ANALYSIS OF A DISCRETE TIME QUEUEING-INVENTORY MODEL WITH BACK-ORDER OF ITEMS

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ABSTRACT

This paper analyses a discrete-time (s, S) *queueing inventory model with service time and back-order in inventory. The arrival of customers is assumed to be the Bernoulli process. Service time follows a geometric distribution. As soon as the inventory level reaches a pre-assigned level due to demands, an order for replenishment is placed. Replenishment time also follows a geometric distribution. When the inventory level reduces to zero due to the service of customers or non-replenishment of items, a maximum of* k *customers are allowed in the system and the remaining customers are assumed to be completely lost till the replenishment. Matrix-Analytic Method (MAM) is used to analyze the model. Stability conditions, various performance measures of the system, waiting-time distribution and reorder-time distribution are obtained. Numerical experiments are also incorporated.*

KEYWORDS

Discrete Time Queueing Inventory, Back-order, Cost Analysis, Matrix Analytic Method

1 INTRODUCTION

The first reported work on discrete-time queue was done by Meisling (1958). In that work, the researcher analyzed a single-server queueing system in which interarrival time and service time are geometrically distributed and as a limiting process, the results in a continuous system are derived. In queueing inventory, stock out generates penalty costs due to the loss and disappointment of customers. This will create perturbed demand and is considered in Schwartz (1966). Gross and Harris (1973) described the progress of an (s, S) inventory system with complete back-ordering and state-dependent lead times. The arrival is assumed to be a Poisson process and the lead time depends on the count of outstanding orders. Archibald (1981) discussed the continuous review (s, S) Policies which minimize the average stationary cost in an inventory system with constant lead time, fixed order cost, linear holding cost per unit time, linear penalty cost per unit short, discrete compound Poisson demand, lost sales and back-ordering. Krishnamoorthy and Islam (2004) analyzed an (s, S) Inventory model where demands form a Poisson process. When the inventory level approaches zero due to service, upcoming arrivals are transferred to a pool of finite capacity. Deepak et al. (2004) considered a queueing system in which work gets postponed due to the finiteness of the buffer. When the buffer of finite capacity is full, further demands are shifted to a pool of customers. A potential customer discovers the full buffer, and will opt for the pool with some probability, or else it will be lost forever.

Manuel (2007) analysed a continuous perishable (s, S) inventory model in which the arrival is under a Markovian arrival process. The expiry of items in the stock and the lead time follow independent exponential distributions. Demands that arrive during a period when the items are out of stock enter either a pool of finite capacity or are lost forever. Sivakumar (2007) studied a continuous perishable inventory model in which the demand is following a Markovian arrival process. The inventoried items have lifetimes that are assumed to follow an exponential distribution. The demands that occur during stock-out periods either enter a pool that has a finite capacity or leaves the system. Any demand that arrives when the pool is full and the inventory level is zero, is also assumed to be lost.

Sivakumar (2009) studied a continuous review perishable (s, S) inventory model in which the arrival is in accordance with a Markovian arrival process. First In First Out discipline is used for the selection of customers from the pool when the inventory level is above a pre-assigned positive value N, which is at most the reorder level. The combined probability distribution of inventory level and the number of demands, the system characteristics and expected total cost are obtained in the steady-state. Sivakumar (2012) also considered a discrete-time inventory system in which demands follow a Markovian arrival process. The replenishment of inventory is according to an (s, S) policy. The lead time follows a phase-type distribution. The demands that take place in stock-out periods either enter a pool or go away from the system with a pre-assigned probability. When the pool has no space and the inventory level is dry, further demands that occur are considered to be lost. For a discussion of discrete-time queueing models, one can refer to Alfa (2002, 2001) and Meisling (1958). The present paper generalizes a work reported in the Ph.D. thesis of Deepthi (2013). The work in this paper is analysed by using the Matrix-Analytic Method discussed in Neuts (1994).

The model in this paper has many applications in real-life situations. For instance, consider an automobile showroom that accepts orders and delivers the vehicles whenever there are vehicles in stock. Here the stock of vehicles can be considered inventory. If the items are exhausted due to service and non-replenishment, orders of at most k are accepted and remaining demands are assumed to be lost.

The rest of the paper is organized as follows. Section 2 provides mathematical modeling and analysis. The stability condition is derived in section 3. Steady-state probability vector and algorithmic analysis are discussed in sections 4 and 5 respectively. Some relevant performance measures are included in section 6. Section 7 analyses the waiting-time distribution of the potential customer. Reorder time distribution is incorporated in section 8. Section 9 illustrates numerical experiments.

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2 Mathematical Modeling and Analysis

The following are the assumptions and notations used in this model.

Assumptions

(i) Inter-arrival times follow a geometric distribution with parameter p

(ii) Service time follows a geometric distribution with parameter q

(iii) Up to k customers are allowed in the system when the inventory level is zero

(iv) Lead time is geometrically distributed with parameter r

Notations

$$N(n): \text{Number of customers in queue at an epoch } n$$
$$I(n): \text{Inventory level at the epoch } n$$

Then $\{(N(n), I(n)); n = 0, 1, 2, 3, ..\}$ is a Discrete Time Markov Chain(DTMC) with state space $\{(i,j); i \geq 0, 0 < j \leq S\} \cup \{(i,0) : 0 \leq i \leq k,\}$. Now, the transition probability matrix of the process has the form

$$P = \begin{array}{cc}
 & \begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & \dots & k-1 & k & k+1 & k+2 & \dots & . & \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ k-1 \\ k \\ k+1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} &
\begin{pmatrix}
C_1 & C_0 & 0 & & & & & & & & & \\
B_2 & B_1 & B_0 & & & & & & & & & \\
0 & B_2 & B_1 & B_0 & & & & & & & & \\
 & & & & & & & & & & & \\
 & \vdots & \vdots & \ddots & \ddots & \ddots & & & & & & \\
 & & & & B_2 & B_1 & B_0 & & & & & \\
 & & & & . & B_2 & D_1 & D_0 & & & & \\
 & & & & & & D_2 & A_1 & A_0 & & & \\
 & & & & & & K & A_2 & A_1 & A_0 & & \\
 & & & & & & K & & A_2 & A_1 & A_0 & \\
 & & & & & & K & & & A_2 & A_1 & A_0 \\
 & & & & & & \vdots & & & \ddots & \ddots & \ddots
\end{pmatrix}
\end{array}$$

where the blocks $C_0, C_1, B_0, B_1, B_2, D_0, D_1, D_2, K, A_0, A_1,$ and A_2 are given by

$$C_0 = \begin{array}{cc}
 & \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\
\begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} &
\begin{pmatrix}
p\bar{r} & & & & & & pr \\
 & p\bar{r} & & & & & pr \\
 & & \ddots & & & & \\
 & & & p\bar{r} & & & pr \\
 & & & & p & & \\
 & & & & & \ddots & \\
 & & & & & & p
\end{pmatrix}
\end{array}$$

$$C_1 = \begin{array}{cc}
 & \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\
\begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} &
\begin{pmatrix}
\bar{p}\bar{r} & & & & & & \bar{p}r \\
 & \bar{p}\bar{r} & & & & & \bar{p}r \\
 & & \ddots & & & & \\
 & & & \bar{p}\bar{r} & & & \bar{p}r \\
 & & & & \bar{p} & & \\
 & & & & & \ddots & \\
 & & & & & & \bar{p}
\end{pmatrix}
\end{array}$$

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$$B_0 = \begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} p\bar{r} & & & & & & pr \\ & p\bar{q}\bar{r} & & & & & p\bar{q}r \\ & & \ddots & & & & \\ & & & p\bar{q}\bar{r} & & & p\bar{q}r \\ & & & & p\bar{q} & & \\ & & & & & \ddots & \\ & & & & & & p\bar{q} \end{pmatrix} \end{array}$$

$$B_1 = \begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} \bar{p}\bar{r} & & & & & & \bar{p}r \\ pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & & & \bar{p}\bar{q}r + pqr \\ & & \ddots & & & & \\ & & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & \bar{p}\bar{q}r + pqr \\ & & & pq & \bar{p}\bar{q} & & \\ & & & & & \ddots & \\ & & & & & pq & \bar{p}\bar{q} \end{pmatrix} \end{array}$$

$$B_2 = \begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} 0 & & & & & & \\ \bar{p}q\bar{r} & & & & & & \bar{p}qr \\ & \ddots & & & & & \\ & & \bar{p}q\bar{r} & & & & \bar{p}qr \\ & & & \bar{p}q & & & \\ & & & & \ddots & & \\ & & & & & \bar{p}q & 0 \end{pmatrix} \end{array}$$

$$D_0 = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} 0 & & & & & & 0 \\ p\bar{q}\bar{r} & & & & & & p\bar{q}r \\ & & \ddots & & & & \\ & & & p\bar{q}\bar{r} & & & p\bar{q}r \\ & & & & p\bar{q} & & \\ & & & & & \ddots & \\ & & & & & & p\bar{q} \end{pmatrix} \end{array}$$

$$D_1 = \begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} \bar{r} & & & & & & r \\ pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & & & \bar{p}\bar{q}r + pqr \\ & & \ddots & & & & \\ & & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & \bar{p}\bar{q}r + pqr \\ & & & pq & \bar{p}\bar{q} & & \\ & & & & & \ddots & \\ & & & & & pq & \bar{p}\bar{q} \end{pmatrix} \end{array}$$

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$$D_2 = \begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ s+1 \\ s+2 \\ \vdots \\ S \end{array} & \begin{pmatrix} q\bar{r} & & & & & & qr \\ & \bar{p}q\bar{r} & & & & & \bar{p}qr \\ & & \ddots & & & & \\ & & & \bar{p}q\bar{r} & & & \bar{p}qr \\ & & & & \bar{p}q & & \\ & & & & & \ddots & \\ & & & & & \bar{p}q & 0 \end{pmatrix} \end{array}$$

$$K = \begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} q\bar{r} & & & & & & qr \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix} \end{array}$$

$$A_0 = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} p\bar{q}\bar{r} & & & & & & p\bar{q}r \\ & p\bar{q}\bar{r} & & & & & p\bar{q}r \\ & & \ddots & & & & \\ & & & p\bar{q}\bar{r} & & & p\bar{q}r \\ & & & & p\bar{q} & & \\ & & & & & \ddots & \\ & & & & & & p\bar{q} \end{pmatrix} \end{array}$$

$$A_1 = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} \bar{p}\bar{q}\bar{r} & & & & & & \bar{p}\bar{q}r \\ pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & & & \bar{p}\bar{q}r + pqr \\ & \ddots & \ddots & & & & \vdots \\ & & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & \bar{p}\bar{q}r + pqr \\ & & & pq & \bar{p}\bar{q} & & \\ & & & & \ddots & \ddots & \\ & & & & & pq & \bar{p}\bar{q} \end{pmatrix} \end{array}$$

$$A_2 = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} 0 & & & & & & \\ \bar{p}q\bar{r} & & & & & & \bar{p}qr \\ & \ddots & & & & & \vdots \\ & & \bar{p}q\bar{r} & & & & \bar{p}qr \\ & & & \bar{p}q & & & \\ & & & & \ddots & & \\ & & & & & \bar{p}q & 0 \end{pmatrix} \end{array}$$

3 STABILITY AND STEADY-STATE ANAYSIS

Theorem 1. *The above Markov chain is stable if and only if* $p < q$

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Proof. Consider the matrix $A = A_0 + A_1 + A_2$. Then

$$A = \begin{array}{cc} & \begin{array}{ccccccc} 1 & \dots & s & s+1 & & \dots & S \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} q\bar{r} & \bar{q}\bar{r} & & & & & r \\ & \ddots & \ddots & & & & \\ & & q\bar{r} & \bar{q}\bar{r} & & & r \\ & & & q & \bar{q} & & \\ & & & & \ddots & \ddots & \\ & & & & & q & \bar{q} \end{pmatrix} \end{array}$$

Let πbe the steady state probability vector of A.Then the given Markov chain is stable if and only if $\boldsymbol{\pi}(A_1 + 2A_2)\mathbf{e} > 1$,where $\mathbf{e}$ is the column vector of ones of order S . On simplification, we get the condition $p < q$.

STEADY STATE PROBABILITY VECTOR

Let $\mathbf{x} = (x_0, x_1, \dots, x_{k-1}, x_k, \dots)$ be the steady state probability vector of P, where $x_i = \begin{cases} x_{i,j} & 0 \leq i \leq k, 0 \leq j \leq S \\ x_{i,j} & i > k, 1 \leq j \leq S \end{cases}$

Under the stability condition, x_i, for $i \geq k+1$, is given by

$$x_{k+1+r} = x_{k+1}R^r, \; r \geq 0$$

where R is the least non-negative root of the equation

$$R^2 A_2 + RA_1 + A_0 = R$$

with a value less than one for spectral radius. The vectors $x_0, x_1, \dots, x_{k+1}$ are given by solving

$$\left.\begin{array}{l} x_0 C_1 + x_1 B_2 = x_0 \\ x_{J-1}B_0 + x_j B_1 + x_{j+1}B_2 = x_j; (1 \leq j \leq k-1) \\ x_{k-1}B_0 + x_K D_1 + x_{k+1}[D_2 + RK(I-R)^{-1}] = x_k \\ x_k D_0 + x_{k+1}(A_1 + RA_2) = x_{k+1} \end{array}\right\} \tag{1}$$

subject to the normalizing condition

$$\left[\sum_{i=0}^{k} x_i + x_{k+1}(I-R)^{-1}\right]\mathbf{e} = 1 \tag{2}$$

EVALUATION OF THE TRUNCATION MATRIX R

The rate matrix R is given by $R = \lim_{n\to\infty} R_n$, where $R_{n+1} = \left(R_n^2 A_2 + A_0\right)(I - A_1)^{-1}$ and $R_0 = 0$. The iteration is usually stopped when $\mid (R_{n+1} - R_n) \mid_{ij} < \epsilon, \forall$ i,j

COMPUTATION OF BOUNDARY PROBABILITIES

Now the system (1) can be solved using the block Gauss-Seidel iterative method. The vectors $x_0, x_1, \dots, x_{k+1s}$ in the $(n+1)$th iteration are given by

$$\begin{array}{l} x_0(n+1) = x_1(n)B_2(I - C_1)^{-1} \\ x_i(n+1) = [x_{i+1}(n)B_2 + x_{i-1}(n+1)B_0](I - B_1)^{-1}; (1 \leq i \leq k-1) \\ x_k(n+1) = (x_{k-1}(n+1)B_0 + x_{k+1}(n)[D_2 + RK(I-R)^{-1}])[I - D_1]^{-1} \\ x_{k+1}(n+1) = x_k(n+1)D_0(I - A_1 - RA_2)^{-1} \end{array}$$

Each iteration is subject to the normalizing condition (2).

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4 SYSTEM PERFORMANCE MEASURES

In order to consider some performance measures of the system under steady state, we take $x_{i,0} = 0$ for $i > k$

(i) Expected level of inventory, ELI, is given by

$$ELI = \sum_{j=1}^{S} \sum_{i=0}^{\infty} j x_{i,j}$$

(ii) Expected number of customers, EC, is obtained by

$$EC = \sum_{j=1}^{S} \sum_{i=0}^{\infty} i x_{i,j}$$

(iii) Expected departure after completing the service, EDS is given by

$$EDS = q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{i,j}$$

(iv) The Expected reorder rate, ERR, is given by

$$ERR = q \sum_{i=1}^{\infty} x_{i,s+1}$$

(v) Expected replenishment rate

$$ERR = r \sum_{j=0}^{s} \sum_{i=0}^{\infty} x_{i,j}$$

(vi) Probability that the inventory level zero, PI_0, is given by

$$PI_0 = \sum_{i=0}^{\infty} x_{i,0}$$

(vii) Expected loss rate of customers

$$ELR = p x_{k0} + q\bar{r} \sum_{i=k+1}^{\infty} (i-k) x_{i+1,1}$$

(viii) Expected number of demands waiting in the system during stock out period, EW_0 is given by

$$EW_0 = \sum_{i=1}^{k} i x_{i,0}$$

(ix) Expected reordering quantity, ERQ is given by

$$ERQ = \sum_{j=0}^{s} (S-j) y_j, \text{where } y_j \text{ is the probability that inventory level is j when}$$

replenishment takes place

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4.1 WAITING TIME DISTRIBUTION

Here we assume the queue discipline as First In First Out (FIFO). Let V_n be the number of customers in the queue ahead of the arriving customer. Then $((V_n, I(n))n = 1, 2, \dots)$ is a discrete-time Markov process with state space
$\{(i, j), 0 \leq i < \infty, 0 \leq j \leq S\}$, under the assumption that k is large. The transition probability matrix is given by

$$P(W_q) = \begin{bmatrix} 1 & 0 & & & \\ t & T_1 & 0 & & \\ & T_2 & T_1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \end{bmatrix}, \quad T_1 = \begin{bmatrix} \bar{r} & & & & & r \\ & \bar{q}\bar{r} & & & & \bar{q}r \\ & & \ddots & & & \vdots \\ & & & \bar{q}\bar{r} & & \bar{q}r \\ & & & & \bar{q} & \\ & & & & & \ddots \\ & & & & & \bar{q} \end{bmatrix}_{(S+1)\times(S+1)},$$

$$T_2 = \begin{bmatrix} 0 & & & & & 0 \\ \bar{q}\bar{r} & & & & & \bar{q}r \\ & \ddots & & & & \vdots \\ & & \bar{q}\bar{r} & & & \bar{q}r \\ & & & \bar{q} & & \\ & & & & \ddots & \\ & & & & \bar{q} & 0 \end{bmatrix}_{(S+1)\times(S+1)} \qquad t = \begin{bmatrix} 0 \\ q \\ \vdots \\ q \end{bmatrix}_{(S+1)\times 1}$$

Suppose there are i customers in the system in front of an arriving customer and j number of items in the inventory is available to the arriving customer. Let T be the waiting time in queue. Then the upper limit for T with a certain probability is calculated as follows.

Consider $z_n = e_{ij}[P(W_q)]^n$, where e_{ij} is infinite row vector whose $(1 + i + j + Si)^{th}$ element is one remaining entries are zeros. Then $P(T \leq n) = z_n(1)$,the first entry of z_n

4.2 REORDER TIME DISTRIBUTION

In this section, we calculate the average time taken to fall inventory from S to s. If the current inventory level is $s + i$, then re-order will takes place only when the service of i customers is completed. Note that the distribution of time taken to reach inventory level to s from S is a discrete phase-type distribution having $(S - s)(S - s + 3)/2$ phases and the transition probability matrix given by

$$P^* = \begin{bmatrix} 1 & 0 \\ t & T \end{bmatrix}$$

$$\text{T}= \begin{array}{c} \\ s+1 \\ s+2 \\ \vdots \\ S-N-1 \\ S-1 \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} s+1 & s+2 & \dots & S-N-1 & S-1 & S \end{array} \\ \begin{pmatrix} H_1^1 & & & & & \\ H_2^2 & H_1^2 & & & & \\ & \ddots & \ddots & & & \\ & & & & & \\ & & & H_2^{S-s-1} & H_1^{S-s-1} & \\ & & & & H_2^{S-s} & H_1^{S-s} \end{pmatrix} \end{array}$$

$$\text{where } H_1^i = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ i-1 \\ i \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & \dots & i-1 & i \end{array} \\ \begin{pmatrix} \bar{p} & p & & & \\ & \bar{p}\bar{q} & p\bar{q} & & \\ & & \ddots & \ddots & \\ & & & p\bar{q} & p\bar{q} \\ & & & & q \end{pmatrix}_{(i+1)\times(i+1)} \end{array}$$

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$$H_2^i = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & \ldots & & & i-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ \\ i-1 \\ i \end{array} & \begin{pmatrix} 0 & 0 & & & & \\ \bar{p}q & pq & & & & \\ & & \ddots & \ddots & & \\ & & & & & \\ & & & & \bar{p}q & pq \\ & & & & & q \end{pmatrix}_{(i+1)\times i} \end{array} \quad \text{and} \quad t_{=} \begin{bmatrix} 0 \\ q_1 \\ \vdots \\ 0 \end{bmatrix}_{[(S-s)(S-s+3)/2]\times 1}$$

Therefore expected time to fall the inventory level from S to s due to service is $\alpha(I-T)^{-1}\mathbf{1}$, where $\mathbf{1}$ is the column vector of 1's of order $[(S-s)(S-s+3)/2]$. If α is the $[(S-s)(S-s+3)/2]$ row vector whose $[((S-s)*(S-s+1)+2)/2]^{th}$ entry is 1 and remaining entries are zero, then expected time of reorder is given by

$$ETR = \alpha(I-T)^{-1}\mathbf{1}$$

5 NUMERICAL EXPERIMENTS

5.1 COST FUNCTION

Define the expected total cost of the system per unit time as

$$ETC = C_0ERO + C_1(ERQ)(ERR) + C_2(ELI) + C_3(EW_0) + C_4(ELC)$$

where,

C_0 : The setup cost/order
C_1 : Procurement cost/unit
C_2 : Holding cost of inventory/unit/unit time
C_3 : Customers holding cost when inventory level is zero/customer/unit time
C_4 : Cost due to loss of customers/unit/unit time

5.2 GRAPHICAL ILLUSTRATIONS

In this paper, we obtained various performance measures. The change in the parameters such as arrival rate, service rate, replenishment rate, number of back-order, etc. may affect these performance measures.

Figures 1, 2, and 3 illustrate the variation of ETC with p, q and r by keeping all other parameters constant as indicated in the figure. The optimum value of ETC is obtained at $p = 0.5665$ in figure 1, $q = 0.885$ in figure 2 and $r = 0.1175$ in figure 3, corresponding optimum values of the expected total cost are 13.5276, 13.4706 and 14.6625 and which are indicated in the figures 1, 2 and 3 respectively.

In figure 4, we analyzed the variation of ETW using the expression for expected waiting time(ETW) that we derived in the above section. By varying q and r, we analysed the variations of ETW, assuming that arriving customers find 10 customers ahead of him and the level of inventory 2. We can see that ETW decreases with the increase of q and r.

https://doi.org/10.17993/3cemp.2022.110250.50-62

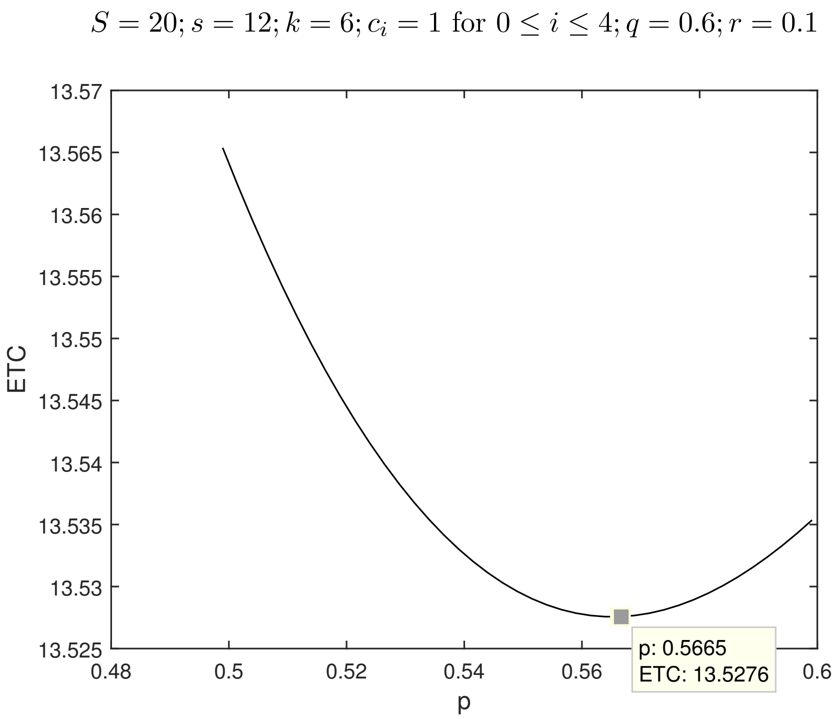

Figure 1 – Variation of expected total cost with p

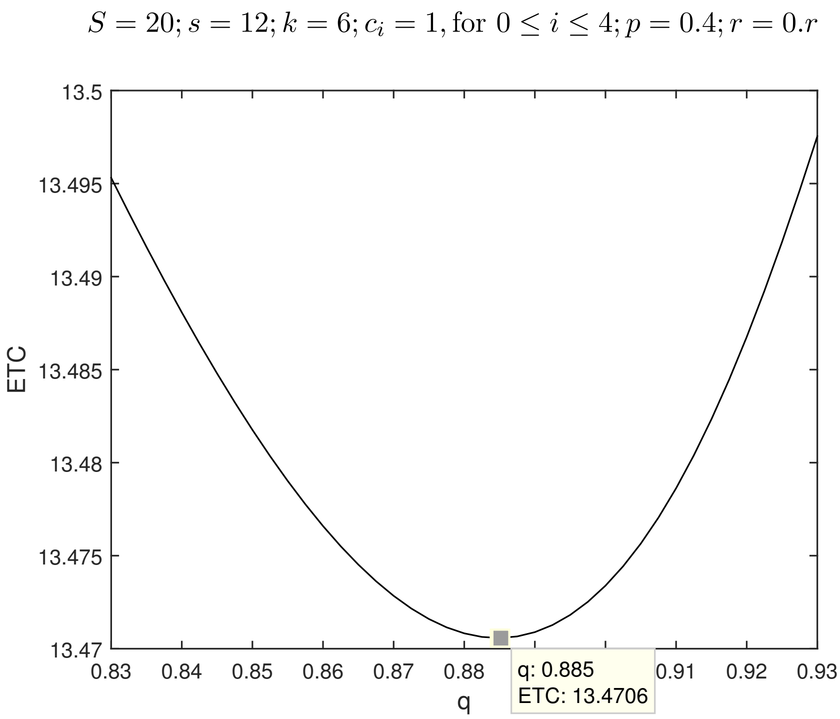

Figure 2 – Variation of expected total cost with q

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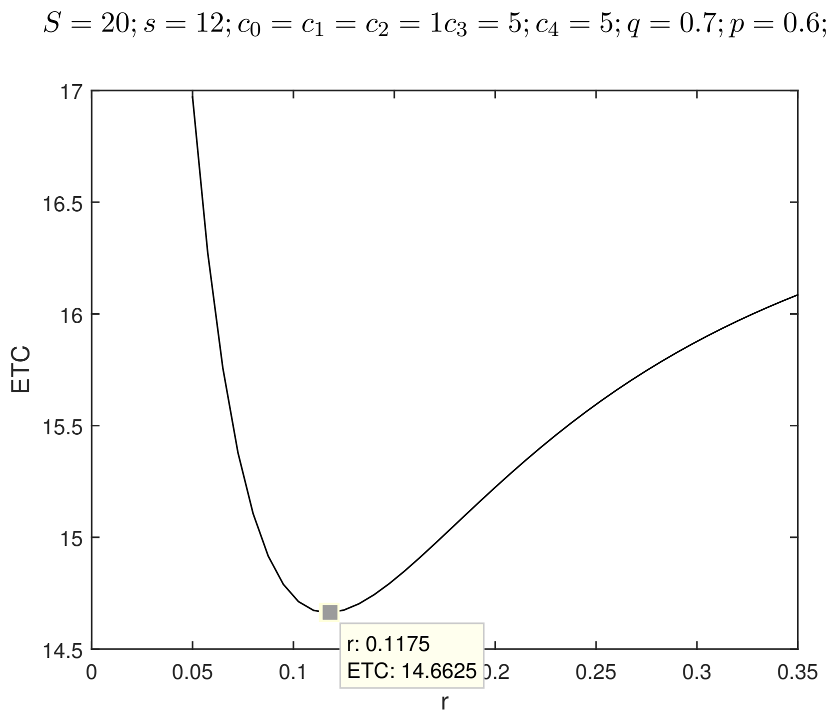

Figure 3 – Variation of expected total cost with r

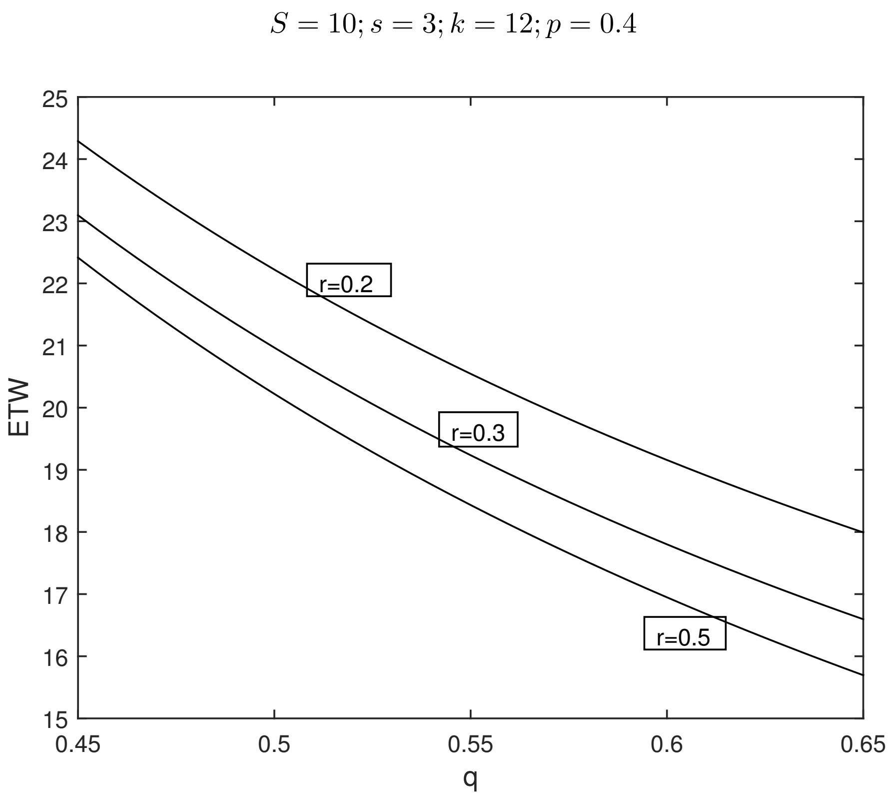

Figure 4 – Variation of expected waiting time with q

https://doi.org/10.17993/3cemp.2022.110250.50-62

CONCLUSIONS

In this paper, the attempt was to analyze a discrete-time inventory model with service time and back-order in inventory. Stability condition, waiting time distribution and reorder time distribution are analyzed. Numerical experiments are incorporated into the model to highlight the effect of variation in system parameters. The work can be further extended by considering the Discrete Markov Arrival Process(DMAP) and discrete phase type service distribution.

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