TWO-COMMODITY PERISHABLE INVENTORY SYSTEM WITH PARTIAL BACKLOG DEMANDS

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ABSTRACT

This article examines a two-commodity continuous review perishable inventory system. The demands are arrived at for each product by independent Markovian arrival processes (MAP). Lifetimes follow an exponential distribution. The commodities are assumed to be substitutable. If both commodities have reached zero, demand is backlogged up to predetermined levels. This article's novelty has been a local purchase, which is made to clear the backlog instantaneously when demand reaches a predetermined level. In the steady-state, the joint probability distribution of inventory levels of both commodities is obtained. Several metrics of system performance in steady-state are derived and also provided as numerical examples to explain the optimum values of the system's parameters.

KEYWORDS

Two-commodity Inventory system, Substitutable items, Joint ordering Policy, Markov arrival demands, Partial backlog.

1 INTRODUCTION

Over the last few decades, researchers have been fascinated with the study of a two-commodity inventory system. It has more importance because these systems are more sophisticated than single commodity inventory systems due to the large number of items held and their coordinated behaviors. Also, many organisations have increasingly used multi-commodity inventory systems. However, the correlation of the reorder points for each item is the major challenge in multi-product inventory systems. Unlike systems that deal with a single commodity, the reordering methods in these systems are more complicated. So, replenishment orders for groups of products must be well coordinated. Initially, research focused on inventory models with independently defined reorder points. The individual ordering policy includes calculating the best order quantity and reordering duration for each item. This ordering policy implementation provides the system with significant flexibility in picking the appropriate inventory models for each item and separately modifying the policy. However, joint ordering policies are preferred over individual ordering policies when the products share the same storage space and transportation facilities. Joint replenishment has several advantages because the joint ordering policy allows for the simultaneous replenishment of several commodities, quantity discounts, and significant savings in ordering and purchasing expenses. The joint replenishment was proposed by Balintfy and developed by Silver. More details about joint replenishment can be seen in Anbazhagan et al. (2012, 2015), Senthil Kumar, and Sivakumar. Various models with two-commodity readers can read in Anbazhagan and Arivarignan, Benny et al., Krishnamoorthy et al., and Ozkar et al.

In the earlier literature on inventory systems, it has generally been recognized that inventory models built under the presumption of a product's lifetime being indefinite until its storage, i.e., an item once placed in a storeroom stays unmodified and entirely functional for supplying future demand. However, this is not the case. When constructing inventory models, one aspect for consideration is an item's perishability, as commodities do not necessarily retain their properties when held for future use. In general, perishability is the outcome of stock depletion, which consists of obsolescence, breakage, decay, losing usefulness, and many other factors. Some examples of perishable objects are meals, evaporative fluids, chemicals, drugs, and radioactive substances. For more details about perishable product readers can refer Karthikeyan and Sudesh, Nahmias, Sivakumar et al., Smrutirekha Debataa et al., Umay and Bahar, Yadavalli et al. (2010, 2015), Zhang et al.

Several research articles examine inventory systems in which required products are directly provided from stock if the item is available. Demand that appears during stock-out times results in either lost sales or a backlog (demand satisfied immediately after the arrival of ordered items). Initially, it is believed that there is a total backlog of unfilled demand. In actuality, many customers are willing to wait until the end of the shortage period to pick up their orders, while others are not. As a result, it is presumed that any predefined quantity of demand (partial backlog) that appeared during the stock-out time is satisfied. For more details about backlog concept readers an refer Adak Sudip and Mahapatra, Cárdenas-Barrón Leopoldo et al., Khan et al., Kurt et al., San José et al., Stanley et al. and Tai et al. Generally, customer satisfaction generates a lot of profit for the system. So the shopkeeper does the maximum amount of work to satisfy the customers. In a practical situation, the local purchase is made by the shopkeeper when the shop runs out of stock and that item's replenishment has been delayed. We can see this act in clothing stores, supermarkets, and all the retailers' shops.

In this article we assume that demands during the stock-out periods are backlogged. We further assume that when the number of backlogged demands reaches a prefixed level a local purchase is made to clear the backlog instantaneously so that the inventory level of the corresponding commodity becomes zero. In the following sections, We have obtained the joint probability distribution for the inventory levels of both commodities in the steady state case in section 3. Various system performance measures in the steady state are derived in section 4 and the cost analysis and the results are illustrated numerically in section 5 and 6.

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2 THE MODEL

We consider a two-commodity inventory system with the maximum capacity S_i units for $i-$th commodity $(i = 1, 2)$. The demands for $i-$th commodity is of unit size. The demands for commodity-1 arrive according to a Markovian arrival process *(MAP)* with representation (D_0, D_1) where D's are of order $m_1 \times m_1$. The underlying Markov chain $J_1(t)$ of the MAP has the generator $D(= D_0 + D_1)$ and a stationary row vector λ_1 of length m_1. Independently of this process, demands for commodity-2 arrive according to a MAP with representation (F_0, F_1) where F's are of order $m_2 \times m_2$. The underlying Markov chain $J_2(t)$ of this MAP has the generator $F(= F_0 + F_1)$ and a stationary row vector λ_2 of length m_2. The items are perishable in nature. The life time of each commodity is assumed to be distributed as exponential with parameter $\gamma_i, (i = 1, 2)$. The two-commodities serve as substitute for each other, that is, a demand for a commodity that is sold out, is satisfied with the other commodity when still in stock. If both the commodities are out of stock, any arriving demands are backlogged. The backlog is allowed up to the level $N_i(< \infty)$ for the $i-$th commodity $(i = 1, 2)$. Whenever the backlog level reaches $N_i, (i = 1, 2)$ an order for N_i items is placed which is replenished instantaneously. The reorder level for the i-th commodity is fixed at $s_i(1 \leq s_i \leq S_i)$ with an ordering quantity for the i-th commodity is $Q_i(= S_i - s_i > s_i + N_i + 1)$ items when both inventory levels are less than or equal to their respective reorder levels. The requirement $S_i - s_i > s_i + N_i + 1$ ensures that after the replenishment the inventory levels of both commodities will be always above the respective reorder levels; otherwise it may not be possible to place reorder (according to this policy) which leads to perpetual shortage. More explicitly if $L_i(t)$ represents inventory level of i-th commodity at time t, then a reorder for both commodities is made when $L_1(t) \leq s_1$ and $L_2(t) \leq s_2$. The lead time is assumed to be distributed as negative exponential with parameter $\beta(> 0)$.

Notations

$[A]_{ij}$	: The element/submatrix at (i, j)-th position of A.
$\mathbf{0}$	: Zero matrix.
I	: An identity matrix.
I_k	: An identity matrix of order k.
$A \otimes B$	: Kronecker product of matrices A and B.
$A \oplus B$	: Kronecker sum of matrices A and B.
$\boldsymbol{e}$	: A column vector of $1'$s of appropriate dimension.

3 ANALYSIS

From the assumptions made on the input and output processes it can be shown that the quadruple $(L_1, L_2, J_1, J_2) = \{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ is a Markov process with state space given by

$$\begin{aligned} E &= \{(i, k, j_1, j_2) | i = 1, 2, \ldots, S_1, k = 0, 1, \ldots, S_2, j_1 = 1, 2, \ldots, m_1, j_2 = 1, 2, \ldots, m_2\} \\ &\cup \{(i, k, j_1, j_2) | i = 0, k = -(N_2 - 1), -(N_2 - 2), \ldots, S_2, j_1 = 1, 2, \ldots, m_1, j_2 = 1, 2, \ldots, m_2\} \\ &\cup \{(i, k, j_1, j_2) | i = -(N_1 - 1), -(N_1 - 2), \ldots, -1, k = -(N_2 - 1), -(N_2 - 2), \ldots, 0, \\ &\quad j_1 = 1, 2, \ldots, m_1, j_2 = 1, 2, \ldots, m_2\}. \end{aligned}$$

Define the following ordered sets :

$$\begin{aligned} \mathbf{i} &= (\ (i, 0), (i, 1), \ldots, (i, S_2)\ )\,, \\ < i > &= (\ (i, -N_2 + 1), (i, -N_2 + 2), \ldots, (i, S_2)\ )\,, \\ [i] &= (\ (i, -N_2 + 1), (i, -N_2 + 2), \ldots, (i, 0)\ )\,, \\ (i, j) &= (\ (i, j, 1), (i, j, 2), \ldots, (i, j, m_1)\ )\,, \\ (i, j, k) &= (\ (i, j, k, 1), (i, j, k, 2), \ldots, (i, j, k, m_2)\ )\,, \end{aligned}$$

Then the state space is ordered as $([-N_1+1],[-N_1+2],\dots,[-1],<0>,\mathbf{1},$ $\mathbf{2},\dots,\mathbf{S_1})$. The infinitesimal generator of P of the Markov process (L_1,L_2,J_1,J_2) has the following block partitioned form :

$$[P]_{ij}=\begin{cases} B_i, & j=i-1, \quad i=0,1,\dots,S_1,\\ \widehat{B}, & j=i-1, \quad i=-(N_1-2),-(N_1-3),\dots,-1,\\ \widetilde{B}, & j=i+(N_1-1), \quad i=-(N_1-1),\\ C, & j=i+Q_1, \quad i=1,2,\dots,s_1,\\ \widehat{C}, & j=i+Q_1, \quad i=0,\\ \widetilde{C}, & j=i+Q_1, \quad i=-(N_1-1),-(N_1-2),\dots,-1,\\ A_i, & j=i, \quad i=0,1,\dots,S,\\ \widehat{A}, & j=i, \quad i=-(N_1-1),-(N_1-2),\dots,-1,\\ \mathbf{0}, & \text{otherwise,}\end{cases}$$

where

$$[C]_{kl}=\begin{cases}\beta I_{m_1}\otimes I_{m_2}, & l=k+Q_2, \quad k=0,1,\dots,s_2,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

$$[\widehat{C}]_{kl}=\begin{cases}\beta I_{m_1}\otimes I_{m_2} & l=k+Q_2, \quad k=-(N_2-1),-(N_2-2),\dots,s_2,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

$$[\widetilde{C}]_{kl}=\begin{cases}\beta I_{m_1}\otimes I_{m_2}, & l=k+Q_2, \quad k=-(N_2-1),-(N_2-2),\dots,0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

For $i=2,3,\dots,S_1$,

$$[B_i]_{kl}=\begin{cases}D_1\otimes I_{m_2}+i\gamma_1 I_{m_1}\otimes I_{m_2}, & l=k, \quad k=1,2,\dots,S_2,\\ D_1\oplus F_1+i\gamma_1 I_{m_1}\otimes I_{m_2}, & l=k, \quad k=0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

For $i=1$,

$$[B_i]_{kl}=\begin{cases}D_1\otimes I_{m_2}+i\gamma_1 I_{m_1}\otimes I_{m_2}, & l=k, \quad k=1,2,\dots,S_2,\\ D_1\oplus F_1+i\gamma_1 I_{m_1}\otimes I_{m_2}, & l=k, \quad k=0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

For $i=0$,

$$[B_i]_{kl}=\begin{cases}D_1\otimes I_{m_2}, & l=k, \quad k=-(N_2-1),(N_2-2),\dots,0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

$$\left[\hat{B}\right]_{kl}=\begin{cases}D_1\otimes I_{m_2}, & l=k, \quad k=-(N_2-1),(N_2-2),\dots,0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

$$\left[\tilde{B}\right]_{kl}=\begin{cases}D_1\otimes I_{m_2}, & l=k, \quad k=-(N_2-1),(N_2-2),\dots,0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

For $i=1,2,\dots s_1$,

$$[A_i]_{kl}=\begin{cases}I_{m_1}\otimes F_1+k\gamma_2 I_{m_1}\otimes I_{m_2}, & l=k-1, \quad k=1,2,\dots,S_2,\\ D_0\oplus F_0-(i\gamma_1+\beta)I_{m_1}\otimes I_{m_2}, & l=k, \quad k=0,\\ D_0\oplus F_0-(i\gamma_1+\beta+k\gamma_2)I_{m_1}\otimes I_{m_2}, & l=k, \quad k=1,2,\dots,s_2\\ D_0\oplus F_0-(i\gamma_1+k\gamma_2)I_{m_1}\otimes I_{m_2}, & l=k, \quad k=s_2+1,s_2+2,\dots,S_2\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

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For $\quad i = s_1+1, s_1+2, \dots S_1,$

$$[A_i]_{kl} = \begin{cases} I_{m_1} \otimes F_1 + k\gamma_2 I_{m_1} \otimes I_{m_2}, & l=k-1, \quad k=1,2,\dots,S_2, \\ D_0 \oplus F_0 - i\gamma_1 I_{m_1} \otimes I_{m_2}, & l=k, \quad k=0, \\ D_0 \oplus F_0 - (i\gamma_1 + k\gamma_2) I_{m_1} \otimes I_{m_2}, & l=k, \quad k=1,2,\dots,S_2 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $\quad i = 0,$

$$[A_i]_{kl} = \begin{cases} D_1 \oplus F_1 + k\gamma_2 I_{m_1} \otimes I_{m_2}, & l=k-1, \quad k=1,2,\dots,S_2, \\ I_{m_1} \otimes F_1, & l=k-1, \quad k=-(N_2-2),-(N_2-3),\dots,-1,0, \\ & (or) \\ & l=k+N_2-1, \quad k=-(N_2-1), \\ D_0 \oplus F_0 - \beta I_{m_1} \otimes I_{m_2}, & l=k, \quad k=-(N_2-1),-(N_2-2),\dots,0, \\ D_0 \oplus F_0 - (\beta + k\gamma_2) I_{m_1} \otimes I_{m_2}, & l=k, \quad k=1,2,\dots,s_2, \\ D_0 \oplus F_0 - k\gamma_2 I_{m_1} \otimes I_{m_2}, & l=k, \quad k=s_2+1,s_2+2,\dots,S_2, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$\left[\widehat{A}\right]_{kl} = \begin{cases} I_{m_1} \otimes F_1, & l=k-1, \quad k=-(N_2-2),-(N_2-3),\dots,-1,0, \\ & (or) \\ & l=k+N_2-1, \quad k=-(N_2-1), \\ D_0 \oplus F_0 - \beta I_{m_1} \otimes I_{m_2}, & l=k, \quad k=-(N_2-1),-(N_2-2),\dots,0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

It may be noted that the matrices $A_i, i=1,2,\dots,S_1, B_i, i=2,3,\dots,S_1$ and C are of size $(S_2+1)m_1m_2 \times (S_2+1)m_1m_2$, B_1 is of size $(S_2+1)m_1m_2 \times (S_1+N_2)m_1m_2$, B_0 is of size $(S_2+N_2)m_1m_2 \times N_2m_1m_2$, $\widehat{B}$ is of size $N_2m_1m_2 \times N_2m_1m_2$, $\widetilde{B}$ is of size $N_2m_1m_2 \times (S_2+N_2)m_1m_2$, $\widehat{C}$ is of size $(S_2+N_2)m_1m_2 \times (S_2+1)m_1m_2$, $\widetilde{C}$ is of size $N_2m_1m_2 \times (S_2+1)m_1m_2$, A_0 is of size $(S_2+N_2)m_1m_2 \times (S_2+N_2)m_1m_2$ and $\widehat{A}$ is of size $N_2m_1m_2 \times N_2m_1m_2$.

3.1 STEADY STATE ANALYSIS

It can be seen from the structure of P that the homogeneous Markov process $\{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ on the finite state space E is irreducible. Hence the limiting distribution $\phi_{(i,k,j_1,j_2)} =$

$$\lim_{t\to\infty} Pr\left[L_1(t)=i, L_2(t)=k, J_1(t)=j_1, J_2(t)=j_2 | L_1(0), L_2(0), J_1(0), J_2(0)\right]$$

exists. Let

$$\begin{aligned} \phi_{(i,k,j_1)} &= (\phi_{(i,k,j_1,1)}, \phi_{(i,k,j_1,2)}, \dots, \phi_{(i,k,j_1,m_2)}), j_1 = 1,2,\dots,m_1, \\ \phi_{(i,k)} &= \left(\phi_{(i,k,1)}, \phi_{(i,k,2)}, \dots, \phi_{(i,k,m_1)}\right), k=-N_2+1,-N_2+2,\dots,S_2, \\ \boldsymbol{\phi}^{(i)} &= \begin{cases} (\phi_{(i,0)}, \phi_{(i,1)}, \dots, \phi_{(i,S_2)}), & \text{if } i=1,2,\dots,S_1, \\ (\phi_{(i,-N_2+1)}, \phi_{(i,-N_2+2)}, \dots, \phi_{(i,S_2)}), & \text{if } i=0, \\ (\phi_{(i,-N_2+1)}, \phi_{(i,-N_2+2)}, \dots, \phi_{(i,0)}), & \text{if } i=-N_1+1,-N_1+2,\dots,-1. \end{cases} \\ \text{and} & \\ \boldsymbol{\Phi} &= (\boldsymbol{\phi}^{(-N_1+1)}, \boldsymbol{\phi}^{(-N_1+2)}, \dots, \boldsymbol{\phi}^{(S_1-1)}, \boldsymbol{\phi}^{(S_1)}). \end{aligned}$$

Then the vector of limiting probabilities $\boldsymbol{\Phi}$ satisfies

$$\boldsymbol{\Phi} P = \mathbf{0} \quad \text{and} \quad \boldsymbol{\Phi}\mathbf{e} = 1. \tag{1}$$

The first equation of the above yields the following set of equations:

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$$\begin{aligned}
\boldsymbol{\phi}^{(i+1)}\widehat{B}+\boldsymbol{\phi}^{(i)}\widehat{A}&=\mathbf{0}, && i=-N_1+1,-N_1+2,\dots,-2,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}\widehat{A}&=\mathbf{0}, && i=-1,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-N_1+1)}\widetilde{B}&=\mathbf{0}, && i=0,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i&=\mathbf{0}, && i=1,2,\dots,Q_1-N_1,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-Q_1)}\widetilde{C}&=\mathbf{0}, && i=Q_1-N_1+1,Q_1-N_1+2,\dots,Q_1-1,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-Q_1)}\widehat{C}&=\mathbf{0}, && i=Q_1, \qquad (2)\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-Q_1)}C&=\mathbf{0}, && i=Q_1+1,Q_1+2\dots,S_1-1,\\
\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-Q_1)}C&=\mathbf{0}, && i=S_1.
\end{aligned}$$

The equations (except (2)) can be recursively solved to get

$$\boldsymbol{\phi}^{(i)}=\boldsymbol{\phi}^{(Q_1)}\theta_i,\quad i=-N_1+1,-N_1+2,\dots,S_1,$$

where

$$\theta_i=\begin{cases}
-\theta_{i+1}\widehat{B}\widehat{A}^{-1}, & i=-(N_1-1),-(N_1-2),\dots,-2,\\
-\theta_{i+1}B_0\widehat{A}^{-1}, & i=-1,\\
-\left(\theta_{i+1}B_{i+1}+\theta_{i-N_1+1}\widetilde{B}\right)A_i^{-1}, & i=0,\\
-\theta_{i+1}B_{i+1}A_i^{-1}, & i=1,2,\dots,Q_1-N_1,\\
-\left(\theta_{i+1}B_{i+1}+\theta_{i-Q_1}\widetilde{C}\right)A_i^{-1}, & i=Q_1-N_1+1,Q_1-N_1+2,\dots,Q_1-1,\\
I, & i=Q_1,\\
-\left(\theta_{i+1}B_{i+1}+\theta_{i-Q_1}C\right)A_i^{-1}, & i=Q_1+1,Q_1+2,\dots,S_1-1,\\
-\theta_{i-Q_1}CA_i^{-1}, & i=S_1.
\end{cases}$$

Substituting the values of θ_i in equation (2) and in the normalizing condition we get the value of $\boldsymbol{\phi}^{(Q_1)}$.

4 SYSTEM PERFORMANCE MEASURES

In this section we derive some stationary performance measures of the system. Using these measures, we can construct the total expected cost per unit time.

4.1 MEAN INVENTORY LEVEL

Let η_{I_i} denote the mean inventory level of $i-$th commodity in the steady state $(i=1,2)$. Since $\phi_{(i,j)}$ is the steady state probability vector for inventory level of first commodity is i and the second commodity is j, we have

$$\eta_{I_1} = \sum_{i=1}^{S_1}\sum_{k=0}^{S_2} i\phi_{(i,k)}\mathbf{e}.$$

and

$$\eta_{I_2} = \sum_{i=0}^{S_1}\sum_{k=1}^{S_2} k\phi_{(i,k)}\mathbf{e}.$$

4.2 MEAN REORDER RATE

A reorder for both commodities is made when the joint inventory level, drops to either (s_1,s_2) or $(s_1,j), j<s_2$ or $(i,s_2), i<s_1$. Let ζ_R denote the mean joint reorder rate for both commodities in the

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steady state and it is given by

$$\begin{aligned}\eta_R &= \frac{1}{\lambda_1}\sum_{k=0}^{s_2}\phi_{(s_1+1,k)}\left(D_1\otimes I_{m_2}\right)\mathbf{e}+\frac{1}{\lambda_2}\sum_{i=0}^{s_1}\phi_{(i,s_2+1)}\left(I_{m_1}\otimes F_1\right)\mathbf{e}\\ &+\frac{1}{\lambda_1}\phi_{(0,s_2+1)}\left(I_{m_1}\otimes F_1\right)\mathbf{e}+\frac{1}{\lambda_2}\phi_{(s_1+1,0)}\left(D_1\otimes I_{m_2}\right)\mathbf{e}\\ &+(s_1+1)\gamma_1\sum_{k=0}^{s_2}\phi_{(s_1+1,k)}\mathbf{e}+(s_2+1)\gamma_2\sum_{i=0}^{s_1}\phi_{(i,s_2+1)}\mathbf{e}.\end{aligned}$$

Let η_{R_i} denote the mean individual reorder rate for commodity-i in the steady state $(i=1,2)$. When the inventory level of commodity-1 is $-(N_1-1)$, a demand for commodity-1 will trigger the individual reorder for commodity-1. Hence we get

$$\eta_{R_1} = \frac{1}{\lambda_1}\sum_{k=-N_2+1}^{0}\phi_{(-N_1+1,k)}\left(D_1\otimes I_{m_2}\right)\mathbf{e}.$$

Similar arguments lead to

$$\eta_{R_2} = \frac{1}{\lambda_2}\sum_{i=-N_1+1}^{0}\phi_{(i,-N_2+1)}\left(I_{m_1}\otimes F_1\right)\mathbf{e}.$$

4.3 AVERAGE BACKLOG

Let η_{B_i} denote the mean backlog of commodity-i in the steady state $(i=1,2)$. Then we have

$$\eta_{B_1} = \sum_{i=-N_1+1}^{-1}\sum_{k=-N_2+1}^{0}|i|\phi_{(i,k)}\mathbf{e}.$$

and

$$\eta_{B_2} = \sum_{i=-N_1+1}^{0}\sum_{k=-N_2+1}^{-1}|k|\phi_{(i,k)}\mathbf{e}.$$

4.4 MEAN PERISHABLE RATE

Let the mean perishable rate of commodity-i in the steady state de denoted by $\zeta_{F_i}, (i=1,2)$. Then we have

$$\eta_{F_1} = \sum_{i=1}^{S_1}\sum_{k=0}^{S_2}i\gamma_1\phi_{(i,k)}\mathbf{e}.$$

and

$$\eta_{F_2} = \sum_{i=0}^{S_1}\sum_{k=1}^{S_2}k\gamma_2\phi_{(i,k)}\mathbf{e}.$$

5 COST ANALYSIS

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

$$\begin{aligned}TC(S_1,S_2,s_1,s_2,N_1,N_2) &= c_{h_1}\eta_{I_1}+c_{h_2}\eta_{I_2}+c_r\eta_R+c_{r_1}\eta_{R_1}+c_{r_2}\eta_{R_2}\\ &+c_{b_1}\eta_{B_1}+c_{b_2}\eta_{B_2}+c_{p_1}\eta_{F_1}+c_{p_2}\eta_{F_2},\end{aligned}$$

where

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c_r : Setup cost per order.
c_{r_i} : Setup cost for the i-th commodity under local purchase (i=1,2).
c_{h_i} : Holding cost for the i-th commodity per unit time, $i = 1, 2$.
c_{p_i} : Perishable cost per unit item per unit time of i-th commodity (i=1,2).
c_{b_i} : Cost per unit backlog for the i-th commodity per unit time, $i = 1, 2$.

By substituting the values for η's we can compute the value of $TC(S_1, S_2, s_1, s_2, N_1, N_2)$.

Since the evaluation of the ϕ's involve recursive computations, it is quite difficult to show the convexity of the total expected cost rate. However we present the following example to demonstrate the computability of the results derived in our work, and to illustrate the existence of local optima when the total cost function is treated as a function of only two variables.

6 NUMERICAL ILLUSTRATION

We consider the following numerical example : The demand for first commodity is given by (D_0, D_1) where

$$D_0 = \begin{pmatrix} -50 & 0 \\ 0 & -5 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 39 & 11 \\ 3.9 & 1.1 \end{pmatrix}.$$

The demand for second commodity is given by (F_0, F_1) where

$$F_0 = \begin{pmatrix} -20 & 0 \\ 0 & -2 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 19 & 1 \\ 1.9 & 0.1 \end{pmatrix}.$$

In the following tables, the optimal cost for each row is shown in underlined and the optimal cost for each column is shown in bold.
Let $\gamma_1 = 1, \gamma_2 = 1, \beta = 25, s_1 = 2, s_2 = 2, N_1 = 3, N_2 = 3, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 75, c_{r_1} = 2, c_{r_2} = 2, c_{b_1} = 1, c_{b_2} = 1, c_{p_1} = 2, c_{p_2} = 1$.

$$\text{Let} \qquad \overline{TC}(S_1, S_2) = TC(S_1, S_2, 2, 2, 3, 3).$$

From table 1, the numerical values shows that $\overline{TC}(S_1, S_2)$ is a convex function in (S_1, S_2) and the

Table 1 – Total Expected Cost Rate of S_1 and S_2

S_1 \ S_2	10	11	12	13	14
13	9.872429	9.630808	$\underline{9.596855}$	9.743616	9.775500
14	9.709404	9.520833	$\underline{9.501561}$	9.633814	9.684700
15	9.594168	9.451205	$\underline{9.446881}$	9.569767	9.634135
16	9.517610	$\underline{9.413976}$	**9.424364**	**9.541746**	**9.616449**
17	9.472896	$\underline{\mathbf{9.403231}}$	9.427757	9.542487	9.625701
18	**9.454763**	$\underline{9.414471}$	9.452342	9.566473	9.657093
19	9.459070	$\underline{9.444205}$	9.494506	9.609474	9.706772

(possibly local) optimum occurs at $(S_1, S_2) = (17, 11)$.
Let $\gamma_1 = 0.01, \gamma_2 = 0.8, \beta = 18, S_2 = 20, s_2 = 3, N_1 = 3, N_2 = 3, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 0.55, c_{r_1} = 0.45, c_{r_2} = 0.5, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.4$.

$$\text{Let} \qquad \overline{TC}(S_1, s_1) = TC(S_1, 20, s_1, 3, 3, 3).$$

From table 2, the numerical values shows that $\overline{TC}(S_1, s_1)$ is a convex function in (S_1, s_1) and the

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Table 2 – Total Expected Cost Rate of S_1 and s_1

S_1 \ s_1	4	5	6	7	8
49	5.076500	5.075467	5.080127	5.088899	5.100611
50	**5.076459**	5.075080	5.079444	5.087956	5.099435
51	5.076589	5.074872	5.078945	5.087203	5.098452
52	5.076888	**5.074837**	5.078626	5.086635	5.097661
53	5.077351	5.074974	**5.078484**	5.086248	5.097055
54	5.077976	5.075278	5.078514	5.086039	5.096631
55	5.078760	5.075747	5.078714	**5.086004**	5.096385
56	5.079700	5.076377	5.079080	5.086140	**5.096314**
57	5.080793	5.077165	5.079609	5.086443	5.096414

(possibly local) optimum occurs at $(S_1, s_1) = (52, 5)$.

let $\gamma_1 = 0.01, \gamma_2 = 0.9, \beta = 10, S_2 = 20, s_2 = 2, S_1 = 20, s_1 = 2, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 21, c_{r_1} = 15, c_{r_2} = 18, c_{b_1} = 5, c_{b_2} = 5, c_{p_1} = 0.8, c_{p_2} = 0.75$.

$$\text{Let} \qquad \overline{TC}(N_1, N_2) = TC(20, 20, 2, 2, N_1, N_2).$$

From table 3, the numerical values shows that $\overline{TC}(N_1, N_2)$ is a convex function in (N_1, N_2) and the

Table 3 – Total Expected Cost Rate of N_1 and N_2

N_1 \ N_2	3	4	5	6	7
4	10.565306	10.543407	10.533310	10.531192	10.535652
5	10.509154	10.491453	10.486547	10.490709	10.502146
6	10.476386	10.465856	10.468392	**10.481047**	**10.501967**
7	10.456879	10.454703	**10.465625**	10.487919	10.520371
8	**10.453162**	**10.449303**	10.468540	10.500486	10.545501
9	10.461411	10.459465	10.492293	10.541267	10.609235

(possibly local) optimum occurs at $(N_1, N_2) = (8, 4)$.

Let $\gamma_1 = 0.1, \gamma_2 = 0.8, \beta = 18, S_1 = 20, s_1 = 3, s_2 = 2, N_1 = 3; c_{h_1} = 0.1, c_{h_2} = 0.1, c_r = 0.11, c_{r_1} = 0.1, c_{r_2} = 0.1, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.1$.

$$\text{Let} \qquad \overline{TC}(S_2, N_2) = TC(20, S_2, 3, 2, 3, N_2).$$

From table 4, the numerical values shows that $\overline{TC}(S_2, N_2)$ is a convex function in (S_2, N_2) and the

Table 4 – Total Expected Cost Rate of S_2 and N_2

S_2 \ N_2	5	6	7	8	9
39	8.654678	8.653849	8.654060	8.654295	8.654545
40	**8.517849**	**8.517004**	**8.517197**	**8.517414**	**8.517645**
41	8.689863	8.680004	8.680181	8.680380	8.680593
42	8.843738	8.842865	8.843027	8.843210	8.843407
43	9.005788	9.005604	9.005752	9.005920	9.006102

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(possibly local) optimum occurs at $(S_2, N_2) = (40, 6)$.

Let $\gamma_1 = 0.1, \gamma_2 = 0.8, \beta = 18, S_1 = 20, s_1 = 3, N_2 = 3, N_1 = 3; c_{h_1} = 0.1, c_{h_2} = 0.1, c_r = 0.11, c_{r_1} = 0.1, c_{r_2} = 0.1, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.1$.

$$\text{Let} \qquad \overline{TC}(S_2, s_2) = TC(20, S_2, 3, s_2, 3, 3).$$

From table 5, the numerical values shows that $\overline{TC}(S_2, s_2)$ is a convex function in (S_2, s_2) and the

Table 5 – Total Expected Cost Rate of S_2 and s_2

S_2 \ s_2	2	3	4	5	6
39	8.656617	8.655266	8.656616	8.657680	8.658492
40	**8.519803**	**8.518349**	**8.519626**	**8.520642**	**8.521425**
41	8.699830	8.681279	8.682486	8.683454	8.684207
42	8.846717	8.844073	8.845214	8.846135	8.846858
43	9.007478	9.006747	9.007824	9.008700	9.009393

(possibly local) optimum occurs at $(S_2, s_2) = (40, 3)$.
Let $\gamma_1 = 0.01, \gamma_2 = 0.8, \beta = 18, S_2 = 20, s_1 = 2, s_2 = 3, N_2 = 3; c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 0.55, c_{r_1} = 0.45, c_{r_2} = 0.5, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.4$.

$$\text{Let} \qquad \overline{TC}(S_1, N_1) = TC(S_1, 20, 2, 3, N_1, 3).$$

From table 6, the numerical values shows that $\overline{TC}(S_1, N_1)$ is a convex function in (S_1, N_1) and the

Table 6 – Total Expected Cost Rate of S_1 and N_1

S_1 \ N_1	5	6	7	8	9
49	5.034028	5.023524	5.021157	5.023724	5.029325
50	**5.033895**	5.023100	5.020510	5.022892	5.028327
51	5.033935	5.022856	5.020048	5.022249	5.027524
52	5.034146	**5.022789**	5.019767	5.021791	5.026910
53	5.034523	5.022894	**5.019662**	5.021515	5.026482
54	5.035065	5.023168	5.019732	**5.021415**	5.026235
55	5.035768	5.023608	5.019971	5.021490	**5.026165**
56	5.036629	5.024210	5.020377	5.021734	5.026268

(possibly local) optimum occurs at $(S_1, N_1) = (53, 7)$.
The Figure 1 grants the impact of the perishable rate γ_1, on the total expected cost rate TC via four curves which relate to β = 18.5,18.6,18.7,18.8. Since figure 1, we perceive that the total cost value decreases when the perishable rate γ_1 and the replenishment rate β increases.

The Figure 2 grants the impact of the perishable rate γ_2, on the total expected cost rate TC via three curves which relate to β = 19,20 and 21. Since figure 2, we perceive that the total cost value decreases when the perishable rate γ_2 and the replenishment rate β increases.
In tables 7 and 8, we show that the impact of the cost values on the optimal values (S_1^*, s_1^*) and the corresponding total expected cost rate. Towards this end, we first fix the parameters and cost value as $S_2 = 20, s_2 = 3, N_1 = 3, N_2 = 3, \beta = 18, \gamma_1 = 0.01, \gamma_2 = 0.8, c_{r_1} = 0.45, c_{r_2} = 0.5$.

Table 7 – Impact of Cost Values

C_{h_2}				0.01					
C_{p_2}				0.4			0.5		
C_{b_2}				0.09	0.1	0.11	0.09	0.1	0.11
C_r	C_{h_1}	C_{p_1}	C_{b_1}						
0.4	0.01	0.10	0.09	55 6 5.0730	55 6 5.0733	55 6 5.0736	55 6 6.1763	55 6 6.1765	55 6 6.1768
			0.10	55 6 5.0734	55 6 5.0737	55 6 5.0740	55 6 6.1767	55 6 6.1770	55 6 6.1773
			0.11	55 6 5.0735	55 6 5.0738	55 6 5.0741	55 6 6.1768	55 6 6.1771	55 6 6.1774
		0.20	0.09	55 6 5.1261	55 6 5.1264	55 6 5.1267	55 6 6.2202	55 6 6.2205	55 6 6.2208
			0.10	55 6 5.1266	54 6 5.1269	54 6 5.1272	54 6 6.2206	54 6 6.2209	54 6 6.2212
			0.11	54 6 5.1271	54 6 5.1273	54 6 5.1276	54 6 6.2211	54 6 6.2214	54 6 6.2217
	0.02	0.10	0.09	54 6 5.5517	54 6 5.5520	54 6 5.5523	53 6 6.6793	53 6 6.6795	53 6 6.6798
			0.10	54 6 5.5522	54 6 5.5524	54 6 5.5527	53 6 6.6797	53 6 6.6800	53 6 6.6873
			0.11	54 6 5.5526	54 5 5.5529	54 5 5.5532	53 5 6.6817	53 5 6.6819	53 5 6.6893
		0.20	0.09	54 6 5.6281	53 5 5.6284	53 5 5.6287	53 5 6.6822	53 5 6.6835	53 5 6.6900
			0.10	53 6 5.6296	53 5 5.6299	53 5 5.6309	53 5 6.6826	53 5 6.6837	53 5 6.6912
			0.11	53 6 5.6371	53 5 5.6373	53 5 5.6376	53 5 6.6829	53 5 6.6838	52 5 6.6915
0.5	0.01	0.10	0.09	55 6 5.6462	55 6 5.6468	55 6 5.6471	55 6 6.6983	55 6 6.6985	55 6 6.6988
			0.10	55 6 5.6771	55 6 5.6777	55 6 5.6780	55 6 6.7067	55 6 6.7079	55 6 6.7083
			0.11	55 6 5.6784	55 6 5.6787	55 6 5.6790	55 6 6.7177	55 6 6.7178	55 6 6.7179
		0.20	0.09	55 6 5.6861	55 6 5.6884	55 6 5.6887	55 6 6.7202	55 6 6.7265	55 6 6.7268
			0.10	55 6 5.6886	54 6 5.6889	54 6 5.6890	54 6 6.7566	54 6 6.7569	54 6 6.7570
			0.11	54 6 5.6887	54 6 5.6890	54 6 5.6891	54 6 6.7571	54 6 6.7574	54 6 6.7577
	0.02	0.10	0.09	54 6 5.6892	54 6 5.6894	54 6 5.6896	53 6 6.7692	53 6 6.7698	53 6 6.7791
			0.10	54 6 5.6893	54 6 5.6902	54 6 5.6907	53 6 6.7857	53 6 6.7860	53 6 6.7893
			0.11	54 6 5.6896	54 5 5.6909	54 5 5.6912	53 5 6.7919	53 5 6.7942	53 5 6.7958
		0.20	0.09	54 5 5.7281	53 5 5.7284	53 5 5.7287	53 5 6.8012	53 5 6.8015	53 5 6.8018
			0.10	53 6 5.7296	53 5 5.7299	53 5 5.7309	53 5 6.8026	53 5 6.8029	53 5 6.8032
			0.11	53 6 5.8371	53 5 5.8373	53 5 5.8376	53 5 6.8036	53 5 6.8044	52 5 6.8047

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Table 8 – Impact of Cost Value

C_{h_2}				0.02					
C_{p_2}				0.4			0.5		
C_{b_2}				0.09	0.1	0.11	0.09	0.1	0.11
C_r	C_{h_1}	C_{p_1}	C_{b_1}						
0.4	0.01	0.10	0.09	54 6 5.2189	54 6 5.2191	54 6 5.2194	54 6 6.2197	54 6 6.2213	54 6 6.2232
			0.10	54 6 5.2196	54 6 5.2199	54 6 5.3101	54 6 6.2367	54 6 6.2370	54 6 6.2373
			0.11	54 6 5.2234	54 6 5.2237	54 6 5.3240	54 6 6.2767	54 6 6.2770	54 6 6.2773
		0.20	0.09	54 6 5.2281	54 6 5.2284	54 6 5.3287	54 6 6.3902	54 6 6.3913	54 6 6.3956
			0.10	53 6 5.3296	53 6 5.3299	53 6 5.3302	53 6 6.4106	53 6 6.4109	53 6 6.4112
			0.11	53 6 5.3311	53 6 5.3323	53 6 5.3346	53 6 6.4152	53 6 6.4163	53 6 6.4177
	0.02	0.10	0.09	53 6 5.6517	53 6 5.6520	53 6 5.6523	52 6 6.7793	52 6 6.7795	52 6 6.7798
			0.10	53 6 5.6522	53 6 5.6524	53 6 5.6527	52 6 6.7797	52 6 6.7800	52 6 6.7873
			0.11	53 5 5.6526	53 5 5.6529	53 5 5.6532	52 5 6.7817	52 5 6.7819	52 5 6.7883
		0.20	0.09	53 5 5.7281	53 5 5.7284	53 5 5.7287	52 5 6.7912	52 5 6.7915	52 5 6.7918
			0.10	52 5 5.7296	52 5 5.7299	52 5 5.7309	52 5 6.8026	52 5 6.8029	52 5 6.8112
			0.11	52 5 5.7371	52 5 5.7373	52 5 5.7376	52 5 6.8116	52 5 6.8134	52 5 6.8147
0.5	0.01	0.10	0.09	54 6 5.8762	54 6 5.8768	54 6 5.8771	54 6 6.8783	54 6 6.8785	54 6 6.8788
			0.10	54 6 5.8771	54 6 5.8777	54 6 5.8780	54 6 6.9767	54 6 6.9790	54 6 6.9793
			0.11	54 6 5.8784	54 6 5.8787	54 6 5.8790	54 6 6.9777	54 6 6.9870	54 6 6.9873
		0.20	0.09	54 6 6.1761	54 6 6.1784	54 6 6.1787	54 6 7.2202	54 6 7.2565	54 6 7.2568
			0.10	53 6 6.1786	53 6 6.1789	53 6 6.1792	53 6 7.2566	53 6 7.2569	53 6 7.2572
			0.11	53 6 6.1791	53 6 6.1793	53 6 6.1796	53 6 7.2671	53 6 7.2674	53 6 7.2677
	0.02	0.10	0.09	53 6 6.5627	53 6 6.5670	53 6 6.5823	52 6 7.6797	52 6 7.6894	52 6 7.6898
			0.10	53 6 6.5792	53 6 6.5802	53 6 6.5907	52 6 7.6857	52 6 7.6920	52 6 7.6923
			0.11	53 5 6.5906	53 5 6.5929	53 5 6.5942	52 5 7.6919	52 5 7.6942	52 5 7.6958
		0.20	0.09	53 5 6.7281	53 5 6.7284	53 5 6.7287	52 5 7.7312	52 5 7.7315	52 5 7.7318
			0.10	52 5 6.7296	52 5 6.7299	52 5 6.7309	52 5 7.7426	52 5 7.7429	52 5 7.7512
			0.11	52 5 6.8371	52 5 6.8373	52 5 6.8376	52 5 7.7516	52 5 7.7534	52 5 7.7547

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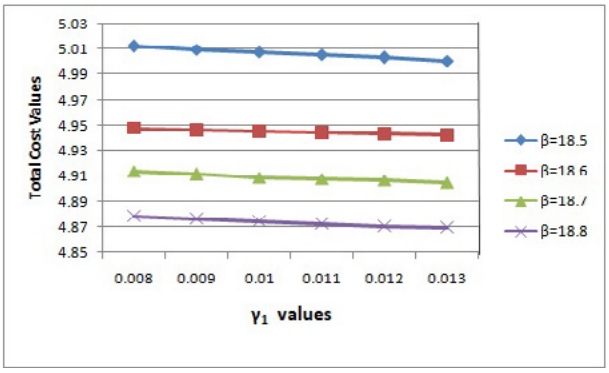

Figure 1 – TC versus γ_1

$\gamma_2 = 0.8, S_1 = 52, s1 = 5, S_2 = 20, s_2 = 3, N_1 = 3, N_2 = 3, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 0.55, c_{r_1} = 0.45, c_{r_2} = 0.5, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.4$..

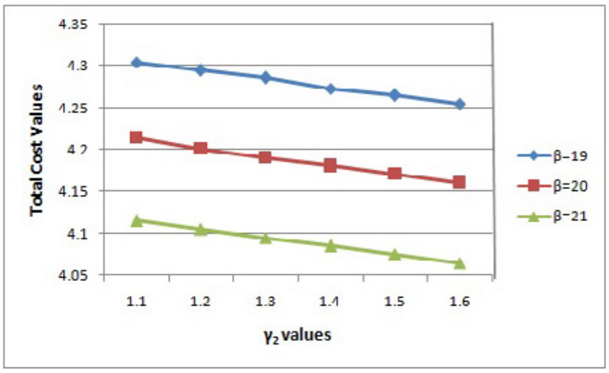

Figure 2 – TC versus γ_2

$\gamma_1 = 0.01, S_1 = 52, s_1 = 5, S_2 = 20, s_2 = 3, N_1 = 3, N_2 = 3, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 0.55, c_{r_1} = 0.45, c_{r_2} = 0.5, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.4$.

7 CONCLUSION

In this article, we examined the substitutable perishable inventory system. Specifically, we analyzed the structure of the system performance that takes place when a local purchase is made to clear the backlog instantaneously if both commodities have reached zero and demand is backlogged up to predetermined levels. Arriving customers follow a Markovian arrival process. The commodities are assumed to be substitutable. If both commodities have reached zero, demand is backlogged up to predetermined levels. Graphical results of perishable rates and replenishment rates had been presented. This shows that if the perishable and replenished rate increases then the total cost would increases. The results of the contribution were illustrated using numerical patterns to estimate the convexity of the overall cost rate of this system. The impact of cost values on total expected cost rate were shown. In the future, our proposed model can be expanded by various reordering policies and described by real data values.

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