APPLICATIONS OF FIXED POINT THEOREMS TO SOLUTIONS OF OPERATOR EQUATIONS IN BANACH SPACES

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ABSTRACT

In this paper we use Browder's and Gohde's fixed point theorem, Kirk's fixed point theorem and the Sadovskii fixed point theorem to obtain solutions of operator equations in Banach spaces.

KEYWORDS

fixed point, Banach spaces

1 INTRODUCTION

Perhaps the most famous fixed-point theorem is the Banach's contraction principle which has several applications. Motivated by this we have considered in this review article, applications of other well-known fixed-point theorems in various kinds of Banach spaces. This article should be of interest to mathematicians working in the fields of fixed-point theory and functional analysis.
In Section 1 we apply the Browder's and Göhde's fixed point theorem for the existence of solutions of operator equations involving asymptotically nonexpansive mappings in uniformly convex Banach spaces. In Section 2 we apply Kirk's fixed point theorem for the existence of solutions of the operator equation $x - Tx = f$ in reflexive Banach spaces and in Section 3 we apply the Sadovskii fixed point theorem for existence of solutions of the operator equation $x - Tx = f$ in arbitrary Banach spaces.

2 Application of Browder's and Göhde's fixed point theorem

Definition 1. *[1] A mapping T from a metric space (X, d) into another metric space (Y, ρ) is said to satisfy Lipschitz condition on X if there exists a constant $L > 0$ such that*

$$\rho(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. If L is the least number for which Lipschitz condition holds, then L is called Lipschitz constant. If $L = 1$, the mapping is said to be nonexpansive.

Definition 2. *[2] Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if for each $n \in \mathbb{N}$ there exists a positive constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$||T^n x - T^n y|| \leq k_n ||x - y||$$

for all $x, y \in K$.

The Browder's and Göhde's fixed point theorem is as follows:

Theorem 1. *[3] Let X be a uniformly convex Banach space and C a nonempty, closed, convex and bounded subset of X. Then every nonexpansive mapping $T : C \to C$ has a fixed point in C.*

We now state the main theorem of Section 1.

Theorem 2. *Let X be a uniformly convex Banach space and K a nonempty subset of X. Let $T : K \to K$ be an asymptotically nonexpansive mapping and $f_n \in K$, then the operator equation*

$$k_n x = T^n x + f_n$$

where $n \in \mathbb{N}$ and k_n is the Lipschitz constant of the iterates T^n, has a solution if and only if, for any $x_1 \in K$, the sequence of iterates $\{x_n\}$ in K defined by

$$k_n x_{n+1} = T^n x_n + f_n$$

$n \in \mathbb{N}$ is bounded.

Proof. For every $n \in \mathbb{N}$, let T_{f_n} be defined to be a mapping from K into K by

$$T_{f_n}(u) = \frac{1}{k_n}[T^n u + f_n].$$

Then $u_n \in K$ is a solution of

$$x = \frac{1}{k_n}[T^n x + f_n]$$

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if and only if u_n is a fixed point of T_{f_n}. Since T is asymptotically nonexpansive it follows that T_{f_n} is nonexpansive for all $n \in \mathbb{N}$.

$$||T_{f_n}(x) - T_{f_n}(y)|| = \frac{1}{k_n}||T^n(x) - T^n(y)|| \leq ||x - y||.$$

Suppose T_{f_n} has a fixed point $u_n \in K$. Then

$$||x_{n+1} - u_n|| = ||\frac{1}{k_n}[T^n x_n + f_n] - u_n|| = ||T_{f_n}(x_n) - T_{f_n}(u_n)|| \leq ||x_n - u_n||,$$

T_{f_n} being nonexpasive. Since $\{||x_n - u_n||\}$ is non-increasing, hence $\{x_n\}$ is bounded. Conversely, suppose that$\{x_n\}$ is bounded. Let $d = diam(\{x_n\})$ and

$$B_d[x] = \{y \in K : ||x - y|| \leq d\}$$

for each $x \in K$. Set

$$C_n = \bigcap_{i \geq n} B_d[x_i] \subset K.$$

Hence C_n is a nonempty, convex set for each $n \in \mathbb{N}$. Now we claim that $T_{f_n}(C_n) \subset C_{n+1}$. Let $y \in B_d[x_n]$ which implies $||y - x_n|| \leq d$. Since T_{f_n} is nonexpansive, we get

$$||T_{f_n}(y) - T_{f_n}(x_n)|| \leq d$$

$$||\frac{1}{k_n}[T^n(y) + f_n] - \frac{1}{k_n}[T^n(x_n) + f_n]|| \leq d$$

or

$$||\frac{1}{k_n}[T^n(y) + f_n] - x_{n+1}|| \leq d$$

or

$$\frac{1}{k_n}[T^n(y) + f_n] \in B_d[x_{n+1}]$$

giving

$$T_{f_n}(y) \in B_d[x_{n+1}]$$

proving that $T_{f_n}(C_n) \subset C_{n+1}$.

Let $C = \overline{\bigcup_{n \in \mathbb{N}} C_n}$. Since C_n increases with n, C is a closed, convex and bounded subset of K. We can easily see that T_{f_n} maps C into C.

$$T_{f_n}(C) = T_{f_n}(\overline{\bigcup_{n \in \mathbb{N}} C_n}) \subseteq \overline{T_{f_n}(\bigcup_{n \in \mathbb{N}} C_n)} = \overline{\bigcup_{n \in \mathbb{N}} T_{f_n}(C_n)} \subseteq \overline{\bigcup_{n \in \mathbb{N}} C_{n+1}} = C.$$

Applying the Browder's and Göhde's theorem to T_{f_n} and C we get a fixed point of T_{f_n} in C. Since $C \subset K$, we obtain a fixed point of T_{f_n} in K.

3 Application of Kirk's Fixed Point Theorem

Let us recall some definitions and results that we shall require for the proof of the Main Theorem of Section 2.

Definition 3. *[2] Let (X, ρ) and (M, d) be metric spaces. A mapping $f : X \to M$ is said to be nonexpansive if for each $x, y \in X$,*

$$d(f(x), f(y)) \leq \rho(x, y).$$

Definition 4. *[1] A convex subset K of a Banach space X is said to have normal structure if each bounded, convex subset S of K with diam $S > 0$ contains a nondiametral point.*

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The following theorem gives application of the Browder-Göhde-Kirk's theorem for the existence of solutions of the operator equation

$$x - Tx = f.$$

It is known that every uniformly convex Banach space is reflexive. We generalize the theorem below to reflexive Banach spaces using Kirk's fixed point theorem

Theorem 3. *[3] Let X be a uniformly convex Banach space, f an element in X and $T : X \to X$ a nonexpansive mapping, then the operator equation*

$$x - Tx = f$$

has a solution x if and only if for any $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ in X defined by $x_{n+1} = Tx_n + f$, $n \in \mathbb{N}_0$ is bounded.

Definition 5. *[1] A Banach space X is said to satisfy the Opial condition if whenever a sequence $\{x_n\}$ in X converges weakly to $x_0 \in X$, then*

$$\lim_{n\to\infty} \inf \|x_n - x_0\| < \lim_{n\to\infty} \inf \|x_n - x\|$$

for all $x \in X$, $x \neq x_0$.

Lemma 1. *[3] Let X be a reflexive Banach space with the Opial condition. Then X has normal structure.*

Lemma 2. *[3] A closed subspace of a reflexive Banach space is reflexive.*

Now we state the Kirk's fixed point theorem.

Theorem 4. *[3] Let X be a Banach space and C a nonempty weakly compact, convex subset of X with normal structure, then every nonexpansive mapping $T : C \to C$ has a fixed point.*

We state the main theorem of Section 2.

Theorem 5. *Let X be a reflexive Banach space satisfying Opial condition. Let $f \in X$ and $T : X \to X$ be a nonexpansive mapping. Then the operator equation*

$$x - Tx = f$$

has a solution x if and only if for any $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ in X defined by $x_{n+1} = Tx_n + f$, $n \in \mathbb{N}_0$ is bounded.

Proof. Let T_f be the mapping from X into X given by

$$T_f(u) = Tu + f.$$

Then u is a solution of

$$x - Tx = f$$

if and only if u is a fixed point of T_f. Clearly T_f is nonexpansive. Suppose T_f has a fixed point $u \in X$. Then for all $n \in \mathbb{N}$,

$$\|x_{n+1} - u\| \leq \|x_n - u\|.$$

Hence $\{x_n\}$ is bounded.

Conversely, suppose that $\{x_n\}$ is bounded. Let $d = diam(\{x_n\})$ and

$$B_d[x] = \{y \in X : \ \|x - y\| \leq d\}$$

for each $x \in X$. Set $C_n = \bigcap_{i \geq n} B_d[x_i]$. Then C_n is a nonempty convex set for each n, and

$$T_f(C_n) \subset C_{n+1}.$$

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Let C be the closure of the union of C_n for $n \in \mathbb{N}$,

$$C = \overline{\bigcup_{n \in \mathbb{N}} C_n}.$$

Since C_n increases with n, C is a closed, convex and bounded subset of X. It is known that [1] bounded, closed and convex subsets of reflexive Banach spaces are weakly compact, hence we get that C is weakly compact.

Now since

$$T_f(C) = T_f(\overline{\bigcup C_n}) \subseteq \overline{T_f(\bigcup C_n)} = \overline{\bigcup T_f(C_n)} \subseteq \overline{\bigcup C_{n+1}} = C,$$

we get that T_f maps C into itself. By Lemma 1.3.6, C is a reflexive Banach space. Now X satisfies Opial condition and C being a closed subset of X, will also satisfy Opial condition. Hence by Lemma 1.3.5, C has normal structure. Finally, applying Kirk's fixed point theorem we get that T_f has a fixed point in C which proves the theorem.

4 Application of Sadovskii Fixed Point Theorem

We recall some definitions

Definition 6. *[1] Let (M, ρ) denote a complete metric space and let $\mathfrak{B}$ denote the collection of nonempty and bounded subsets of M. Define the Kuratowski measure of noncompactness $\alpha : \mathfrak{B} \to \mathbb{R}^+$ by taking for $A \in \mathfrak{B}$,*
$\alpha(A)$=inf$\{\epsilon > 0$ A is contained in the union of a finite number of sets in $\mathfrak{B}$ each having diameter less than $\epsilon\}$.
If M is a Banach space the function α has the following properties for $A, B \in \mathfrak{B}$
1. $\alpha(A) = 0 \Leftrightarrow \overline{A}$ is compact,
2. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

Definition 7. *[2] Let K be a subset of a metric space M. A mapping $T : K \to M$ is said to be condensing if T is bounded and continuous and if*

$$\alpha(T(D)) < \alpha(D)$$

for all bounded subsets D of M for which $\alpha(D) > 0$.

We state the Sadovskii fixed point theorem.

Theorem 6. *[2] Let K be a nonempty, bounded closed and convex subset of a Banach space and let $T : K \to K$ be a condensing mapping, then T has a fixed point.*

The main result of section 3 is the following:

Theorem 7. *Let X be an arbitrary Banach space, let $f \in X$ and $T : X \to X$ be a condensing mapping, then the operator equation*

$$x - Tx = f$$

has a solution if and only if for any $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ in X, defined by $x_{n+1} = Tx_n + f$, $n \in \mathbb{N}_0$ is bounded.

Proof. Let the mapping $T_f : X \to X$ be defined by

$$T_f(u) = Tu + f.$$

Then u is a solution of the operator equation

$$x - Tx = f$$

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if and only if u is a fixed point of T_f.
Since T is bounded and continuous, T_f is also bounded and continuous. Using the properties of the Kuratowski measure of noncompactness, for all bounded subsets D of X, we have

$$\alpha(T_f(D)) = \alpha(T(D) + \{f\}) \leq \alpha(T(D)) + \alpha(\{f\}).$$

Since $\{f\}$ is compact, $\overline{\{f\}}$ is compact, implying $\alpha(\{f\}) = 0$, giving

$$\alpha(T_f(D)) \leq \alpha(T(D)) < \alpha(T(D)).$$

Since T is condensing mapping and it follows that T_f is a condensing mapping.
Suppose T_f has a fixed point u in X. Then for all $n \in \mathbb{N}$, since T_f is a continuous mapping being condensing, we get

$$\|x_{n+1} - u\| = \|Tx_n + f - u\| = \|T_f(x_n) - T_f(u)\| \leq \|x_n - u\|.$$

Hence $\{x_n\}$ is bounded.

Conversely, suppose that $\{x_n\}$ is bounded. Let $d = diam(\{x_n\})$ and for each $x \in X$

$$B_d[x] = \{y \in X : \|x - y\| \leq d\}.$$

Set $C_n = \bigcap_{i \geq n} B_d[x_i]$, then C_n is a nonempty convex set for each n. Using that T is a continuous mapping and the given Picard iteration, we have

$$\begin{aligned} y \in B_d[x_n] &\Rightarrow \|y - x_n\| \leq d \\ &\Rightarrow \|Ty - Tx_n\| \leq d \\ &\Rightarrow \|Ty - [x_{n+1} - f]\| \leq d \\ &\Rightarrow \|(Ty + f) - x_{n+1}\| \leq d \\ &\Rightarrow (Ty + f) \in B_d[x_{n+1}]. \end{aligned}$$

Applying this, we get the following

$$\begin{aligned} T_f(C_n) &= T_f(\bigcap_{i \geq n} B_d[x_i]) \\ &\subseteq \bigcap_{i \geq n} T_f(B_d[x_i]) \\ &= \bigcap_{i \geq n} \{T_f(y) : \|y - x_i\| \leq d\} \\ &= \bigcap_{i \geq n} \{(Ty + f) : \|y - x_i\| \leq d\} \\ &\subseteq \bigcap_{i \geq n+1} B_d[x_i] = C_{n+1}. \end{aligned}$$

Let us define

$$C = \overline{\bigcup_{n \in \mathbb{N}} C_n}.$$

Since C_n increases with n,

$$C_n \subset C_{n+1} \subset C_{n+2} \subset ....,$$

it follows that C is a closed, convex and bounded subset of X. Now we have

$$T_f(C) = T_f\left(\overline{\bigcup_{n \in \mathbb{N}} C_n}\right) \subseteq \overline{T_f\left(\bigcup_{n \in \mathbb{N}} C_n\right)} = \overline{\bigcup_{n \in \mathbb{N}} T_f(C_n)} \subseteq \overline{\bigcup_{n \in \mathbb{N}} C_{n+1}} = C$$

giving $T_f : C \to C$ since T_f is continuous mapping.

Finally, applying the Sadovskii fixed point theorem to T_f and C, we obtain that T_f has a fixed point in C which proves the theorem.

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