FG- COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED S^* METRIC SPACES

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Reception: 12/09/2022 **Acceptance:** 27/09/2022 **Publication:** 29/12/2022

Suggested citation:

Prajisha E. and Shaini P. (2022). FG- coupled fixed point theorems in partially ordered S* metric spaces.*3C TIC. Cuadernos de desarrollo aplicados a las TIC*, *11*(2), 81-97. https://doi.org/10.17993/3ctic.2022.112.81-97

ABSTRACT

This is a review paper based on a recent article on FG- coupled fixed points [17], in which the authors established FG- coupled fixed point theorems in partially ordered complete S^* metric space. The results were illustrated by suitable examples, too. An S^* metric is an n-tuple metric from n-product of a set to the non negative reals. The theorems in [17] generalizes the main results of Gnana Bhaskar and Lakshmikantham [5].

KEYWORDS

FG- Coupled Fixed Point, Mixed Monotone Property, Partially Ordered Set, S^* Metric

1 INTRODUCTION

In 1906, Maurice Frechet introduced the concept of metric as a generalization of distance. He defined a metric on a set as a function from the bi-product of the set to the non-negative reals that satisfy certain axioms. Later, several authors generalized the concept of metrics by either changing the domain or co-domain of the metric function or by varying the properties of the metric function [3,4,8,11,15,16]. An n-tuple metric called S^* metric is the latest development in this direction. Since the existence of fixed points is depending on the function as well as on its domain, studies started on fixed point theory by considering those generalized metric spaces. Now a lot of fixed point and coupled fixed point results are available under different types of metric spaces [2,6,7,9,13,14]. In [1] Abdellaoui, M.A. and Dahmani, Z. introduced S^* metric and they have proved fixed point results in S^* metric spaces. But the same concept can be seen in [10], under a different name. In [10] Mujahid Abbas, Bashir Ali, and Yusuf I Suleiman coined the name A- metric for this concept, and they have proved common coupled fixed point theorems with an illustrative example.
Recently, the concept of FG- coupled fixed points was introduced as a generalization of the concept of coupled fixed points in [12]. Some of the famous coupled fixed point theorems are generalized to FG- coupled fixed point theorems in [12,18,19].
In [17], the authors established FG- coupled fixed point theorems in the setting of partially ordered complete S^* metric spaces. This is a review paper of [17].

Some useful definitions and results are as follows:

Definition 1. *[1, 10] An* ***S^* metric*** *on a nonempty set X is a function* $S^* : X^n \to [0,\infty)$ *satisfying:*

(i) $S^*(x_1, x_2, \cdots, x_n) \geq 0,$

(ii) $S^*(x_1, x_2, \cdots, x_n) = 0$ *if and only if* $x_1 = x_2 = \cdots = x_n,$

(iii) $S^*(x_1, x_2, \cdots, x_n) \leq S^*(x_1, \cdots, x_1, a) + S^*(x_2, \cdots, x_2, a) + \cdots + S^*(x_n, \cdots, x_n, a)$

for any $x_1, x_2, \cdots, x_n, a \in X$*. The pair* (X, S^*) *is called* ***S^* metric space****.*

Lemma 1. *[1, 10] Suppose that* (X, S^*) *is an S^* metric space. Then for all* $x_1, x_2 \in X$*, we have* $S^*(x_1, x_1, \cdots, x_1, x_2) = S^*(x_2, x_2, \cdots, x_2, x_1)$

Definition 2. *[1, 10] We say that the sequence* $\{x_p\}_{p\in\mathbb{N}}$ *of the space X is* ***convergent*** *to x if* $S^*(x_p, x_p, \cdots, x_p, x) \to 0$ *as* $p \to \infty$*. We write* $\lim_{p\to\infty} x_p = x$

Definition 3. *[1, 10] We say that the sequence* $\{x_p\}_{p\in\mathbb{N}}$ *of the space X is of* ***Cauchy*** *if for each* $\epsilon > 0$*, there exist* $p_0 \in \mathbb{N}$ *such that for any* $p, q \geq p_0$*,*
$S^*(x_p, \cdots, x_p, x_q) < \epsilon$

The space (X, S^*) is **complete** if all its Cauchy sequences are convergent.

Lemma 2. *[1, 10] Let* (X, S^*) *be an S^* metric space. If* $\{x_p\}_{p\in\mathbb{N}}$ *in X converges to x, then x is unique.*

Definition 4. *[12] Let X and Y be any two non-empty sets and* $F : X \times Y \to X$ *and* $G : Y \times X \to Y$ *be two mappings. An element* $(x, y) \in X \times Y$ *is said to be an FG- coupled fixed point if* $F(x, y) = x$ *and* $G(y, x) = y$*.*

Definition 5. *[12] Let* $(X, \preceq_{P_1})$ *and* $(Y, \preceq_{P_2})$ *be two partially ordered sets and* $F : X \times Y \to X$ *and* $G : Y \times X \to Y$ *be two mappings. We say that F and G have mixed monotone property if F and G are increasing in first variable and monotone decreasing second variable, i.e., if for all* $(x, y) \in X \times Y$*,* $x_1, x_2 \in X, x_1 \preceq_{P_1} x_2$ *implies* $F(x_1, y) \preceq_{P_1} F(x_2, y)$ *and* $G(y, x_2) \preceq_{P_2} G(y, x_1)$ *and* $y_1, y_2 \in Y, y_1 \preceq_{P_2} y_2$ *implies* $F(x, y_2) \preceq_{P_1} F(x, y_1)$ *and* $G(y_1, x) \preceq_{P_2} G(y_2, x)$*.*

 https://doi.org/10.17993/3ctic.2022.112.81-97

Note 1. *[12] Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two mappings, then for $n \geq 1$, $F^n(x,y) = F(F^{n-1}(x,y), G^{n-1}(y,x))$ and $G^n(y,x) = G(G^{n-1}(y,x), F^{n-1}(x,y))$, and $F^0(x,y) = x$ and $G^0(y,x) = y$ for all $x \in X$ and $y \in Y$.*

Note 2. Let $(X, \preceq_{P_1})$ and $(Y, \preceq_{P_2})$ be two partially ordered sets, then we define the partial order $\leq_{12}$ on $X \times Y$ and the partial order $\leq_{21}$ on $Y \times X$ as follows:
For all $x, u \in X$ and $y, v \in Y$

$$\begin{aligned}(x,y) \leq_{12} (u,v) \quad &\Leftrightarrow \quad x \preceq_{P_1} u \text{ and } v \preceq_{P_2} y \\ (y,x) \leq_{21} (v,u) \quad &\Leftrightarrow \quad y \preceq_{P_2} v \text{ and } u \preceq_{P_1} x\end{aligned}$$

2 Main Results in [17]

Mainly two FG-coupled fixed point theorems are discussed in [17], first one deals with the existence of FG-coupled fixed point and the second deals with both the existence and uniqueness of FG-coupled fixed point. They are as follow:

Theorem 1. *[17] Let $(X, S_x^*, \preceq_{P_1})$ and $(Y, S_y^*, \preceq_{P_2})$ be two partially ordered complete S^* metric spaces and $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two mappings with mixed monotone property and satisfy the following:*

$$\begin{aligned}&S_x^*\big(F(x,y), \cdots, F(x,y), F(u,v)\big) \\ &\leq a_1 S_x^*(x, \cdots, x, u) + a_2 S_x^*\big(x, \cdots, x, F(x,y)\big) + a_3 S_x^*\big(x, \cdots, x, F(u,v)\big) \\ &\quad + a_4 S_x^*\big(u, \cdots, u, F(x,y)\big) + a_5 S_x^*\big(u, \cdots, u, F(u,v)\big), \ \forall (x,y) \leq_{12} (u,v)\end{aligned} \tag{1}$$

and

$$\begin{aligned}&S_y^*\big(G(y,x), \cdots, G(y,x), G(v,u)\big) \\ &\leq b_1 S_y^*(y, \cdots, y, v) + b_2 S_y^*\big(y, \cdots, y, G(y,x)\big) + b_3 S_y^*\big(y, \cdots, y, G(v,u)\big) \\ &\quad + b_4 S_y^*\big(v, \cdots, v, G(y,x)\big) + b_5 S_y^*\big(v, \cdots, v, G(v,u)\big), \ \forall (y,x) \leq_{21} (v,u)\end{aligned} \tag{2}$$

for the non negative $a_i, b_i, i = 1,2,3,4,5$ with
$a_1 + a_2 + na_3 + a_5 < 1$, $b_1 + b_2 + nb_4 + b_5 < 1$, $a_3 + a_5 < 1$, $b_2 + b_4 < 1$.
Also suppose that either

(I) F and G are continuous or

(II) X and Y have the following properties:

(i) if $\{z_k\}$ is an increasing sequence in X with $z_k \to z$, then $z_k \preceq_{P_1} z$ for all $k \in \mathbb{N}$

(ii) if $\{w_k\}$ is a decreasing sequence in Y with $w_k \to w$, then $w \preceq_{P_2} w_k$ for all $k \in \mathbb{N}$.

If there exist $x_0 \in X$ and $y_0 \in Y$ with $(x_0, y_0) \leq_{12} \big(F(x_0,y_0), G(y_0,x_0)\big)$, then there exist an FG-coupled fixed point.

Proof. Given $x_0 \in X$ and $y_0 \in Y$ such that $(x_0, y_0) \leq_{12} \big(F(x_0,y_0), G(y_0,x_0)\big)$. If $(x_0,y_0) = \big(F(x_0,y_0), G(y_0,x_0)\big)$ then (x_0,y_0) is an FG- coupled fixed point.
Otherwise we have

$$(x_0,y_0) <_{12} \big(F(x_0,y_0), G(y_0,x_0)\big)$$

Then by the definition of the partial order on $X \times Y$ we have either
$x_0 \preceq_{P_1} F(x_0,y_0)$ and $G(y_0,x_0) \prec_{P_2} y_0$ **or** $x_0 \prec_{P_1} F(x_0,y_0)$ and $G(y_0,x_0) \preceq_{P_2} y_0$.
Without loss of generality we assume that

$$x_0 \preceq_{P_1} F(x_0,y_0) \ \text{ and } \ G(y_0,x_0) \prec_{P_2} y_0 \tag{3}$$

 https://doi.org/10.17993/3ctic.2022.112.81-97

Let $x_1 = F(x_0, y_0)$ and $y_1 = G(y_0, x_0)$.
By (3) we have,

$$x_0 \preceq_{P_1} x_1 \text{ and } y_1 \prec_{P_2} y_0$$

By the mixed monotone property of F and G we have

$$\begin{aligned} F(x_0, y_0) &\preceq_{P_1} F(x_1, y_0) \\ &\preceq_{P_1} F(x_1, y_1) \end{aligned} \tag{4}$$

and

$$\begin{aligned} G(y_1, x_1) &\preceq_{P_2} G(y_0, x_1) \\ &\preceq_{P_2} G(y_0, x_0) \end{aligned} \tag{5}$$

Let $x_2 = F(x_1, y_1)$ and $y_2 = G(y_1, x_1)$.
By (4) and (5) we have,

$$x_1 \preceq_{P_1} x_2 \text{ and } y_2 \preceq_{P_2} y_1$$

Continuing this process by using the mixed monotone property of F and G and by using the definition of partial order on $X \times Y$ we get sequences $\{x_m\}$ and $\{y_m\}$ in X and Y respectively as: for all $m \in \mathbb{N} \cup \{0\}$

$$x_{m+1} = F(x_m, y_m) \text{ and } y_{m+1} = G(y_m, x_m) \tag{6}$$

with the property that for all $m \in \mathbb{N} \cup \{0\}$

$$x_m \preceq_{P_1} x_{m+1} \text{ and } y_{m+1} \preceq_{P_2} y_m \tag{7}$$

That is by the definition of partial order on $X \times Y$ and $Y \times X$ we have,

$$(x_m, y_m) \leq_{12} (x_{m+1}, y_{m+1})$$

and

$$(y_m, x_m) \geq_{21} (y_{m+1}, x_{m+1})$$

Claim: For all $k \in \mathbb{N}$

$$S_x^*(x_k, \cdots, x_k, x_{k+1}) \leq \alpha^k\ S_x^*(x_0, \cdots, x_0, x_1) \tag{8}$$

and

$$S_y^*(y_k, \cdots, y_k, y_{k+1}) \leq \beta^k\ S_y^*(y_0, \cdots, y_0, y_1) \tag{9}$$

where

$$\alpha = \frac{a_1 + a_2 + (n-1)a_3}{1 - a_3 - a_5} < 1 \text{ and } \beta = \frac{b_1 + b_5 + (n-1)b_4}{1 - b_2 - b_4} < 1 \tag{10}$$

Now, we prove the claim by the method of mathematical induction.
When $k = 1$ we have,

$$\begin{aligned}
&S_x^*(x_1, \cdots, x_1, x_2) \\
&= S_x^*\big(F(x_0, y_0), \cdots, F(x_0, y_0), F(x_1, y_1)\big) \\
&\leq a_1 S_x^*(x_0, \cdots, x_0, x_1) + a_2 S_x^*\big(x_0, \cdots, x_0, F(x_0, y_0)\big) + a_3 S_x^*\big(x_0, \cdots, x_0, F(x_1, y_1)\big) \\
&\qquad + a_4 S_x^*\big(x_1, \cdots, x_1, F(x_0, y_0)\big) + a_5 S_x^*\big(x_1, \cdots, x_1, F(x_1, y_1)\big) \\
&= a_1 S_x^*(x_0, \cdots, x_0, x_1) + a_2 S_x^*(x_0, \cdots, x_0, x_1) + a_3 S_x^*(x_0, \cdots, x_0, x_2) \\
&\qquad + a_5 S_x^*(x_1, \cdots, x_1, x_2) \\
&= (a_1 + a_2) S_x^*(x_0, \cdots, x_0, x_1) + a_3 S_x^*(x_0, \cdots, x_0, x_2) + a_5 S_x^*(x_1, \cdots, x_1, x_2) \\
&\leq (a_1 + a_2) S_x^*(x_0, \cdots, x_0, x_1) + a_3 \big[(n-1) S_x^*(x_0, \cdots, x_0, x_1) \\
&\qquad + S_x^*(x_2, \cdots, x_2, x_1)\big] + a_5 S_x^*(x_1, \cdots, x_1, x_2) \\
&= (a_1 + a_2) S_x^*(x_0, \cdots, x_0, x_1) + a_3 \big[(n-1) S_x^*(x_0, \cdots, x_0, x_1) \\
&\qquad + S_x^*(x_1, \cdots, x_1, x_2)\big] + a_5 S_x^*(x_1, \cdots, x_1, x_2) \\
&= (a_1 + a_2 + (n-1)a_3)\ S_x^*(x_0, \cdots, x_0, x_1) + (a_3 + a_5) S_x^*(x_1, \cdots, x_1, x_2)
\end{aligned}$$

 https://doi.org/10.17993/3ctic.2022.112.81-97

which implies that

$$(1-a_3-a_5)S_x^*(x_1,\cdots,x_1,x_2) \ \le \ (a_1+a_2+(n-1)a_3)\ S_x^*(x_0,\cdots,x_0,x_1)$$

That is,

$$S_x^*(x_1,\cdots,x_1,x_2) \ \le \ \frac{a_1+a_2+(n-1)a_3}{1-a_3-a_5}\ S_x^*(x_0,\cdots,x_0,x_1)$$

Similarly we have,

$$S_y^*(y_1,\cdots,y_1,y_2) \ \le \ (b_1+b_5+(n-1)b_4)\ S_y^*(y_0,\cdots,y_0,y_1)+(b_2+b_4)S_y^*(y_1,\cdots,y_1,y_2)$$

which implies that

$$(1-b_2-b_4)\ S_y^*(y_1,\cdots,y_1,y_2) \ \le \ (b_1+b_5+(n-1)b_4)\ S_y^*(y_0,\cdots,y_0,y_1)$$

That is,

$$S_y^*(y_1,\cdots,y_1,y_2) \ \le \ \frac{b_1+b_5+(n-1)b_4}{1-b_2-b_4}\ S_y^*(y_0,\cdots,y_0,y_1)$$

Thus the claim is true for $k=1$.
Now assume the claim for $k\le m$ and check for $k=m+1$.
Consider,

$$\begin{aligned}
&S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&=S_x^*\big(F(x_m,y_m),\cdots,F(x_m,y_m),F(x_{m+1},y_{m+1})\big)\\
&\le a_1S_x^*(x_m,\cdots,x_m,x_{m+1})+a_2S_x^*\big(x_m,\cdots,x_m,F(x_m,y_m)\big)\\
&\qquad +a_3S_x^*\big(x_m,\cdots,x_m,F(x_{m+1},y_{m+1})\big)+a_4S_x^*\big(x_{m+1},\cdots,x_{m+1},F(x_m,y_m)\big)\\
&\qquad\qquad\qquad\qquad +a_5S_x^*\big(x_{m+1},\cdots,x_{m+1},F(x_{m+1},y_{m+1})\big)\\
&=a_1S_x^*(x_m,\cdots,x_m,x_{m+1})+a_2S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\qquad +a_3S_x^*(x_m,\cdots,x_m,x_{m+2})+a_5S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&=(a_1+a_2)S_x^*(x_m,\cdots,x_m,x_{m+1})+a_3S_x^*(x_m,\cdots,x_m,x_{m+2})+a_5S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&\le(a_1+a_2)S_x^*(x_m,\cdots,x_m,x_{m+1})+a_3\big[(n-1)S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\qquad +S_x^*(x_{m+2},\cdots,x_{m+2},x_{m+1})\big]+a_5S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&=(a_1+a_2)S_x^*(x_m,\cdots,x_m,x_{m+1})+a_3\big[(n-1)S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\qquad +S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\big]+a_5S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&=(a_1+a_2+(n-1)a_3)\ S_x^*(x_m,\cdots,x_m,x_{m+1})+(a_3+a_5)S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-a_3-a_5)S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&\le(a_1+a_2+(n-1)a_3)\ S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\le(a_1+a_2+(n-1)a_3)\left[\frac{a_1+a_2+(n-1)a_3}{1-a_3-a_5}\right]^m S_x^*(x_0,\cdots,x_0,x_1)
\end{aligned}$$

Therefore,

$$S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\le\left[\frac{a_1+a_2+(n-1)a_3}{1-a_3-a_5}\right]^{m+1}\ S_x^*(x_0,\cdots,x_0,x_1)$$

Similarly we have,

$$\begin{aligned}
S_y^*(y_{m+1},\cdots,y_{m+1},y_{m+2}) \ &\le \ (b_1+b_5+b_4(n-1))S_y^*(y_m,\cdots,y_m,y_{m+1})\\
&\quad +(b_2+b_4)S_y^*(y_{m+1},\cdots,y_{m+1},y_{m+2})
\end{aligned}$$

 https://doi.org/10.17993/3ctic.2022.112.81-97

which implies that

$$\begin{aligned}&(1-b_2-b_4)S_y^*(y_{m+1},\cdots,y_{m+1},y_{m+2})\\&\le(b_1+b_5+b_4(n-1))S_y^*(y_m,\cdots,y_m,y_{m+1})\\&\le(b_1+b_5+b_4(n-1))\Big[\frac{b_1+b_5+(n-1)b_4}{1-b_2-b_4}\Big]^m S_y^*(y_0,\cdots,y_0,y_1)\end{aligned}$$

Therefore,

$$S_y^*(y_{m+1},\cdots,y_{m+1},y_{m+2})\le\Big[\frac{b_1+b_5+(n-1)b_4}{1-b_2-b_4}\Big]^{m+1}S_y^*(y_0,\cdots,y_0,y_1)$$

Thus the claim is true for all $k\in\mathbb{N}$.
Next we prove that $\{x_m\}$ is a Cauchy sequence in X and $\{y_m\}$ is a Cauchy sequence in Y.
Let $p,q\in\mathbb{N}$ with $p<q$.
Consider,

$$\begin{aligned}&S_x^*(x_p,\cdots,x_p,x_q)\\&\le(n-1)S_x^*(x_p,\cdots,x_p,x_{p+1})+S_x^*(x_q,\cdots,x_q,x_{p+1})\\&=(n-1)S_x^*(x_p,\cdots,x_p,x_{p+1})+S_x^*(x_{p+1},\cdots,x_{p+1},x_q)\\&\le(n-1)S_x^*(x_p,\cdots,x_p,x_{p+1})+(n-1)S_x^*(x_{p+1},\cdots,x_{p+1},x_{p+2})\\&\qquad+S_x^*(x_q,\cdots,x_q,x_{p+2})\\&=(n-1)\Big[S_x^*(x_p,\cdots,x_p,x_{p+1})+S_x^*(x_{p+1},\cdots,x_{p+1},x_{p+2})\Big]\\&\qquad+S_x^*(x_{p+2},\cdots,x_{p+2},x_q)\\&\qquad\cdots\\&\qquad\cdots\\&\qquad\cdots\\&\le(n-1)\Big[S_x^*(x_p,\cdots,x_p,x_{p+1})+S_x^*(x_{p+1},\cdots,x_{p+1},x_{p+2})\\&\quad+\cdots+S_x^*(x_{q-2},\cdots,x_{q-2},x_{q-1})\Big]+S_x^*(x_q,\cdots,x_q,x_{q-1})\\&=(n-1)\sum_{i=p}^{q-2}S_x^*(x_i,\cdots,x_i,x_{i+1})+S_x^*(x_{q-1},\cdots,x_{q-1},x_q)\\&\le(n-1)\sum_{i=p}^{q-2}\alpha^iS_x^*(x_0,\cdots,x_0,x_1)+\alpha^{q-1}S_x^*(x_0,\cdots,x_0,x_1)\end{aligned}$$

$$\begin{aligned}&=(n-1)S_x^*(x_0,\cdots,x_0,x_1)\sum_{i=p}^{q-2}\alpha^i+\alpha^{q-1}S_x^*(x_0,\cdots,x_0,x_1)\\&\le(n-1)\frac{\alpha^p}{1-\alpha}S_x^*(x_0,\cdots,x_0,x_1)+\alpha^{q-1}S_x^*(x_0,\cdots,x_0,x_1)\\&\to 0\ as\ p,\ q\to\infty\text{ since }\alpha<1\end{aligned}$$

Thus, $\{x_m\}$ is a Cauchy sequence in X.

https://doi.org/10.17993/3ctic.2022.112.81-97

Similarly we have,

$$\begin{aligned}
S_y^*(y_p,\cdots,y_p,y_q) &\le (n-1)\sum_{i=p}^{q-2} S_y^*(y_i,\cdots,y_i,y_{i+1}) + S_y^*(y_{q-1},\cdots,y_{q-1},y_q)\\
&\le (n-1)\sum_{i=p}^{q-2}\beta^i\ S_y^*(y_0,\cdots,y_0,y_1)+\beta^{q-1}S_y^*(y_0,\cdots,y_0,y_1)\\
&= (n-1)\ S_y^*(y_0,\cdots,y_0,y_1)\sum_{i=p}^{q-2}\ \beta^i + \beta^{q-1}S_y^*(y_0,\cdots,y_0,y_1)\\
&\le (n-1)\frac{\beta^p}{1-\beta}S_y^*(y_0,\cdots,y_0,y_1) + \beta^{q-1}S_y^*(y_0,\cdots,y_0,y_1)\\
&\to 0\ as\ p,\ q\to\infty \text{ since } \beta<1
\end{aligned}$$

Thus, $\{y_m\}$ is a Cauchy sequence in Y.
Since X and Y are complete S^* metric spaces, there exist $x \in X$ and $y \in Y$ such that

$$\lim_{p\to\infty} x_p = x \ \text{ and }\ \lim_{p\to\infty} y_p = y \tag{11}$$

Case (I): First assume that F and G are continuous.
Therefore by (6) and (11) we have,

$$x = \lim_{p\to\infty} x_{p+1} = \lim_{p\to\infty} F(x_p,y_p) = F(x,y)$$

and

$$y = \lim_{p\to\infty} y_{p+1} = \lim_{p\to\infty} G(y_p,x_p) = G(y,x)$$

That is $F(x,y)=x$ and $G(y,x)=y$.
Thus (x,y) is an FG- coupled fixed point.
Case (II): Suppose that X and Y have the properties (i) and (ii) respectively.
By (7) we have, $\{x_m\}$ is an increasing sequence in X and $\{y_m\}$ is a decreasing sequence in Y and by (11) we have $\lim_{m\to\infty} x_m = x$ and $\lim_{m\to\infty} y_m = y$
Therefore by the hypothesis we have for all $m\in\mathbb{N}$

$$x_m \preceq_{P_1} x \ \text{ and }\ y \preceq_{P_2} y_m$$

Therefore by the definition of partial order on $X\times Y$ and $Y\times X$ we have

$$(x_m,y_m) \le_{12} (x,y) \ \text{ and }\ (y,x)\le_{21}(y_m,x_m)$$

Consider,

$$\begin{aligned}
&S_x^*\big(x,\cdots,x,F(x,y)\big)\\
&\le (n-1)S_x^*(x,\cdots,x,x_{m+1})+S_x^*(F(x,y),\cdots,F(x,y),x_{m+1})\\
&= (n-1)S_x^*(x,\cdots,x,x_{m+1}) + S_x^*\big(F(x,y),\cdots,F(x,y),F(x_m,y_m)\big)\\
&= (n-1)S_x^*(x,\cdots,x,x_{m+1})+S_x^*\big(F(x_m,y_m),\cdots,F(x_m,y_m),F(x,y)\big)\\
&\le (n-1)S_x^*(x,\cdots,x,x_{m+1}) + a_1S_x^*(x_m,\cdots,x_m,x) + a_2S_x^*\big(x_m,\cdots,x_m,F(x_m,y_m)\big)\\
&\quad +a_3S_x^*\big(x_m,\cdots,x_m,F(x,y)\big)+a_4S_x^*\big(x,\cdots,x,F(x_m,y_m)\big)+a_5S_x^*\big(x,\cdots,x,F(x,y)\big)\\
&= (n-1)S_x^*(x,\cdots,x,x_{m+1}) + a_1S_x^*(x_m,\cdots,x_m,x) + a_2S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\quad +a_3S_x^*\big(x_m,\cdots,x_m,F(x,y)\big) + a_4S_x^*(x,\cdots,x,x_{m+1}) + a_5S_x^*\big(x,\cdots,x,F(x,y)\big)\\
&\le (n-1)S_x^*(x,\cdots,x,x_{m+1}) + a_1S_x^*(x_m,\cdots,x_m,x) + a_2S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\quad +a_3\big[(n-1)S_x^*(x_m,\cdots,x_m,x) + S_x^*(F(x,y),\cdots,F(x,y),x)\big]\\
&\quad +a_4S_x^*(x,\cdots,x,x_{m+1}) + a_5S_x^*\big(x,\cdots,x,F(x,y)\big)
\end{aligned}$$

 https://doi.org/10.17993/3ctic.2022.112.81-97

By taking the limit as $m \to \infty$ on both sides, and by using (11) and Lemma 1 we get

$$S_x^*\big(x,\cdots,x,F(x,y)\big) \leq (a_3+a_5)\ S_x^*\big(x,\cdots,x,F(x,y)\big)$$

since $a_3+a_5<1$ we get $S_x^*\big(x,\cdots,x,F(x,y)\big)=0$
Thus we get

$$F(x,y)=x \tag{12}$$

Similarly,

$$\begin{aligned}
&S_y^*\big(y,\cdots,y,G(y,x)\big)\\
&\leq (n-1)S_y^*(y,\cdots,y,y_{m+1}) + S_y^*(G(y,x),\cdots,G(y,x),y_{m+1})\\
&\leq (n-1)S_y^*(y,\cdots,y,y_{m+1}) + S_y^*\big(G(y,x),\cdots,G(y,x),G(y_m,x_m)\big)\\
&\leq (n-1)S_y^*(y,\cdots,y,y_{m+1}) + b_1S_y^*(y,\cdots,y,y_m) + b_2S_y^*\big(y,\cdots,y,G(y,x)\big)\\
&\quad +b_3S_y^*\big(y,\cdots,y,G(y_m,x_m)\big)+b_4S_y^*\big(y_m,\cdots,y_m,G(y,x)\big)\\
&\qquad\qquad +b_5S_y^*\big(y_m,\cdots,y_m,G(y_m,x_m)\big)\\
&\leq (n-1)S_y^*(y,\cdots,y,y_{m+1}) + b_1S_y^*(y,\cdots,y,y_m) + b_2S_y^*\big(y,\cdots,y,G(y,x)\big)\\
&\quad +b_3S_y^*(y,\cdots,y,y_{m+1})+b_4S_y^*\big(y_m,\cdots,y_m,G(y,x)\big)+b_5S_y^*(y_m,\cdots,y_m,y_{m+1})\\
&\leq (n-1)S_y^*(y,\cdots,y,y_{m+1}) + b_1S_y^*(y,\cdots,y,y_m) + b_2S_y^*\big(y,\cdots,y,G(y,x)\big)\\
&\quad +b_3S_y^*(y,\cdots,y,y_{m+1})+b_4\big[(n-1)S_y^*(y_m,\cdots,y_m,y) + S^*(G(y,x),\cdots,G(y,x),y)\big]\\
&\qquad\qquad +b_5S_y^*(y_m,\cdots,y_m,y_{m+1})
\end{aligned}$$

By taking the limit as $m \to \infty$ on both sides, using (11) and Lemma 1 we get

$$S_y^*\big(y,\cdots,y,G(y,x)\big) \leq (b_2+b_4)\ S_y^*\big(y,\cdots,y,G(y,x)\big)$$

since $b_2+b_4<1$ we get $S_y^*\big(y,\cdots,y,G(y,x)\big)=0$
Thus we get

$$G(y,x)=y \tag{13}$$

Therefore by (12) and (13), (x,y) is an FG- coupled fixed point.
Hence the proof.

By taking $n=2$, $X=Y$, $F=G$, $a_2=b_2=k$, $a_5=b_5=l$ and the remaining $a_i,b_i=0$, we get a coupled fixed fixed point theorem for Kannan type mapping. We give the result as a corollary as follows:

Corollary 1. *Let $(X,d,\preceq)$ be a partially ordered complete metric space and $F: X\times X\to X$ be a mapping having the mixed monotone property on X satisfying:*

$$d\big(F(x,y),F(u,v)\big) \leq k\ d\big(x,F(x,y)\big) + l\ d\big(u,F(u,v)\big)\ \forall\ (x,y)\geq(u,v)$$

for non negative k,l with $k+l<1$
Suppose that either

(I) F is continuous or

(II) X satisfy the following:

(i) if $\{x_k\}$ is an increasing sequence in X with $x_k\to x$, then $x_k\preceq x$ for all $k\in\mathbb{N}$

(ii) if $\{y_k\}$ is a decreasing sequence in X with $y_k\to y$, then $y\preceq y_k$ for all $k\in\mathbb{N}$.

If there exist $x_0,y_0\in X$ such that $(x_0,y_0)\leq\big(F(x_0,y_0),F(y_0,x_0)\big)$ then F has a coupled fixed point.

By taking $n=2$, $X=Y$, $F=G$, $a_3=b_3=k$, $a_4=b_4=l$ and the remaining $a_i,b_i=0$, we get a coupled fixed point theorem for Chatterjea type mapping. We give the result as a corollary as follows:

 https://doi.org/10.17993/3ctic.2022.112.81-97

Corollary 2. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $F : X \times X \to X$ be a mapping having the mixed monotone property on X satisfying:*

$$d\big(F(x,y), F(u,v)\big) \leq k\ d\big(x, F(u,v)\big) + l\ d\big(u, F(x,y)\big)\ \forall\ (x,y) \geq (u,v)$$

for $k, l \in [0, \frac{1}{2})$
Suppose that either

(I) F is continuous or

(II) X satisfy the following:

(i) if $\{x_k\}$ is an increasing sequence in X with $x_k \to x$, then $x_k \preceq x$ for all $k \in \mathbb{N}$

(ii) if $\{y_k\}$ is a decreasing sequence in X with $y_k \to y$, then $y \preceq y_k$ for all $k \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $(x_0, y_0) \leq \big(F(x_0,y_0), F(y_0,x_0)\big)$ then F has a coupled fixed point.

Remark 1. *By putting different values to the constants $a_i, b_i;\ i = 1,2,3,4,5$ which satisfy the conditions mentioned in Theorem 1 we get various FG- coupled fixed point theorems.*

Remark 2. *By varying the constants $a_i, b_i;\ i = 1,2,3,4,5$ which satisfy the conditions mentioned in Theorem 1 and by taking $X = Y$ and $F = G$ we get different coupled fixed point theorems.*

Example 1. *Let $X = [0,1]$ and $Y = [-1,0]$*
Consider the metric S^* defined on both X and Y as

$$S^*(a_1, \cdots, a_n) = \sum_{i=1}^{n} \sum_{i<j} |a_i - a_j|$$

For $x, u \in X$, consider the partial order $\leq$ as $x \leq u \ \Leftrightarrow\ x = u$
and for $y, v \in Y$, define partial order $\leq$ as $y \leq v \ \Leftrightarrow\ y = v$.
Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be defined as

$$F(x,y) = \frac{x-y}{2} \quad \text{and} \quad G(y,x) = \frac{2y-x}{3}$$

As per the partial order defined on X and Y it can be easily verified that F and G are mixed monotone mappings and satisfy the conditions (1) and (2).
Here $\{(x,-x) : x \in [0,1]\}$ is the set of all FG- coupled fixed points.

Theorem 2. *[17] Let $(X, S^*_x, \preceq_{P_1})$ and $(Y, S^*_y, \preceq_{P_2})$ be two partially ordered complete S^* metric spaces and $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two mappings with mixed monotone property satisfying:*

$$S^*_x\big(F(x,y), \cdots, F(x,y), F(u,v)\big) \leq aS^*_x(x, \cdots, x, u) + bS^*_y(y, \cdots, y, v),\ \forall (x,y) \leq_{12} (u,v) \tag{14}$$

and

$$S^*_y\big(G(y,x), \cdots, G(y,x), G(v,u)\big) \leq aS^*_y(y, \cdots, y, v) + bS^*_x(x, \cdots, x, u),\ \forall (y,x) \leq_{21} (v,u) \tag{15}$$

for non negative a, b with $a + b < 1$. Also suppose that either

(I) F and G are continuous or

(II) X and Y have the following properties:

(i) if $\{z_k\}$ is an increasing sequence in X with $z_k \to z$, then $z_k \preceq_{P_1} z$ for all $k \in \mathbb{N}$

(ii) if $\{w_k\}$ is a decreasing sequence in Y with $w_k \to w$, then $w \preceq_{P_2} w_k$ for all $k \in \mathbb{N}$.

 https://doi.org/10.17993/3ctic.2022.112.81-97

If there exist $x_0 \in X$ and $y_0 \in Y$ with $(x_0, y_0) \leq_{12} \big(F(x_0, y_0), G(y_0, x_0)\big)$, then there exist an FG-coupled fixed point in $X \times Y$.
Moreover unique FG- coupled fixed point exists if

(III) for every $(x, y), (x_1, y_1) \in X \times Y$ there exist a $(u, v) \in X \times Y$ that is comparable to both (x, y) and (x_1, y_1).

Proof. Following as in the prof of Theorem 1 we can construct an increasing sequence $\{x_m\}_{m\in\mathbb{N}}$ in X and a decreasing sequence $\{y_m\}_{m\in\mathbb{N}}$ in Y defined as:

$$x_{m+1} = F(x_m, y_m) \quad \text{and} \quad y_{m+1} = G(y_m, x_m) \tag{16}$$

with the property that

$$(x_m, y_m) \leq_{12} (x_{m+1}, y_{m+1})$$

and

$$(y_m, x_m) \leq_{21} (y_{m+1}, x_{m+1})$$

By (16) we have

$$x_{m+1} = F(x_m, y_m) = F^{m+1}(x_0, y_0) \quad \text{and} \quad y_{m+1} = G(y_m, x_m) = G^m(y_0, x_0) \tag{17}$$

Claim: For $p \in \mathbb{N}$,

$$S_x^*(x_p, \cdots, x_p, x_{p+1}) \leq (a+b)^p \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right] \tag{18}$$

and

$$S_y^*(y_p, \cdots, y_p, y_{p+1}) \leq (a+b)^p \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right] \tag{19}$$

Now we prove the claim by the method of mathematical induction.
When $p = 1$ we have,

$$\begin{aligned} S_x^*(x_1, \cdots, x_1, x_2) &= S_x^*\big(F(x_0, y_0), \cdots, F(x_0, y_0), F(x_1, y_1)\big) \\ &\leq a\ S_x^*(x_0, \cdots, x_0, x_1) + b\ S_y^*(y_0, \cdots, y_0, y_1) \\ &\leq (a+b)\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)] \end{aligned}$$

and

$$\begin{aligned} S_y^*(y_1, \cdots, y_1, y_2) &= S_y^*(y_2, \cdots, y_2, y_1) \\ &= S_y^*\big(G(y_1, x_1), \cdots, G(y_1, x_1), G(y_0, x_0)\big) \\ &\leq a\ S_y^*(y_1, \cdots, y_1, y_0) + b\ S_x^*(x_1, \cdots, x_1, x_0) \\ &= a\ S_y^*(y_0, \cdots, y_0, y_1) + b\ S_x^*(x_0, \cdots, x_0, x_1) \\ &\leq (a+b)\ \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right] \end{aligned}$$

Therefore the claim is true for $p = 1$.
Now assume the claim for $p \leq m$ and check for $p = m + 1$.
Consider,

$$\begin{aligned} S_x^*(x_{m+1}, \cdots, x_{m+1}, x_{m+2}) &= S_x^*\big(F(x_m, y_m), \cdots, F(x_m, y_m), F(x_{m+1}, y_{m+1})\big) \\ &\leq\ a\ S_x^*(x_m, \cdots, x_m, x_{m+1}) + b\ S_y^*(y_m, \cdots, y_m, y_{m+1}) \\ &\leq a\ (a+b)^m\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)] \\ &\quad +b\ (a+b)^m\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)] \\ &= (a+b)^{m+1}\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)] \end{aligned}$$

Similarly,

$$S_y^*(y_{m+1}, \cdots, y_{m+1}, y_{m+2}) \quad \leq \quad (a+b)^{m+1}\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)]$$

https://doi.org/10.17993/3ctic.2022.112.81-97

Thus the claim is true for all $p \in \mathbb{N}$.
Next we prove that $\{x_p\}_{p\in\mathbb{N}}$ and $\{y_p\}_{p\in\mathbb{N}}$ are Cauchy sequences in X and Y respectively.
Consider for $p, q \in \mathbb{N}$ with $p < q$,

$$\begin{aligned}
&S^*_x(x_p, \cdots, x_p, x_q)\\
&\le (n-1)S^*_x(x_p, \cdots, x_p, x_{p+1}) + S^*_x(x_q, \cdots, x_q, x_{p+1})\\
&= (n-1)S^*_x(x_p, \cdots, x_p, x_{p+1}) + S^*_x(x_{p+1}, \cdots, x_{p+1}, x_q)\\
&\le (n-1)S^*_x(x_p, \cdots, x_p, x_{p+1}) + (n-1)S^*_x(x_{p+1}, \cdots, x_{p+1}, x_{p+2})\\
&\qquad\qquad + S^*_x(x_q, \cdots, x_q, x_{p+2})\\
&= (n-1)\Big[S^*_x(x_p, \cdots, x_p, x_{p+1}) + S^*_x(x_{p+1}, \cdots, x_{p+1}, x_{p+2})\Big]\\
&\qquad\qquad + S^*_x(x_{p+2}, \cdots, x_{p+2}, x_q)\\
&\qquad\qquad \cdots\\
&\qquad\qquad \cdots\\
&\qquad\qquad \cdots\\
&\le (n-1)\Big[S^*_x(x_p, \cdots, x_p, x_{p+1}) + S^*_x(x_{p+1}, \cdots, x_{p+1}, x_{p+2})\\
&\quad + \cdots + S^*_x(x_{q-2}, \cdots, x_{q-2}, x_{q-1})\Big] + S^*_x(x_q, \cdots, x_q, x_{q-1})\\
&= (n-1) \sum_{i=p}^{q-2} S^*_x(x_i, \cdots, x_i, x_{i+1}) + S^*_x(x_{q-1}, \cdots, x_{q-1}, x_q)\\
&\le (n-1) \sum_{i=p}^{q-2} (a+b)^i \; [S^*_x(x_0, \cdots, x_0, x_1) + S^*_y(y_0, \cdots, y_0, y_1)]\\
&\qquad\qquad + (a+b)^{q-1} \; [S^*_x(x_0, \cdots, x_0, x_1) + S^*_y(y_0, \cdots, y_0, y_1)]
\end{aligned}$$

$$\begin{aligned}
&= (n-1) \; [S^*_x(x_0, \cdots, x_0, x_1) + S^*_y(y_0, \cdots, y_0, y_1)] \sum_{i=p}^{q-2} (a+b)^i\\
&\quad + (a+b)^{q-1} \; [S^*_x(x_0, \cdots, x_0, x_1) + S^*_y(y_0, \cdots, y_0, y_1)]\\
&\le (n-1) \; \frac{(a+b)^p}{1-a-b} \; [S^*_x(x_0, \cdots, x_0, x_1) + S^*_y(y_0, \cdots, y_0, y_1)]\\
&\quad + (a+b)^{q-1} \; [S^*_x(x_0, \cdots, x_0, x_1) + S^*_y(y_0, \cdots, y_0, y_1)]\\
&\to 0 \; as \; p, q \to \infty, \quad \text{since } a+b < 1.
\end{aligned}$$

Thus, $\{x_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in X.
Similarly we have,

$$\begin{aligned}
S^*_y(y_p, \cdots, y_p, y_q) \;\; &\le \;\; (n-1) \; \frac{(a+b)^p}{1-a-b} \; [S^*_x(x_0, \cdots, x_0, x_1) + S^*_y(y_0, \cdots, y_0, y_1)]\\
&\quad + (a+b)^{q-1} \; [S^*_x(x_0, \cdots, x_0, x_1) + S^*_y(y_0, \cdots, y_0, y_1)]\\
&\quad \to 0 \; as \; p, q \to \infty, \quad \text{since } a+b < 1.
\end{aligned}$$

Thus, $\{y_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in Y.
Since X and Y are complete S^* metric spaces, there exist $x \in X$ and $y \in Y$ such that

$$\lim_{p\to\infty} x_p = x \quad \text{and} \quad \lim_{p\to\infty} y_p = y \tag{20}$$

As in the Theorem 1, by assuming the continuity of F and G we can prove that $(x, y) \in X \times Y$ is an FG- coupled fixed point.
Now, suppose that X and Y have the properties (i) and (ii) respectively.
Since $\{x_m\}$ is increasing in X and $\{y_m\}$ is decreasing in Y and by using (20) we have $\forall\, m \in \mathbb{N}$ $x_m \preceq_{P_1} x$

https://doi.org/10.17993/3ctic.2022.112.81-97

and $y_m \succeq_{P_2} y$
That is by the definition of partial order on $X \times Y$ and $Y \times X$ we have

$$(x_m, y_m) \leq_{12} (x, y) \text{ and } (y_m, x_m) \geq_{21} (y, x)$$

Now consider,

$$\begin{aligned}
&S_x^*\big(x, \cdots, x, F(x,y)\big)\\
&\leq (n-1)\ S_x^*\big(x, \cdots, x, F(x_p, y_p)\big) + S_x^*\big(F(x,y), \cdots, F(x,y), F(x_p, y_p)\big)\\
&= (n-1)\ S_x^*\big(x, \cdots, x, F(x_p, y_p)\big) + S_x^*\big(F(x_p,y_p), \cdots, F(x_p,y_p), F(x, y)\big)\\
&\leq (n-1)\ S_x^*(x, \cdots, x, x_{p+1}) + a\ S_x^*(x_p, \cdots, x_p, x) + b\ S_y^*(y_p, \cdots, y_p, y)\\
&\to 0 \ as\ p \to \infty
\end{aligned}$$

Thus, $F(x, y) = x$.
Similarly,

$$\begin{aligned}
&S_y^*\big(y, \cdots, y, G(y,x)\big)\\
&\leq (n-1)\ S_y^*\big(y, \cdots, y, G(y_p, x_p)\big) + S_y^*\big(G(y,x), \cdots, G(y,x), G(y_p, x_p)\big)\\
&= (n-1)\ S_y^*(y, \cdots, y, y_{p+1}) + S_y^*\big(G(y,x), \cdots, G(y,x), G(y_p, x_p)\big)\\
&\leq (n-1)\ S_y^*(y, \cdots, y, y_{p+1}) + a\ S_y^*(y, \cdots, y, y_p) + b\ S_x^*(x, \cdots, x, x_p)\\
&\to 0 \ as\ p \to \infty
\end{aligned}$$

Thus, $G(y, x) = y$.
That is, $F(x, y) = x$ and $G(y, x) = y$.
Hence (x, y) is an FG- coupled fixed point.

Next we prove the uniqueness of FG- coupled fixed point.
Claim: for any two points $(x_1, y_1),\ (x_2, y_2) \in X \times Y$ which are comparable and for all $k \in \mathbb{N}$

$$S_x^*\big(F^k(x_1,y_1), \cdots, F^k(x_1,y_1), F^k(x_2,y_2)\big) \leq (a+b)^k \big[S_x^*(x_1, \cdots, x_1, x_2) + S_y^*(y_1, \cdots, y_1, y_2)\big] \tag{21}$$

and

$$S_y^*\big(G^k(y_1,x_1), \cdots, G^k(y_1,x_1), G^k(y_2,x_2)\big) \leq (a+b)^k \big[S_x^*(x_1, \cdots, x_1, x_2) + S_y^*(y_1, \cdots, y_1, y_2)\big] \tag{22}$$

Without loss of generality assume that $(x_1, y_1) \leq_{12} (x_2, y_2)$.
That is by the definition of partial order we have $x_1 \preceq_{P_1} x_2$ and $y_2 \preceq_{P_2} y_1$
By the mixed monotone property of F and G we have

$$\begin{aligned}
F(x_1, y_1) &\preceq_{P_1} F(x_2, y_1)\\
&\preceq_{P_1} F(x_2, y_2)
\end{aligned}$$

and

$$\begin{aligned}
G(y_1, x_1) &\succeq_{p_2} G(y_2, x_1)\\
&\succeq_{p_2} G(y_2, x_2)
\end{aligned}$$

Again by the mixed monotone property of F and G we have

$$\begin{aligned}
F^2(x_1, y_1) &= F\big(F(x_1,y_1), G(y_1,x_1)\big)\\
&\preceq_{P_1} F\big(F(x_2,y_2), G(y_1,x_1)\big)\\
&\preceq_{P_1} F\big(F(x_2,y_2), G(y_2,x_2)\big)\\
&= F^2(x_2, y_2)
\end{aligned}$$

https://doi.org/10.17993/3ctic.2022.112.81-97

and

$$\begin{aligned} G^2(y_1,x_1) &= G\big(G(y_1,x_1),F(x_1,y_1)\big)\\ &\succeq_{P_2} G\big(G(y_2,x_2),F(x_1,y_1)\big)\\ &\succeq_{P_2} G\big(G(y_2,x_2),F(x_2,y_2)\big)\\ &= G^2(y_2,x_2) \end{aligned}$$

Continuing like this we get $\forall\ m \in \mathbb{N}\cup\{0\}$,

$$F^m(x_1,y_1) \preceq_{P_1} F^m(x_2,y_2) \text{ and } G^m(y_1,x_1) \succeq_{P_2} G^m(y_2,x_2)$$

That is by the definition of partial order on $X \times Y$ and $Y \times X$ we have $\forall\ m \in \mathbb{N}\cup\{0\}$

$$\big(F^m(x_1,y_1),G^m(y_1,x_1)\big) \leq_{12} \big(F^m(x_2,y_2),G^m(y_2,x_2)\big)$$

and

$$\big(G^m(y_1,x_1),F^m(x_1,y_1)\big) \geq_{21} \big(G^m(y_2,x_2),F^m(x_2,y_2)\big)$$

Now, we prove the claim by the method of mathematical induction.
When $k=1$ we have,

$$\begin{aligned} &S_x^*\big(F(x_1,y_1),\cdots,F(x_1,y_1),F(x_2,y_2)\big)\\ &\quad\leq a\ S_x^*(x_1,\cdots,x_1,x_2)+b\ S_y^*(y_1,\cdots,y_1,y_2)\\ &\quad\leq (a+b)[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)] \end{aligned}$$

and

$$\begin{aligned} &S_y^*\big(G(y_1,x_1),\cdots,G(y_1,x_1),G(y_2,x_2)\big)\\ &\quad= S_y^*\big(G(y_2,x_2),\cdots,G(y_2,x_2),G(y_1,x_1)\big)\\ &\quad\leq a\ S_y^*(y_2,\cdots,y_2,y_1)+b\ S_x^*(x_2,\cdots,x_2,x_1)\\ &\quad\leq a\ S_y^*(y_1,\cdots,y_1,y_2)+b\ S_x^*(x_1,\cdots,x_1,x_2)\\ &\quad\leq (a+b)[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)] \end{aligned}$$

Therefore claim is true for $k=1$.
Now assume the claim for $k \leq m$ and check for $k=m+1$.
Consider,

$$\begin{aligned} &S_x^*\big(F^{m+1}(x_1,y_1),\cdots,F^{m+1}(x_1,y_1),F^{m+1}(x_2,y_2)\big)\\ &=S_x^*\Big(F\big(F^m(x_1,y_1),G^m(y_1,x_1)\big),\cdots,F\big(F^m(x_1,y_1),G^m(y_1,x_1)\big),F\big(F^m(x_2,y_2),G^m(y_2,x_2)\big)\Big)\\ &\leq aS_x^*\big(F^m(x_1,y_1),\cdots,F^m(x_1,y_1),F^m(x_2,y_2)\big)+bS_y^*\big(G^m(y_1,x_1),\cdots,G^m(y_1,x_1),G^m(y_2,x_2)\big)\\ &\leq a\ (a+b)^m[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)]\\ &\quad+b\ (a+b)^m[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)]\\ &=(a+b)^{m+1}[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)] \end{aligned}$$

Similarly we get,

$$\begin{aligned} S_y^*\big(G^{m+1}(y_1,x_1),\cdots,G^{m+1}(y_1,x_1),G^{m+1}(y_2,x_2)\big) \leq\ &(a+b)^{m+1}[S_x^*(x_1,\cdots,x_1,x_2)\\ &+S_y^*(y_1,\cdots,y_1,y_2)] \end{aligned}$$

Thus the claim is true for all $k \in \mathbb{N}$.
Suppose that $(x,y),(x^*,y^*)$ be any two FG- coupled fixed points.
That is

$$F(x,y)=x \text{ and } G(y,x)=y \tag{23}$$

 https://doi.org/10.17993/3ctic.2022.112.81-97

and

$$F(x^*,y^*)=x^* \quad \text{and} \quad G(y^*,x^*)=y^* \tag{24}$$

Case 1: If (x,y) and (x^*,y^*) are comparable, then

$$\begin{aligned} S_x^*(x,\cdots,x,x^*) &= S_x^*\big(F(x,y),\cdots,F(x,y),F(x^*,y^*)\big) \\ &\leq a\ S_x^*(x,\cdots,x,x^*)+b\ S_y^*(y,\cdots,y,y^*) \end{aligned} \tag{25}$$

and

$$\begin{aligned} S_y^*(y,\cdots,y,y^*) &= S_y^*\big(G(y,x),\cdots,G(y,x),G(y^*,x^*)\big) \\ &\leq a\ S_y^*(y,\cdots,y,y^*)+b\ S_x^*(x,\cdots,x,x^*) \end{aligned} \tag{26}$$

Adding (25) and (26) we get

$$S_x^*(x,\cdots,x,x^*)+S_y^*(y,\cdots,y,y^*) \leq (a+b)\ [S_x^*(x,\cdots,x,x^*)+S_y^*(y,\cdots,y,y^*)]$$

which implies that $S_x^*(x,\cdots,x,x^*)+S_y^*(y,\cdots,y,y^*)=0$ since $a+b<1$.
Therefore we have, $x=x^*$ and $y=y^*$.
Case 2: Suppose (x,y) and (x^*,y^*) are not comparable.
Then by the hypothesis there exist $(u,v)\in X\times Y$ which is comparable to both (x,y) and (x^*,y^*).
Consider,

$$S_x^*(x,\cdots,x,x^*) \quad = \quad S_x^*\big(F^k(x,y),\cdots,F^k(x,y),F^k(x^*,y^*)\big)$$

$$\begin{aligned} &\leq (n-1)\ S_x^*\big(F^k(x,y),\cdots,F^k(x,y),F^k(u,v)\big)+S_x^*\big(F^k(x^*,y^*),\cdots,F^k(x^*,y^*),F^k(u,v)\big) \\ &\leq (n-1)\ (a+b)^k[S_x^*(x,\cdots,x,u)+S_y^*(y,\cdots,y,v)] \\ &\qquad +(a+b)^k[S_x^*(x^*,\cdots,x^*,u)+S_y^*(y^*,\cdots,y^*,v)] \\ &\to 0\ as\ k\to\infty \text{ since } a+b<1 \end{aligned}$$

Thus we have $x=x^*$.
Consider,

$$\begin{aligned} &S_y^*(y,\cdots,y,y^*) \\ &= S_y^*\big(G^k(y,x),\cdots,G^k(y,x),G^k(y^*,x^*)\big) \\ &\leq (n-1)\ S_y^*\big(G^k(y,x),\cdots,G^k(y,x),G^k(v,u)\big)+S_y^*\big(G^k(y^*,x^*),\cdots,G^k(y^*,x^*),G^k(v,u)\big) \\ &\leq (n-1)\ (a+b)^k[S_x^*(x,\cdots,x,u)+S_y^*(y,...,y,v)] \\ &\qquad +(a+b)^k[S_x^*(x^*,\cdots,x^*,u)+S_y^*(y^*,\cdots,y^*,v)] \\ &\to 0\ as\ k\to\infty \text{ since } a+b<1 \end{aligned}$$

Thus by Definition 1 (ii) we have $y=y^*$.
Therefore, $x=x^*$ and $y=y^*$
Hence the proof.

Remark 3. *By taking $n=2$, $a=b=\frac{k}{2}$, $X=Y$ and $F=G$ and assuming condition (I) and (II) of the above theorem we get the Theorems 2.1 and 2.2 of Bhaskar and Lakshmikantham [?] respectively as corollaries to our results.*

Remark 4. *By taking $n=2$, $a=b=\frac{k}{2}$, $X=Y$ and $F=G$ and assuming condition (I) and (III) of the above theorem we get the Theorems 2.4 of Bhaskar and Lakshmikantham [?] as a corollary to our results.*

We illustrate the above theorem with the following example.

 https://doi.org/10.17993/3ctic.2022.112.81-97

Example 2. *Let* $X = [0, \infty)$ *and* $Y = (-\infty, 0]$ *with the usual order in* $\mathbb{R}$
Consider the S^* *metric on both* X *and* Y *as*

$$S^*(a_1, \cdots, a_n) = \sum_{i=1}^{n} \sum_{i<j} |a_i - a_j|$$

Define $F : X \times Y \to X$ *and* $G : Y \times X \to Y$ *by*

$$F(x, y) = \frac{2x - 3y}{7n} \quad \text{and} \quad G(y, x) = \frac{2y - 3x}{7n}$$

For $x, u \in X$ *and* $y, v \in Y$ *with* $x \leq u$ *and* $y \geq v$ *we have*

$$\frac{2x - 3y}{7n} \leq \frac{2u - 3y}{7n}, \quad \frac{2y - 3x}{7n} \geq \frac{2y - 3u}{7n}$$

and

$$\frac{2x - 3y}{7n} \leq \frac{2x - 3v}{7n}, \quad \frac{2y - 3x}{7n} \geq \frac{2v - 3x}{7n}$$

That is,

$$F(x, y) \leq F(u, y), \; G(y, x) \geq G(y, u) \quad \text{and} \quad F(x, y) \leq F(x, v), \; G(y, x) \geq G(v, x)$$

Therefore F *and* G *are mixed monotone mappings.*
Next we show that F *and* G *satisfy the contractive type conditions (14) and (15)*

$$\begin{aligned} S^*(F(x, y), \cdots, F(x, y), F(u, v)) &= (n-1) \left| \frac{2x - 3y}{7n} - \frac{2u - 3v}{7n} \right| \\ &\leq \frac{2}{7n} (n-1)|x - u| + \frac{3}{7n} (n-1)|y - v| \\ &= \frac{2}{7n} S^*(x, \cdots, x, u) + \frac{3}{7n} S^*(y, \cdots, y, v) \end{aligned}$$

and

$$\begin{aligned} S^*(G(y, x), \cdots, G(y, x), G(v, u)) &= (n-1) \left| \frac{2y - 3x}{7n} - \frac{2v - 3u}{7n} \right| \\ &\leq \frac{2}{7n} (n-1)|y - v| + \frac{3}{7n} (n-1)|x - u| \\ &= \frac{2}{7n} S^*(y, \cdots, y, v) + \frac{3}{7n} S^*(x, \cdots, x, u) \end{aligned}$$

Therefore F *and* G *satisfy the contractive type conditions (14) and (15) for* $a = \frac{2}{7n}$ *and* $b = \frac{3}{7n}$.
Here $(0, 0)$ *is the unique* FG*- coupled fixed point.*

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