\gamma\|\nabla u\|_q^q - \int_\Omega |\nabla u|^q \log|\nabla u| dx$$

Then we can observe that g is decreasing since

$$g'(\gamma) = (p-q)\gamma^{p-q-1}\|\Delta u\|_p^p - \frac{1}{\gamma}\|\nabla u\|_q^q < 0$$

Also, $\lim_{\gamma\to 0^+} g(\gamma) = \infty$ and $\lim_{\gamma\to\infty} g(\gamma) = -\infty$.
Hence, a unique γ^* with $g(\gamma^*) = 0$ is guaranteed.
Also, $g(\gamma) > 0$ for $0 < \gamma < \gamma^*$ and $g(\gamma) < 0$ for $\gamma^* < \gamma < \infty$.
Now, since $\frac{d}{d\gamma}J(\gamma u) = \gamma^{q-1}g(\gamma)$ we obtain $\frac{d}{d\gamma}J(\gamma u)|_{\gamma=\gamma^*} = 0$ and also $J(\gamma u)$ is increasing on $0 < \gamma \leq \gamma^*$, decreasing on $\gamma^* \leq \gamma < \infty$ and takes the maximum at $\gamma = \gamma^*$.

(iv) *(iv)* is obvious since $I(\gamma u) = \gamma\frac{d}{d\gamma}J(\gamma u)$.

The above lemmas are useful to prove the main results in the following section.

 https://doi.org/10.17993/3ctic.2022.112.108-122

3 MAIN RESULTS

In this section, we prove the existence of weak local solutions to the problem (1). Further, we show that the weak solution exists globally using the potential well method when the initial energy of the system is subcritical and critical. We show that the solution becomes unbounded in finite time and specifies an upper limit for the blow-up time.

Theorem 1. *(The Local existence)*
Let $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}$ and $2 < p < q < p(1+\frac{2}{N+2})$. Then a $T > 0$ and a unique weak solution $u(t)$ of problem(1) agreeing the energy inequality

$$\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + J(u(t)) \leq J(u_0) \quad ,0 \leq t \leq T \tag{7}$$

and $u(0) = u_0$ exists.

Proof. **Existence**
Let $\{w_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $W_0^{2,p}(\Omega)$. We use the approximation,

$$u_k(x,t) = \sum_{i=1}^{k} a_{k,i}(t)w_i(x), \qquad k = 1,2,\ldots$$

where $a_{k,i}(t) : [0,T] \to \mathbb{R}$ accepts the below ODE.

$$\begin{aligned}(u_{kt}, w_i) + (\nabla u_{kt}, \nabla w_i) + (|\Delta u_k|^{p-2}\Delta u_k, \Delta w_i) + (|\nabla u_k|^{q-2}\nabla u_k, \nabla w_i)\\ = (|\nabla u_k|^{q-2}\nabla u_k \log|\nabla u_k|, \nabla w_i)\end{aligned} \tag{8}$$

$i = 1,2,\ldots,k$ and

$$u_k(x,0) = \sum_{i=1}^{k} a_{k,i}(0)w_i(x) \to u_0(x) \text{ in } W_0^{2,p}(\Omega)\backslash\{0\}$$

By Peano's theorem, the above ODE has a solution $a_{k,i}$ and we can find a $T_k > 0$ with $a_{k,i} \in C^1([0,T_k])$, which implies $u_k \in C^1([0,T_k]; W_0^{2,p}(\Omega))$.
Now by multiplying (8) by $a_{k,i}(t)$, summing it for $i = 1,2,\ldots,k$ and integrating with respect to t from 0 to t we obtain,

$$\frac{1}{2}\|u_k\|_{H_0^1}^2 + \int_0^t (\|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q)dt = \frac{1}{2}\|u_k(0)\|_{H_0^1}^2 + \int_0^t\int_\Omega |\nabla u_k|^q \log|\nabla u_k| dxdt$$

That is,

$$\psi_k(t) = \psi_k(0) + \int_0^t\int_\Omega |\nabla u_k|^q \log|\nabla u_k| dxdt \tag{9}$$

where

$$\psi_k(t) = \frac{1}{2}\|u_k\|_{H_0^1}^2 + \int_0^t (\|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q)dt \tag{10}$$

We obtain the following by employing lemma(1), Gagliardo-Nirenberg interpolation inequality and Young's inequality.

$$\begin{aligned}\int_\Omega |\nabla u_k|^q \log|\nabla u_k| dx &\leq \int_{\{x\in\Omega:|\nabla u_k|\geq 1\}} |\nabla u_k|^q \log|\nabla u_k| dx\\ &\leq (e\rho)^{-1}\|\nabla u_k\|_{q+\rho}^{q+\rho}\\ &\leq (e\rho)^{-1}C_1^{q+\rho}\|\Delta u_k\|_p^{\theta(q+\rho)}\|u_k\|_2^{(1-\theta)(q+\rho)}\\ &\leq \epsilon\|\Delta u_k\|_p^p + C(\epsilon)\|u_k\|_2^{\frac{p(1-\theta)(q+\rho)}{p-\theta(q+\rho)}}\end{aligned} \tag{11}$$

https://doi.org/10.17993/3ctic.2022.112.108-122

where $\theta = \left(\frac{1}{n} + \frac{1}{2} - \frac{1}{q+\rho}\right)\left(\frac{2}{n} + \frac{1}{2} - \frac{1}{p}\right)^{-1}, \epsilon \in (0,1)$,
$C(\epsilon) = \left(\frac{p\epsilon}{\theta(q+\rho)}\right)^{\frac{\theta(q+\rho)}{\theta(q+\rho)-p}}\left(\frac{p-\theta(q+\rho)}{p}\right)\left((\epsilon\rho)^{-1}C_1^{q+\rho}\right)^{\frac{p}{p-\theta(q+\rho)}}$ and,
ρ is chosen so that $2 < q+\rho < p(1+\frac{2}{n+2})$.
Let $\beta = \frac{p(1-\theta)(q+\rho)}{2(p-\theta(q+\rho))} = \frac{np+(p-n)(q+\rho)}{p(4+n)-(n+2)(q+\rho)}$. Then $\beta > 1$ and

$$\int_\Omega |\nabla u_k|^q \log|\nabla u_k| dx \le \epsilon\|\Delta u_k\|_p^p + C(\epsilon)\|u_k\|_2^{2\beta} \tag{12}$$

Then (9) implies that,

$$\begin{aligned}\psi_k(t) &\le \psi_k(0) + \epsilon\int_0^t \|\Delta u_k\|_p^p dt + C(\epsilon)\int_0^t \|u_k\|_2^{2\beta} dt\\ &\le C_2 + \epsilon\psi_k(t) + C(\epsilon)2^\beta\int_0^t\left(\left(\frac{1}{2}\|u_k\|_{H_0^1}^2\right)^\beta + \left(\int_0^s(\|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q)ds\right)^\beta\right)dt\\ &\le C_2 + \epsilon\psi_k(t) + C_3\int_0^t \psi_k(t)^\beta dt\end{aligned}$$

Hence we get,

$$\psi_k(t) \le C_4 + C_5\int_0^t \psi_k(t)^\beta dt$$

Then the Gronwall-Bellman-Bihari type integral inequality gives a T such that $0 < T < \frac{C_4^{1-\beta}}{C_5(1-\beta)}$ and

$$\psi_k(t) \le C_T \text{ for all } t \in [0,T]. \tag{13}$$

Hence the solution of (8) exists in $[0,T]$ for all k.
Now multiplying (8) by $a'_{k,i}(t)$ and summing for $i = 1,2,\ldots,k$ we get,

$$\begin{aligned}(u_{kt}, u_{kt}) + (\nabla u_{kt}, \nabla u_{kt}) + (|\Delta u_k|^{p-2}\Delta u_k, \Delta u_{kt}) + (|\nabla u_k|^{q-2}\nabla u_k, \nabla u_{kt})\\ = (|\nabla u_k|^{q-2}\nabla u_k \log|\nabla u_k|, \nabla u_{kt})\end{aligned}$$

integrating with respect to t,

$$\int_0^t \|u_{kt}\|_{H_0^1}^2 dt + J(u_k(t)) = J(u_k(0)) \text{ for all } t \in [0,T]. \tag{14}$$

As contrast to that, a constant $C_6 > 0$ satisfying

$$J(u_k(0)) \le C_6 \text{ for all } k. \tag{15}$$

exists since $u_k(0) \to u_0$ and by the continuity of J. Then from (12),(13),(14) and (15) we can see that

$$\begin{aligned}C_6 &\ge \int_0^t \|u_{kt}\|_{H_0^1}^2 dt + \frac{1}{p}\|\Delta u_k\|_p^p + \frac{q+1}{q^2}\|\nabla u_k\|_q^q - \frac{1}{q}\int_\Omega |\nabla u_k|^q \log|\nabla u_k| dx\\ &\ge \int_0^t \|u_{kt}\|_{H_0^1}^2 dt + \left(\frac{1}{p} - \frac{\epsilon}{q}\right)\|\Delta u_k\|_p^p + \frac{q+1}{q^2}\|\nabla u_k\|_q^q - \frac{C(\epsilon)}{q}\|u_k\|_{H_0^1}^{2\beta}\\ &\ge \int_0^t \|u_{kt}\|_{H_0^1}^2 dt + \left(\frac{1}{p} - \frac{\epsilon}{q}\right)\|\Delta u_k\|_p^p + \frac{q+1}{q^2}\|\nabla u_k\|_q^q - \frac{C(\epsilon)}{q}2^\beta C_T^\beta\end{aligned}$$

Let $\tilde{C} = C_6 + \frac{C(\epsilon)2^\beta}{q}C_T^\beta$. Then we gain that

$$\begin{aligned}\int_0^t \|u_{kt}\|_2^2 dt &\le \tilde{C}\\ \int_0^t \|\nabla u_{kt}\|_2^2 dt &\le \tilde{C}\\ \|\Delta u_k\|_p^p &< \tilde{C}\left(\frac{1}{p} - \frac{\epsilon}{q}\right)^{-1}\\ \|\nabla u_k\|_q^q &< \tilde{C}\frac{q^2}{q+1}\end{aligned}$$

 https://doi.org/10.17993/3ctic.2022.112.108-122

Thus we have $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(0,T;W_0^{2,p}(\Omega))$ and $\{u_{kt}\}_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;H_0^1(\Omega))$. Hence there exists a subsequence, however indicated by $\{u_k\}_{k\in\mathbb{N}}$ which agrees,

$$\begin{aligned} u_k &\to u && \text{weakly* in } L^\infty(0,T;W_0^{2,p}(\Omega)) \\ u_{kt} &\to u_t && \text{weakly in } L^2(0,T;H_0^1(\Omega)) \\ u_k &\to u && \text{weakly* in } L^\infty(0,T;W_0^{1,q}(\Omega)) \end{aligned}$$

since

$$u_{kt} \to u_t \qquad \text{weakly in } L^2(0,T;L^2(\Omega))$$

by Aubin-Lions lemma we get,

$$u_k \quad \to \quad u \text{ strongly in } C(0,T;L^2(\Omega))$$

Therefore,

$$|\Delta u_k|^{p-2}\Delta u_k \quad \to \quad \xi_1 \text{ weakly* in } L^\infty(0,T;W_0^{-2,p'}(\Omega))$$

and,

$$|\nabla u_k|^{q-2}\nabla u_k \quad \to \quad \xi_2 \text{ weakly* in } L^\infty(0,T;W_0^{-1,q'}(\Omega))$$

where $W_0^{-2,p'}(\Omega)$ is the dual space of $W_0^{2,p}(\Omega)$ and $W_0^{-1,q'}(\Omega)$ is the dual space of $W_0^{1,q}(\Omega)$. Now from the theory of monotone operators, it concludes,

$$\xi_1 = |\Delta u|^{p-2}\Delta u \qquad \text{and} \qquad \xi_2 = |\nabla u|^{q-2}\nabla u.$$

Now let $\Phi(u) = |u|^{q-2}u\log|u|$. We have

$$\begin{aligned} \nabla u_k &\to \nabla u && \text{weakly* in } L^\infty(0,T;L^2(\Omega)) \\ \nabla u_{kt} &\to \nabla u_t && \text{weakly in } L^2(0,T;L^2(\Omega)) \end{aligned}$$

Therefore,

$$\nabla u_k \to \nabla u \text{ strongly in } C(0,T;L^2(\Omega))$$

and

$$\Phi(\nabla u_k) \to \Phi(\nabla u) \qquad a.e \text{ in } \Omega\times(0,T)$$

We again use Lemma(1) and Gagliardo-Nirenberg interpolation inequality to emerge the below.

$$\begin{aligned} \int_\Omega (\Phi(\nabla u_k))^{q'}dx &\leq \int_{\{x\in\Omega:|\nabla u_k|\leq 1\}} \left(|\nabla u_k|^{q-1}|\log|\nabla u_k||\right)^{q'}dx \\ &\quad + \int_{\{x\in\Omega:|\nabla u_k|\geq 1\}} \left(|\nabla u_k|^{q-1}|\log|\nabla u_k||\right)^{q'}dx \\ &\leq (e(q-1))^{-q'}|\Omega| + (e\mu)^{-q'}\|\nabla u_k\|_r^r \\ &\leq (e(q-1))^{-q'}|\Omega| + (e\mu)^{-q'}C_7^r\|\Delta u_k\|_p^{r\alpha}\|u_k\|_2^{r(1-\alpha)} \\ &< C_8 \end{aligned}$$

where $r = (q-1+\mu)q'$, $q' = \frac{q}{q-1}$ and $\alpha = \left(\frac{1}{n}+\frac{1}{2}-\frac{1}{r}\right)\left(\frac{2}{n}+\frac{1}{2}-\frac{1}{p}\right)^{-1}$. Hence,

$$\Phi(\nabla u_k) \to \Phi(\nabla u) \text{ weakly* in } L^\infty(0,T;L^{q'}(\Omega))$$

Now for a fixed i in (8) letting k tends to ∞ we get,

$$(u_t,w_i) + (\nabla u_t,\nabla w_i) + (|\Delta u|^{p-2}\Delta u,\Delta w_i) + (|\nabla u|^{q-2}\nabla u,\nabla w_i) = (|\nabla u|^{q-2}\nabla u\log|\nabla u|,\nabla w_i)$$

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for all $i = 1, 2, \dots, k$. Then for all $\phi \in W_0^{2,p}(\Omega)$ and for a.e. $t \in [0, T]$,

$$(u_t, \phi) + (\nabla u_t, \nabla \phi) + (|\Delta u|^{p-2}\Delta u, \Delta \phi) + (|\nabla u|^{q-2}\nabla u, \nabla \phi) = (|\nabla u|^{q-2}\nabla u \log |\nabla u|, \nabla \phi)$$

and $u(x, 0) = u_0(x)$ in $W_0 2, p^2(\Omega) \backslash \{0\}$.

Uniqueness

Let u and $\tilde{u}$ be two weak solutions of problem (1). For any $\phi \in H_0^2(\Omega)$, it is noted that,

$$(u_t, \phi) + (\nabla u_t, \nabla \phi) + (|\Delta u|^{p-2}\Delta u, \Delta \phi) + (|\nabla u|^{q-2}\nabla u, \nabla \phi) = (|\nabla u|^{q-2}\nabla u \log |\nabla u|, \nabla \phi)$$

$$(\tilde{u}_t, \phi) + (\nabla \tilde{u}_t, \nabla \phi) + (|\Delta \tilde{u}|^{p-2}\Delta \tilde{u}, \Delta \phi) + (|\nabla \tilde{u}|^{q-2}\nabla \tilde{u}, \nabla \phi) = (|\nabla \tilde{u}|^{q-2}\nabla \tilde{u} \log |\nabla \tilde{u}|, \nabla \phi)$$

On subtraction of one equation from the other and taking $\phi = u - \tilde{u}$, the above yields that

$$\begin{aligned}(\phi_t, \phi) + (\nabla \phi_t, \nabla \phi) &+ \int_\Omega (|\Delta u|^{p-2}\Delta u - |\Delta \tilde{u}|^{p-2}\Delta \tilde{u})(\Delta u - \Delta \tilde{u})dx \\ &+ \int_\Omega (|\nabla u|^{q-2}\nabla u - |\nabla \tilde{u}|^{q-2}\nabla \tilde{u})(\nabla u - \nabla \tilde{u})dx \\ &= \int_\Omega (|\nabla u|^{q-2}\nabla u \log |\nabla u| - |\nabla \tilde{u}|^{q-2}\nabla \tilde{u} \log |\nabla \tilde{u}|)(\nabla u - \nabla \tilde{u})dx\end{aligned}$$

Then by the monotonicity of q-Laplacian $\operatorname{div}(|\nabla u|^{q-2}\nabla u)$ and the p-Biharmonic operator $\Delta(|\Delta u|^{p-2}\Delta u)$ and by the Lipschitz continuity of $|x|^{q-2}x \log |x|$ we get,

$$(\phi_t, \phi)_1 \leq L \int_\Omega (\nabla u - \nabla \tilde{u})^2 dx$$

where $L > 0$ is the Lipschitz constant. Thus we obtain,

$$(\phi_t, \phi)_1 \leq L\|\nabla \phi\|_2^2 \leq L\|\phi\|_{H_0^1}^2$$

By the integration from 0 to t with respect to t we obtain that,

$$\|\phi\|_{H_0^1}^2 - \|\phi(0)\|_{H_0^1}^2 \leq L \int_0^t \|\phi\|_{H_0^1}^2 dt.$$

Since $\phi(0) = u(0) - \tilde{u}(0) = 0$, apply Gronwall's inequality to gain,

$$\|\phi\|_{H_0^1}^2 = 0$$

Therefore, $\phi = 0$ a.e. in $\Omega \times (0, T)$. That is, $u = \tilde{u}$ a.e. in $\Omega \times (0, T)$.

Energy inequality

Let $\chi \in C[0, T]$ be a non-negative function. Then (14) implies

$$\int_0^T \chi(t) \int_0^t \|u_{kt}\|_{H_0^1}^2 ds dt + \int_0^T J(u_k(t))\chi(t)dt = \int_0^T J(u_k(0))\chi(t)dt$$

Since, we have the lower semi-continuity $\int_0^T J(u_k(t))\chi(t)dt$ with respect to the weak topology of $L^2(0, T; W_0^{2,p}(\Omega))$.

$$\int_0^T J(u(t))\chi(t)dt \leq \liminf_{k \to \infty} \int_0^T J(u_k(t))\chi(t)dt$$

also $\int_0^T J(u_k(0))\chi(t)dt \to \int_0^T J(u_0)\chi(t)dt$ as $k \to \infty$. Thus we get,

$$\int_0^T \chi(t) \int_0^t \|u_t\|_{H_0^1}^2 ds dt + \int_0^T J(u(t))\chi(t)dt \leq \int_0^T J(u_0)\chi(t)dt$$

Since $\chi(t)$ is arbitrary,

$$\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + J(u(t)) \leq J(u_0) \quad \text{for } 0 \leq t \leq T.$$

Hence the proof is complete.

Next theorem address the case of the initial energy of the system is sub-critical, i.e, $J(u_0) < d$. We will demonstrate the existence of weak global solutions.

 https://doi.org/10.17993/3ctic.2022.112.108-122

Theorem 2. *(Global Existence for $J(u_0) < d$)*
A unique global weak solution u satisfying the energy estimate,

$$\int_0^t \|u_\tau\|^2_{H_0^1} d\tau + J(u(t)) \leq J(u_0) \quad \textit{for } 0 \leq t < \infty \tag{16}$$

exists for problem(1) if the conditions $J(u_0) < d$ and $I(u_0) > 0$ holds for the initial value $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}$.

Proof. Define $\{w_i\}_{i\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$ as in the proof of Theorem(1). Multiplying (8) by $a'_{k,i}(t)$ and summing over i and integrating with respect to t from 0 to t we identify,

$$\int_0^t \|u_{kt}\|^2_{H_0^1} dt + J(u_k(t)) = J(u_k(0)) \text{ for all } t \in [0, T_{max}) \tag{17}$$

where T_{max} is the maximum time for solution $u_k(x,t)$ to exist.
We have $J(u_k(0)) \to J(u_0)$ as $k \to \infty$ and $J(u_0) < d$. Therefore,

$$\int_0^t \|u_{kt}\|^2_{H_0^1} dt + J(u_k(t)) < d, \quad t \in [0, T_{max}) \tag{18}$$

Since $I(u_0) > 0$ we have $I(u_k(0)) > 0$ for sufficiently large k. We claim that $I(u_k) > 0$ for sufficiently large k. Otherwise we can locate a t_0 such that $I(u_k(t_0)) = 0$, $u_k(t_0) \neq 0$. Then $u_k(t_0) \in \mathcal{N}$ and $J(u_k(t_0)) \geq d$, which is a contradiction to (18).
Therefore $I(u_k) > 0$ for appropriately large k.
Then we get,

$$J(u_k) = \frac{1}{q} I(u_k) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\Delta u_k\|_p^p + \frac{1}{q^2} \|\nabla u_k\|_q^q > 0$$

Therefore,

$$\int_0^t \|u_{kt}\|^2_{H_0^1} dt < d$$

also

$$\left(\frac{1}{p} - \frac{1}{q}\right) \|\Delta u_k\|_p^p + \frac{1}{q^2} \|\nabla u_k\|_q^q < J(u_k) < d$$

Let $K_0 = \min\{\frac{1}{p} - \frac{1}{q}, \frac{1}{q^2}\}$ and $K = d + \frac{d}{K_0}$ then

$$\|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q < d/K_0$$

and

$$\int_0^t \|u_{kt}\|^2_{H_0^1} dt + \|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q < K \tag{19}$$

where $K > 0$. Hence we take $T_{max} = \infty$. Now it is noticeable that problem (1) has a weak global solution by applying identical ideas used to prove the Theorem(1), and the solution u also agrees with the energy inequality

$$\int_0^t \|u_\tau\|^2_{H_0^1} d\tau + J(u(t)) \leq J(u_0), \quad 0 \leq t < \infty.$$

We will explain the global existence of weak solutions in the following theorem for the critical initial energy. That is when $J(u_0) = d$.

Theorem 3. *(Global existence for $J(u_0) = d$)*
Observe the conditions $J(u_0) = d$ and $I(u_0) > 0$ holds for the initial value $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}$. Subsequently problem(1) possesses a unique global weak solution $u \in L^\infty(0,T; W_0^{2,p}(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ for $0 \leq t \leq T$ and it also accepts the energy estimate (16).

 https://doi.org/10.17993/3ctic.2022.112.108-122

Proof. Let $\eta_j = 1 - \frac{1}{j},\ \ j = 1, 2, \ldots$ then $\eta_j \to 1$ when $j \to \infty$. Take into account the below problem:

$$\begin{cases} u_t - \Delta u_t + \Delta(|\Delta u|^{p-2}\Delta u) - \operatorname{div}(|\nabla u|^{q-2}\nabla u) = -\operatorname{div}(|\nabla u|^{q-2}\nabla u \log|\nabla u|) & \text{if } (x,t) \in \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{if } (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = \eta_j u_0(x) = u_0^j & \text{if } x \in \Omega. \end{cases} \tag{20}$$

Since $I(u_0) > 0$, lemma (2)(iv) gives a $\gamma^* > 1$ with $I(\gamma^* u_0) = 0$.
Again from lemma (2)(iii) and (iv) we gain $I(\eta_j u_0) > 0$ and $J(\eta_j u_0) < J(u_0)$ since $\eta_j < 1 < \gamma^*$.
Thus we have $J(u_0^j) < d$ and $I(u_0^j) > 0$.
Then by Theorem(2), for each j problem (20) has a global weak solution $u^j \in L^\infty(0,T; W_0^{p-2}(\Omega))$ with $u_t^j \in L^2(0,T; L^2(\Omega))$ which satisfies the energy inequality,

$$\int_0^t \|u_\tau^j\|_{H_0^1}^2 d\tau + J(u^j(t)) \leq J(u_0^j) \quad \text{for } 0 \leq t < \infty.$$

Thus we have

$$\int_0^t \|u_\tau^j\|_{H_0^1}^2 d\tau + J(u^j) < d \quad \text{for } 0 \leq t < \infty.$$

Now by applying ideas similar to the one used to prove Theorem(1), we obtain a subsequence of $\{u^j\}_{j\in\mathbb{N}}$ converging to a function u, which is a weak solution of problem (1). It also fulfils the energy inequality (16). The solution's uniqueness can also be proved as in Theorem(1).
Hence the proof is over.

The following theorem gives the blow-up of solutions for the subcritical initial energy and an upper bound for blow up time.

Theorem 4. *(Blow up for $J(u_0) < d$)*
Let $u_0 \in H_0^2(\Omega)\backslash\{0\}$, $J(u_0) < d$ and $I(u_0) < 0$. Then the weak solution u of problem (1) blows up in a finite time T_* in the notion, $\lim_{t\to T_*^-} \|u\|_{H_0^1}^2 = \infty$. Furthermore, the upper bound of blow-up time T_* is given by

$$T_* \leq \frac{4(q-1)\|u_0\|_{H_0^1}^2}{q(q-2)^2(d - J(u_0))}.$$

*Proof.*First we prove $J(u(t)) < d$ and $I(u(t)) < 0$ for $t \in [0,T]$, where T indicates the maximum time for which $u(x,t)$ exists.
We have $J(u(t)) < J(u_0) < d$ by (6).
If we can choose a $t_0 \in (0,T)$ with $I(u(t_0)) = 0$ or $J(u(t_0)) = d$, since $J(u(t_0)) < d$, we must have $I(u(t_0)) = 0$.
Which implies $u(t_0) \in \mathcal{N}$ and thus $d \leq J(u(t_0))$, a contradiction.
Hence, $J(u(t)) < d$ and $I(u(t)) < 0$ for $t \in [0,T]$. Now define

$$\mathcal{P}(t) = \int_0^t \|u\|_{H_0^1}^2 dt$$

Then,

$$\mathcal{P}'(t) = \|u\|_{H_0^1}^2$$

and

$$\mathcal{P}''(t) = 2(u, u_t)_1 = -2I(u) > 0$$

Hence for $t > 0$, $\mathcal{P}'(t) \geq \mathcal{P}'(0) = \|u_0\|_{H_0^1}^2 > 0$.
Now fix $t_1 > 0$. Then for $t_1 \leq t < \infty$,

$$\mathcal{P}(t) \geq \mathcal{P}(t_1) \geq t_1\|u_0\|_{H_0^1}^2 > 0$$

By Holder's inequality, we have,

$$\frac{1}{4}(\mathcal{P}'(t) - \mathcal{P}'(0))^2 \leq \int_0^t \|u\|_{H_0^1}^2 dt \int_0^t \|u_t\|_{H_0^1}^2 dt \tag{21}$$

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Since $I(u(t)) < 0$, Lemma 2 *(iv)*, gives a γ^* with $0 < \gamma^* < 1$ and $I(\gamma^* u) = 0$. Therefore,

$$\begin{aligned} d &\leq \left(\frac{1}{p} - \frac{1}{q}\right)(\gamma^*)^p \|\Delta u\|_p^p + \frac{1}{q^2}(\gamma^*)^q \|\nabla u\|_q^q \\ &\leq \left(\frac{1}{p} - \frac{1}{q}\right)\|\Delta u\|_p^p + \frac{1}{q^2}\|\nabla u\|_q^q \end{aligned} \tag{22}$$

Now by using (4),(6) and (22) we see that,

$$\mathcal{P}''(t) \geq 2q(d - J(u_0)) + 2q \int_0^t \|u_t\|_{H_0^1}^2 dt \tag{23}$$

Then from (21) and (23) it follows that

$$\mathcal{P}''(t)\mathcal{P}(t) - \frac{q}{2}(\mathcal{P}'(t) - \mathcal{P}'(0))^2 \geq \mathcal{P}(t)2q(d - J(u_0)) > 0 \text{ for } t \in [t_1, \infty) \tag{24}$$

Now choose $\tilde{T} > 0$ large enough to introduce,

$$\mathcal{Q}(t) = \mathcal{P}(t) + (\tilde{T} - t)\|u_0\|_{H_0^1}^2 \text{ for } t \in [t_1, \tilde{T}]$$

Then $\mathcal{Q}(t) \geq \mathcal{P}(t) > 0$ for $t \in [t_1, \tilde{T}]$, $\mathcal{Q}'(t) = \mathcal{P}'(t) - \mathcal{P}'(0) > 0$ and $\mathcal{Q}''(t) = \mathcal{P}''(t) > 0$.
Hence from (24) we observe,

$$\mathcal{Q}(t)\mathcal{Q}''(t) - \frac{q}{2}(\mathcal{Q}'(t))^2 \geq \mathcal{P}(t)2q(d - J(u_0)) + \mathcal{P}''(t)(\tilde{T} - t)\|u_0\|_{H_0^1}^2 > 0 \tag{25}$$

Now define

$$\mathcal{R}(t) = \mathcal{Q}(t)^{-\frac{q-2}{2}}$$

Then,

$$\mathcal{R}'(t) = -\frac{q-2}{2}\mathcal{Q}(t)^{-\frac{q}{2}}\mathcal{Q}'(t)$$

and

$$\mathcal{R}''(t) = \frac{q-2}{2}\mathcal{Q}(t)^{-\frac{q+2}{2}}\left(\frac{q}{2}(\mathcal{Q}'(t))^2 - \mathcal{Q}(t)\mathcal{Q}''(t)\right) < 0$$

Hence $\mathcal{R}(t)$ is a concave function in $[t_1, \tilde{T}]$ for any sufficiently large $\tilde{T} > t_1$. Also since $\mathcal{R}(t_1) > 0$ and $\mathcal{R}''(t_1) < 0$, there appears a finite time $T_* > t_1 > 0$ having $\lim_{t \to T_*^-} \mathcal{R}(t) = 0$. That yields $\lim_{t \to T_*^-} \mathcal{Q}(t) = +\infty$, which in turn gives $\lim_{t \to T_*^-} \mathcal{P}(t) = +\infty$. Hence we get

$$\lim_{t \to T_*^-} \|u\|_{H_0^1}^2 = +\infty.$$

To obtain an upper limit for blow-up time we define,

$$\mathcal{S}(t) = \mathcal{P}(t) + (T_* - t)\|u_0\|_{H_0^1}^2 + \sigma(t + \varphi)^2 \text{ for } t \in [0, T_*]$$

where the constants $\sigma, \varphi > 0$ will be given later.
Then,

$$\mathcal{S}'(t) = \|u\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2 + 2\sigma(t + \varphi) > 2\sigma(t + \varphi) > 0 \tag{26}$$

also by (23) we get,

$$\mathcal{S}''(t) \geq 2q(d - J(u_0)) + 2q \int_0^t \|u_t\|_{H_0^1}^2 dt + 2\sigma \tag{27}$$

By Schwartz's inequality, we have,

$$\int_0^t \frac{d}{dt}\|u\|_{H_0^1}^2 dt \leq 2 \int_0^t \|u\|_{H_0^1}^2 dt \int_0^t \|u_t\|_{H_0^1}^2 dt \tag{28}$$

 https://doi.org/10.17993/3ctic.2022.112.108-122

Therefore,

$$\begin{aligned}(\mathcal{S}'(t))^2 &= 4\left(\frac{1}{2}\int_0^t \frac{d}{dt}\|u\|^2_{H_0^1}dt + \sigma(t+\varphi)\right)^2\\ &\leq 4\left(\int_0^t \frac{d}{dt}\|u\|^2_{H_0^1}dt + \sigma(t+\varphi)^2\right)\left(\int_0^t \frac{d}{dt}\|u_t\|^2_{H_0^1}dt + \sigma\right)\\ &= 4\left(\mathcal{S}(t) - (T_* - t)\|u_0\|^2_{H_0^1}\right)\left(\int_0^t \frac{d}{dt}\|u_t\|^2_{H_0^1}dt + \sigma\right)\\ &\leq 4\mathcal{S}(t)\left(\int_0^t \frac{d}{dt}\|u_t\|^2_{H_0^1}dt + \sigma\right) \end{aligned} \tag{29}$$

Now by applying (27) and (29) we can see that,

$$\mathcal{S}(t)\mathcal{S}''(t) - \frac{q}{2}(\mathcal{S}'(t))^2 \geq \mathcal{S}(t)(2q(d - J(u_0)) - 2\sigma(q-1))$$

If $\sigma \in \left(0, \frac{q(d-J(u_0))}{q-1}\right)$, then

$$\mathcal{S}(t)\mathcal{S}''(t) - \frac{q}{2}(\mathcal{S}'(t))^2 > 0.$$

Also we have $\mathcal{S}(0) = T_*\|u_0\|^2_{H_0^1} + \sigma\varphi^2 > 0$ and $\mathcal{S}'(0) = 2\sigma\varphi > 0$. Then by Levine's Concavity approach, we obtain the upper bound for blow-up as,

$$T_* \leq \frac{\mathcal{S}(0)}{(\frac{q}{2}-1)\mathcal{S}'(0)} = \frac{T_*\|u_0\|^2_{H_0^1}}{(q-2)\sigma\varphi} + \frac{\varphi}{q-2}$$

Therefore,

$$T_* \leq \frac{\sigma\varphi^2}{(q-2)\sigma\varphi - \|u_0\|^2_{H_0^1}}$$

thus we must have

$$\varphi \in \left(\frac{(q-1)\|u_0\|^2_{H_0^1}}{q(q-2)(d-J(u_0))}, \infty\right)$$

Let $\upsilon = \sigma\varphi \in \left(0, \frac{q(d-J(u_0))\varphi}{q-1}\right)$, then $T_* \leq \frac{\varphi\upsilon}{(q-2)\upsilon - \|u_0\|_{H_0^1}}$.
Now let $h(\varphi, \upsilon) = \frac{\varphi\upsilon}{(q-2)\upsilon - \|u_0\|_{H_0^1}}$. Since h is monotonically decreasing concerning υ, we have

$$\begin{aligned}\inf_{\{(\varphi,\upsilon)\}} h(\varphi,\upsilon) &= \inf_{\{\varphi\}} h\left(\varphi, \frac{q(d-J(u_0))\varphi}{q-1}\right)\\ &= \inf_{\{b\}} k(\varphi)\end{aligned}$$

where,

$$k(\varphi) = h\left(\varphi, \frac{q(d-J(u_0))\varphi}{q-1}\right) = \frac{\varphi^2 q(d-J(u_0))}{q(q-2)(d-J(u_0))\varphi - (q-1)\|u_0\|_{H_0^1}}$$

now since $k(\varphi)$ takes the minimum at $\varphi^* = \frac{2(q-1)\|u_0\|_{H_0^1}}{q(q-2)(d-J(u_0))}$ we can conclude that,

$$T_* \leq k(\varphi^*) = \frac{4(q-1)\|u_0\|^2_{H_0^1}}{q(q-2)^2(d-J(u_0))}.\square$$

The following theorem show that the weak solution of the system blow-up when the initial energy of the system is critical.

Theorem 5. *(Blow up for $J(u_0) = d$)*
Let $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}, J(u_0) = d$ and $I(u_0) < 0$, then the weak solution $u(t)$ of problem (1) blows up in the sense, there appears a $T_* < \infty$ such that $\lim_{t\to T_*^-}\|u\|^2_{H_0^1} = \infty$.

 https://doi.org/10.17993/3ctic.2022.112.108-122

*Proof.*Since $J(u_0) = d > 0$ and $J(u)$ is continuous with respect to t, there appears a t_0 with $J(u(x,t)) > 0$ for $0 < t \leq t_0$. Also, it is easy to see $I(u(t)) < 0$ for every t. Therefore from the energy inequality, $\int_0^{t_0} \|u_\tau\|^2_{H_0^1} d\tau + J(u(t_0)) < J(u_0) = d$, it follows that $J(u(t_0)) < d$.
Now choose $t = t_0$ as initial time, we have $J(u(t_0)) < d$ and $I(u(t_0)) < 0$. Now define

$$\mathcal{P}(t) = \int_{t_0}^{t} \|u\|^2_{H_0^1} \qquad \text{for } t > t_0$$

and the rest of proof resembles the proof of Theorem (4). □

ACKNOWLEDGMENT

The first author acknowledges the Council of Scientific and Industrial Research(CSIR), Govt. of India, for supporting by Junior Research Fellowship(JRF).

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