ON A QUEUEING INVENTORY WITH COMMON LIFE TIME AND REDUCTION SALE CONSEQUENT TO INCREASE IN AGE

Abdul Rof V[1]

Assistant Professor, KAHM Unity Women's College, Manjeri (India). Part-time Research Scholar, Department of Mathematics, Cochin University of Science and Technology. Cochin (India).

abdulrof@cusat.ac.in

https://orcid.org/0000-0002-7479-3651

Achyutha Krishnamoorthy[2]

Centre for Research in Mathematics, CMS College, Kottayam (India). Department of Mathematics, Central University of Kerala, Kasaragod, Kerala, India.

achyuthacusat@gmail.com

https://orcid.org/0000-0002-5911-1325

Reception: 01/08/2022 **Acceptance:** 16/08/2022 **Publication:** 29/12/2022

Suggested citation:

Abdul R. V. and A. Krishnamoorthy (2022). On a Queueing Inventory with Common Life Time and Reduction Sale Consequent to Increase in Age. *3C Empresa. Investigación y pensamiento crítico*, *11*(2), 15-31.
https://doi.org/10.17993/3cemp.2022.110250.15-31

ABSTRACT

Consider an inventoried item for which reduction in sales price is declared as the age of the item increases. Decision to maintain sales price at the same level/reduce, is taken at stages $2, \cdots, k-1, k$. On the items attaining CLT, they are sold at scrap value, provided items are still left in stock. Customer arrival forms a non-homogenous Poisson process, with rate increasing with each sales price reduction. Service time follows exponential distribution.The items are replenished according to (S, s) policy with positive lead time. Each stage of CLT is iid which follows a Phase type distribution with representation(α, S) of order m. The k-fold convolution of this distribution is the CLT of the inventoried items. The problem is modelled as a queueing-inventory problem which is a continuous time Markov chain (CTMC). The stationary distribution of this CTMC is computed and various performance measures are discussed. A cost function is constructed to compute the optimal order quantity and reorder level.The model is compared with queueing inventory model in which the CLT follows Erlang Distribution of order k.

KEYWORDS

Inventory, Lead time, Common Life Time, Reduction Sale

1 INTRODUCTION

The quality of inventory items, especially perishable decreases gradually as their age increases.Seasonal items also have this property.However for that price can be very high during the time which is not the season for that item. It is very common to declare reduction in sales price while the quality of the item decreases. Textile items are real life examples for this phenomenon. The quality of textile items decreases gradually and finally it becomes unusable. Price reduction is an effective strategy to increase the sales volume and to settle the items before they perish.

In this paper we consider a queueing inventory system with S items which has a common life time of k stages. In each stage it is phase type distributed. When it is absorbed from one stage the items reach the next stage and finally reach the k^th stage. When it is absorbed from k^th stage the items attain CLT and become scrapped. The arriving customer gets the inventory in the present stage. The arrival rate of customers is assumed to be increasing in each stage due price reduction. The customers are not allowed in the absence of the inventory. Here we seek answers to two questions. Firstly, how much stock we need to maximize profits, and secondly, when to place an order to deliver items on time to waiting customers.

The common life time inventory models were first introduced by Lian, Z and Neuts, M.F (2005). The common life time was assumed to be of discrete phase type distribution. A perishable queueing inventory system with (Q, r) policy, Poisson demands, identical life times, constant lead times and lost sales with fixed ordering costs was studied by Berke, E and Gurler, L (2008). A cost function is used to find the optimal (Q, r) policy. Krishnamoorthy *et. al* (2006) describes a queueing inventory with reservation, cancellation, common life-time and retrial. Broekmeulen, R and Donselaar, K (2009) discuss a perishable queueing inventory with batch ordering, positive lead time and time varying demand. The replenishment is done using EWA policy (Estimated Withdrawal and Aging) in which the replenishment is done considering the items that will be out-dated within the lead time. Williams, C and Patuwo, B (1999) deal with a perishable inventory queueing system with positive lead time. The optimal reorder quantity is found using numerical computations. Schwarz, M. (2006) derive stationary distributions of joint queue length and inventory processes in explicit product form for various M/M/1-systems with inventory under continuous review and different inventory management policies, and with lost sales. Krishnamoorthy *et al* (2021) provides an overview of the work done in mathematical inventory models with positive service time. Kirubhashankar.*et. al* (2021) analyzes a production inventory model for deteriorating products with constant demand and backlogged shortages which depends on the length of the waiting time for the next order level. Optimum shortage time and total cycle length with the objective of optimizing the total cost are found.

Mehrez, A and Ben-Arieh, D (1991) describes a model that combines several dichotomies, into a single model. It has the objective of deciding the optimal order quantities for a multi-item inventory system over a finite horizon. The demand is probabilistic with service level constraints, and there is an all-unit price break, for orders that exceed a given size. Amirthakodi *et. al* (2015) consider a continuous review perishable inventory system with service facility consisting of finite waiting hall and a single server. The items are replenished based on variable ordering policy. The lead time is assumed to have phase type distribution. develop a deterministic inventory model with quantity discount, pricing and partial back ordering when the product in stock deteriorates with time. Wee, H. M (1999) investigate a base-stock perishable inventory system with Markovian demand and general lead-time and lifetime distributions. Janssen, L *et. al* (2016) presents a review of literature of deteriorating inventory models.

The rest of the paper is distributed as follows. In the second section model description is given. The steady state analysis, stability condition and various performance measures are discussed in the third section. The numerical illustration is given in the fourth section.

2 MODEL DESCRIPTION

Consider a queuing-inventory which has S items. Initially all items are in stage 1. As the items getting older the stages are changed to $2, \cdots, k-1, k$ and finally the CLT of the items is reached and the

 https://doi.org/10.17993/3cemp.2022.110250.15-31

items remain if any, are scrapped. In each stage the CLT is iid that has a phase type distribution with representation (α, S) of order$m$. In the ith stage the distribution is the i-fold convolution of the PH distribution. The inventory is replenished with (S,s) policy with positive lead time which follows an exponential distribution with parameter β.The replenishment order is placed when either the inventory level is dropped to s or CLT of the stocked items attains the level q where $1 \leq q \leq k$. The customers arrive to the system according to a Poisson process of rate $\lambda_i, i = 1, 2, \cdots, k$. A reduction in price is declared when the stage is changed from $1 \to 2, 2 \to 3, 3 \to 4, \ldots\ldots..k-1 \to k$. The service of customers is exponentially distributed with rate $\mu_i, i = 1, 2, \cdots, k$. The service of customers is exponentially distributed with rate $\mu_i, i = 1, 2, \cdots, k$. Customers are not allowed to enterwhen there is no inventory in the system.

Let $N(t)$ be the number of customers, $I(t)$ be the inventory level,$C(t)$, the stage of CLT and $J(t)$ is the phase of CLT at time t.

Then the process $\{(N(t), I(t), C(t), J(t)) : t \geq 0\}$ is a continuous time Markov chain with state space $\{(n, 0) : n \geq 0\}$U$\{(n, i, r, j) : n \geq 0, 1 \leq i \leq S, 1 \leq r \leq k, 1 \leq j \leq m\}U\{\hat{0}\}$

Transition due to arrival:

$(n, i, r, j) \to (n+1, i, r, j)$ with rate $\lambda_r, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq r \leq k, \quad 1 \leq j \leq m$

Transition due to service:

$(n, 1, r, j) \to (n-1, 0)$ with rate $\mu_r, \quad n \geq 1, \quad 1 \leq r \leq k, \quad 1 \leq j \leq m$

$(n, i, r, j) \to (n-1, i-1, r, j)$ with rate $\mu_r, \quad n \geq 1, \quad 2 \leq i \leq S, \quad 1 \leq r \leq k, \quad 1 \leq j \leq m$

Transition due to replenishment:

$(n, 0) \to (n, S, 1, j)$ with rate $\beta\alpha_j, \quad n \geq 0, \quad 1 \leq j \leq m$

$(n, i, r, j) \to (n, S, 1, j)$ with rate $\beta, \quad n \geq 0, \quad 1 \leq i \leq s, \quad 1 \leq r \leq k, \quad 1 \leq j \leq m$

$(n, i, r, j) \to (n, S, 1, j)$ with rate $\beta, \quad n \geq 0, \quad s+1 \leq i \leq S, \quad c \leq r \leq k, \quad 1 \leq j \leq m$

Transition due to phase change of Common Life Time:

$(n, i, k, j) \to (n, 0)$ with rate $t_j, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq j \leq m$

$(n, i, r, j) \to (n, i, r, p)$ with rate $t_{jp}, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq r \leq k-1, \quad 1 \leq j, p \leq m$

Transition due to stage change of Common Life Time:

$(n, i, r, j) \to (n, i, r+1, p)$ with rate $\alpha_p t_j, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq r \leq k-1, \quad 1 \leq j, p \leq m$

$(n, i, k, j) \to (n, 0)$ with rate $t_j, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq j \leq m$

The infinitesimal generator matrix of the process is given by,

$$Q = \begin{bmatrix} B_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

3 STEADY STATE ANALYSIS

Let $A = A_0 + A_1 + A_2$ and let $\Pi = (\pi_0, \pi_1, ..., \pi_S)$ be the steady state probability vector where $\pi_i = ((\pi_i)^1, (\pi_i)^2, \cdots, (\pi_i)^k)$. Here $(\pi_i)^r = (\pi_i(r, 1), \pi_i(r, 2), ...\pi_i(r, m)), 1 \leq i \leq S, 1 \leq r \leq k$

The condtions

 https://doi.org/10.17993/3cemp.2022.110250.15-31

$\pi A = 0$ and $\pi e = 1$, can be re-written as

$$-\pi_0\beta + \sum_{r=1}^{k}(\pi_1)^r\mu_r e_m + \sum_{i=1}^{S}(\pi_i)^k T^0 = 0$$
$$(\pi_i)^{r-1}(T^0\otimes\alpha)(1-\delta_{ir}) + (\pi_i)^{r-1}[(T-(\mu_r+\beta)I] + (\pi_{i+1})^r\mu_r = 0, 1\le i\le s, 1\le r\le k$$
$$(\pi_i)^{r-1}(T^0\otimes\alpha)(1-\delta_{ir}) + (\pi_i)^{r-1}(T-\mu_r I) + (\pi_{i+1})^r\mu_r = 0, s+1\le i\le S-1, 1\le r\le q-1$$
$$(\pi_i)^{r-1}(T^0\otimes\alpha)(1-\delta_{ir}) + (\pi_i)^{r-1}(T-(\mu_r+\beta)I) + (\pi_{i+1})^r\mu_r = 0, s+1\le i\le S-1, q\le r\le S$$
$$\pi_0\alpha + (\sum_{i=1}^{S}\sum_{r=1}^{k}(\pi_i)^r)T = 0$$
$$\sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j) = 1$$

Simplifying the last equation, $\pi_0^{-1} = 1 - \alpha T^{-1}e$
Solving the equations we get, π_i, for each $i = 0, 1, 2, ..., S$.

3.1 STABILITY CONDITION

Theorem:The system is stable if and only if

$$\text{ie, } \sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j)\lambda_r < \sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j)\mu_r$$

Proof The system is stable if $\pi A_0 e < \pi A_2 e$

$$\text{ie, } \sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j)\lambda_r < \sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j)\mu_r$$

3.2 STATIONARY PROBABILITIES

Let $x = (x_0, x_1, ...)$be the stationary probability vector of the system where
$x_i = \{x_i(0) : i \ge 0\}U\{x_i(p,r,j) : i\ge 0, 1\le p\le S, 1\le r\le k, 1\le j\le m\}$
Then $xQ = 0, \quad xe = 1$ which results in the following equations.

$$x_0B_1 + x_1A_2 = 0$$
$$x_{i-1}A_0 + x_iA_1 + x_{i+1}A_2 = 0, \quad i\ge 1$$

Let $x_i = x_0R^i, \quad i \ge 1$. Substituting in $\quad x_{i-1}A_0 + x_iA_1 + x_{i+1}A_2 = 0$, we get,

$$R^2A_2 + RA_1 + A_0 = 0$$

Let $R(0) = 0$ be the initial solution.
From the recurrence relation

$$R(n) = (R^2(n-1)A_2 + A_0)(-A_1^{-1}), \quad n\ge 1$$

we find R .

 https://doi.org/10.17993/3cemp.2022.110250.15-31

3.3 PERFORMANCE MEASURES

1. Expected number of customers in the system $=\sum_{i=0}^{\infty}\sum_{p=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m} ix_i(p,r,j)$
2. Expected number of items in the stage r
   $E_r = \sum_{i=0}^{\infty}\sum_{p=1}^{S}\sum_{j=1}^{m} px_i(p,r,j), r = 1,2,\cdots,k$
3. Expected number of scrapped items $E_s = \sum_{i=0}^{\infty}\sum_{p=1}^{S}\sum_{j=1}^{m} px_i(p,k,j)t_j$.
4. Probability that a newly joining customer is served instantaneously
   $= \sum_{p=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m} x_0(p,r,j)$.
5. Probability that the server is busy$= \sum_{i=1}^{\infty}\sum_{p=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m} x_i(p,r,j)$
   $= 1 - x_0e - \sum_{i=1}^{\infty} x_i(0)$
6. Probability that all items are sold before the realization of CLT $= \sum_{i=0}^{\infty} x_i(0)$
7. Probability that all items in a cycle are scrapped $= \sum_{i=0}^{\infty}\sum_{j=1}^{m} x_i(S,k,j)t_j$
8. Probability that there are no customers in the system $= \sum_{p=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m} x_i(p,r,j)$

3.4 DISTRIDUTION OF WAITING TIME OF A TAGGED CUSTOMER

Consider a customer who joins the system as n^{th} tagged customerand waiting for service where $n > 0$. The rank of the customer reduces to $n-1, n-2, \cdots, 2, 1$ as the customers in the queue join for service. Let $M(t)$ be the rank of the customer, $I(t)$ be the inventory level, C(t) the stage of CLT and J(t) the phase of CLT at time t. Then the process $\{(M(t), I(t), C(t), J(t)) : t \geq 0\}$ is a Markov chain with state space
$\{n, n-1, \cdots, 2, 1\} \times \{0\} \cup \{n, n-1, \cdots, 2, 1\} \times \{1, 2, \cdots, S\} \times \{1, 2, \cdots, k\} \times \{1, 2, \cdots, m\} \cup \{\hat{0}\}$
where $\hat{0}$ is the absorbing state of the Markov Chain which denotes the service epoch of the n^{th} tagged customer.

Let $q_r = Pr\{N(t) = r, I(t) = i, C(t) = p, J(t) = j\} = x_r(0) + \sum_{i=1}^{S}\sum_{p=1}^{k}\sum_{j=1}^{m} x_r(i,p,j)$
Let $\omega_r = \frac{q_r}{\sum_{r=1}^{\infty} q_r}$ Then $\omega = (\omega_r, \omega_{r-1}, \ldots\ldots\ldots\omega_1)$ is the initial probability vector of the distribution.

The infinitesimal generator matrix of the process is,

$$\hat{P} = \begin{bmatrix} W & W^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$W = \begin{bmatrix} W_{n,n} & W_{n,n-1} & & \\ & W_{n-1,n-1} & W_{n-1,n-2} & \\ & & & \\ & & & W_{2,1} \\ & & & W_{1,1} \end{bmatrix}$$

$W_{i,i} = A_0 + A_1, i = 1, \cdots r$ and $W_{i,i-1} = A_2, i = 2, \cdots, r$

and

$$W^0 = \begin{bmatrix} \mathbf{0} \\ \\ A_2 \end{bmatrix}$$

The waiting time distribution is $w(t) = \omega e^{Wx} W^0$ The average waiting time of a tagged customer$=-\omega(W^{-1})e$

https://doi.org/10.17993/3cemp.2022.110250.15-31

3.5 DISTRIBUTION OF THE DURATION OF A CYCLE

The expected cycle length is the duration of time elapsed between two consecutive replenishment orders.Starting from any system state, with the number of items in inventory S (an order has been placed and got materialised), we look at the evolution until the next replenishment takes place, either through realization of CLT or by reaching the inventory level to s. Consider the Markov chain $\{(I(t), N(t), S(t), J(t)) : t \geq 0\}$ with the state space $\{(0, r) : r \geq 0\} \cup \{(r, i, p, j) : 1 \leq r \leq S, 0 \leq i \leq H, 1 \leq p \leq k, 1 \leq j \leq m\} \cup \{\Delta\}$, where$\{\Delta\}$ is the absorbing state of the Markov chain which denotes the realization of the next replenishment by completing one cycle.

Let $p_r = Pr\{N(t) = i, I(t) = r, S(t) = 1\}$
$i\ e$:$p_r = \sum_{i=1}^{H} \sum_{j=1}^{m} x_i(r, 1, j)$
Let $\gamma_r = p_r / \sum_{r=0}^{S} p_r$ Then $\gamma = (\gamma_S, \gamma_{S-1}, \ldots\ldots..\gamma_0)$ is the initial probability vector of the distribution. The infinitesimal generator matrix of the process is,

$$\hat{Q} = \begin{bmatrix} \mathbf{U} & U^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$U = \begin{bmatrix} U_{S,S} & U_{S,S-1} & \mathbf{0} & \cdots & \mathbf{0} & U_{S,0} \\ U_{S-1,S} & U_{S-1,S-1} & U_{S-1,S-2} & \mathbf{0} & \cdots & U_{S-1,0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ U_{s,S} & \mathbf{0} & \cdots & U_{s,s} & \ddots & U_{s,0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ U_{0,S} & \mathbf{0} & \cdots & \cdots & \cdots & -\beta I_{S+1} \end{bmatrix}_{(S+1)\times(S+1)}$$

Here,

for $s + 1 \leq i \leq S$

$$U_{i,i} = \begin{bmatrix} K_{0,0} & K_{0,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & K_{1,1} & K_{0,1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & K_{1,1} & K_{0,1} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & K_{1,1} \end{bmatrix}_{(S+1)\times(S+1)}$$

and for $1 \leq i \leq s$

$$U_{i,i} = \begin{bmatrix} L_{0,0} & K_{0,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & L_{1,1} & K_{0,1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & L_{1,1} & K_{0,1} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & L_{1,1} \end{bmatrix}_{(S+1)\times(S+1)}$$

$$K_{0,0} = \begin{bmatrix} L_1 & M_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & L_2 & M_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & L_{k-1} & M_1 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & L_k \end{bmatrix}_{k\times k}$$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

$$K_{1,1} = \begin{bmatrix} P_1 & M_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & P_2 & M_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & P_{k-1} & M_1 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & P_k \end{bmatrix}_{k\times k}$$

$$K_{0,1} = \begin{bmatrix} N_1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & N_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & N_{k-1} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & N_k \end{bmatrix}_{k\times k}$$

$$L_{0,0} = \begin{bmatrix} U_1 & M_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & U_2 & M_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & U_{k-1} & M_1 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & U_k \end{bmatrix}_{k\times k}$$

$$L_{1,1} = \begin{bmatrix} V_1 & M_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & V_2 & M_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & V_{k-1} & M_1 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & V_k \end{bmatrix}_{k\times k}$$

For $1 \leq i \leq q-1, L_i = T - \lambda_i I_m, U_i = T - (\lambda_i + \mu_i)I_m$
For $q \leq i \leq k, L_i = T - (\lambda_i + \beta)I_m, U_i = T - (\lambda_i + \beta + \mu_i)I_m$
For $1 \leq i \leq k, P_i = T - (\lambda_i + \beta)I_m,\ V_i = T - (\lambda_i + \beta + \mu_i)I_m,\ N_i = \lambda_i I_m$ and $M_1 = T^0 \otimes \alpha$.

For $1 \leq i \leq S$

$$U_{i,i-1} = \begin{bmatrix} \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ J_{1,0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & J_{1,0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & J_{1,0} & \mathbf{0} \end{bmatrix}_{(S+1)\times(S+1)}$$

where

$$J_{1,0} = \begin{bmatrix} H_1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & H_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & H_{k-1} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & H_k \end{bmatrix}_{k\times k}$$

For $1 \leq i \leq k,\ H_i = \mu_i I_m$
For $s+1 \leq i \leq S,\ W_{i,S} = \mathbf{0}$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

$$U_{0,S} = \begin{bmatrix} M_{0,0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & M_{0,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & M_{0,0} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & M_{0,0} \end{bmatrix}_{(S+1)\times(S+1)}$$

$$M_{0,0} = \begin{bmatrix} \beta\alpha & \mathbf{0} & \cdots & \cdots & \mathbf{0} \end{bmatrix}_{1\times k}$$

For $1 \le i \le s$

$$U_{i,S} = \begin{bmatrix} M_{1,1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & M_{1,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & M_{1,1} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & M_{1,1} \end{bmatrix}_{(S+1)\times(S+1)}$$

$$M_{1,1} = \begin{bmatrix} \beta T & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \beta T & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \beta T & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \beta T \end{bmatrix}_{k\times k}$$

For $1 \le i \le S$

$$U_{i,0} = \begin{bmatrix} G_{1,1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & G_{1,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & G_{1,1} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & G_{1,1} \end{bmatrix}_{(S+1)\times(S+1)}$$

$$G_{1,1} = \begin{bmatrix} \mathbf{0} \\ \cdots \\ \mathbf{0} \\ T^0 \end{bmatrix}_{k\times 1}$$

The probability density function of the distribution is, $f(x) = \gamma e^{Ux} U^0$
So the mean of the distribution is the first raw moment, $(\mu_1)^{'} = -\gamma U^{-1} e$
So expected cycle length= $-\gamma U^{-1} e$

4 SPECIAL CASE: ERLANG CLT DISTRIBUTION

Consider an inventory item that has a common life time(CLT) with k stages. At the beginning of stage 1,the age of the item is zero. As the item gets older the age of the item changes to $2, 3, ..., \cdots, k-1, k$.Each stage duration is iid exponention random variable. After k^{th} stage the items remaining, if any, are scrapped.The CLT is modelled as an Erlang Distribution of order k. Customers arrive according to a Poisson process with rates depending on the stage of the CLT, $\lambda_i, i = 1, 2, \cdots, k$. The service rate in each stage is exponential with rate $\mu_i, i = 1, 2, \cdots, k$. A reduction in price is declared when the stage is changed from $1 \to 2, 2 \to 3, 3 \to 4, \ldots\ldots..k-1 \to k$. The stages are changed according to exponential

https://doi.org/10.17993/3cemp.2022.110250.15-31

distribution with rate α. When the inventory level drops to s or the item reaches the stage $q, 1 \leq q \leq k$ an order is placed.he inventory is replenished with positive lead time which is exponentially distributed with rate β.The customers are blocked in the absence of inventory to reduce the system cost.
Let $N(t)$ = Number of customers in the system, $I(t)$ = Inventory level, $C(t)$ is the stage of CLT at time t. Then $\{X(t) : t \geq 0\}$= $\{(N(t), I(t), S(t)) : t \geq 0 \}$ is a Markov process with state space $\{(0,0)\}U\{0,1,2,3,\dots\} \times \{1,2,\dots,S\} \times \{1,2,\cdots,k\}$.

4.1 INFINITESIMAL GENERATOR MATRIX

$$Q = \begin{bmatrix} B_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}$$

Where, each block has order $(kS+1) \times (kS+1)$ and,

$$B_1 = \begin{bmatrix} -\beta & \mathbf{0} & & B_{0,S} \\ B_{1,0} & B_{1,1} & & B_{1,S} \\ & & & \\ B_{s,0} & & B_{s,s} & B_{s,S} \\ & & & \\ B_{S,0} & \mathbf{0} & & B_{S,S} \end{bmatrix}$$

where, for $1 \leq i \leq s$

$$B_{i,i} = \begin{bmatrix} -(\lambda_1+\beta+\alpha) & \alpha & & \\ & -(\lambda_2+\beta+\alpha) & \alpha & \\ & \ddots & \ddots & \\ & & & -(\lambda_k+\beta+\alpha) \end{bmatrix}$$

, for $s+1 \leq i \leq S$

$$B_{i,i} = \begin{bmatrix} -(\lambda_1+\alpha) & \alpha & & & \\ & \ddots & \ddots & & \\ & & -(\lambda_q+\beta+\alpha) & \alpha & \\ & & & & \\ & & & & -(\lambda_k+\beta+\alpha) \end{bmatrix}$$

, for $1 \leq i \leq S$,

$$B_{i,0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha \end{bmatrix}$$

,

$$B_{0,S} = \begin{bmatrix} \beta & 0 & \cdots & 0 \end{bmatrix}$$

https://doi.org/10.17993/3cemp.2022.110250.15-31

for $1 \leq i \leq s$

$$B_{i,S} = \begin{bmatrix} \beta & 0 & \cdots & 0 \\ \beta & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ \beta & 0 & \cdots & 0 \end{bmatrix}$$

,

and for $s+1 \leq i \leq S$

$$B_{i,S} = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ \beta & 0 & \cdots & & 0 \\ \vdots & \cdots & \ddots & & \vdots \\ \beta & 0 & \cdots & & 0 \end{bmatrix}$$

,

$$A_0 = \begin{bmatrix} 0 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & L & \ddots & \cdots & \mathbf{0} \\ \vdots & \ddots & L & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & L \end{bmatrix}$$

$$L = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_k \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ M_1 & \mathbf{0} & \ddots & \cdots & \vdots \\ \mathbf{0} & M & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & M & \mathbf{0} \end{bmatrix}$$

where,

$$M_1 = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}$$

,

$$M = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mu_k \end{bmatrix}$$

https://doi.org/10.17993/3cemp.2022.110250.15-31

$$A_1 = \begin{bmatrix} -\beta & \mathbf{0} & \mathbf{0} & \cdots & \cdots & A_{0,S} \\ A_{1,0} & A_{1,1} & \mathbf{0} & \ddots & \cdots & A_{1,S} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ A_{s,0} & \cdots & \mathbf{0} & A_{s,s} & \ddots & A_{s,S} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ A_{S,0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_{S,S} \end{bmatrix}$$

where, for $1 \le i \le s$

$$A_{i,i} = \begin{bmatrix} -(\lambda_1+\mu_1+\beta+\alpha) & \alpha & \cdots & 0 \\ 0 & -(\lambda_2+\mu_2+\beta+\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha \\ 0 & \cdots & 0 & -(\lambda_k+\mu_k+\beta+\alpha) \end{bmatrix}$$

, for $s+1 \le i \le S$

$$A_{i,i} = \begin{bmatrix} -(\lambda_1+\mu_1+\alpha) & \alpha & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & -(\lambda_q+\mu_q+\beta+\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -(\lambda_k+\mu_k+\beta+\alpha) \end{bmatrix}$$

, for $1 \le i \le S$,

$$A_{i,0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha \end{bmatrix}$$

,

$$A_{0,S} = \begin{bmatrix} \beta & 0 & \cdots & 0 \end{bmatrix}$$

for $1 \le i \le s$

$$A_{i,S} = \begin{bmatrix} \beta & 0 & \cdots & 0 \\ \beta & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ \beta & 0 & \cdots & 0 \end{bmatrix}$$

and

$$A_{i,S} = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ \beta & 0 & \cdots & & 0 \\ \vdots & \cdots & \ddots & & \vdots \\ \beta & 0 & \cdots & & 0 \end{bmatrix}$$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

4.2 STEADY STATE ANALYSIS

Let $A = A_0 + A_1 + A_2$ and $\pi = (\pi_0, \pi_1, \pi_2, ..., \pi_S)$ be the steady state probability distribution of $\{(I(t), S(t)) : t \geq 0\}$, where $\pi_i = (\pi_{i1}, \pi_{i2}, ..., \pi_{ik}), \quad 1 \leq i \leq S.$
Then we have,
$-\pi_0\beta + \pi_1(M_1 + A_{1,0}) + \cdots + \pi_S A_{1,0} = 0$
$\pi_i(A_{1,1} + L) + \pi_{i+1}M = 0, \quad 1 \leq i \leq s$
$\pi_i(A_{S,S} + L) + \pi_{i+1}M = 0, \quad s+1 \leq i \leq S-1$ and
$\pi_0 A_{0,S} + (\pi_1 + \pi_2 + \cdots + \pi_{S-1})A_{1,S} + \pi_S(A_{S,S} + L) = 0$. Simplifying, $\pi_i = \pi_1[(A_{1,1} + L)M^{-1}]^i, \quad 1 \leq i \leq s$
$\pi_i = \pi_1[(A_{1,1} + L)M^{-1}]^s[(A_{S,S} + L)M^{-1}]^{i-s-1}, \quad s+1 \leq i \leq S$
Here π_1 and $\pi_0$can be obtained from the normalizing condition and the first equation.

4.3 STABILITY CONDITION

Theorem: The system is stable if and only if,
$\sum_{i=1}^{S}\sum_{j=1}^{k}\pi_{ij}\lambda_j < \sum_{i=1}^{S}\sum_{j=1}^{k}\pi_{ij}\mu_j$
Proof: The system is stable if and only if

$$\pi A_0 e < \pi A_2 e$$

Substituting we get,

$$\begin{array}{l}[0 \quad \pi_1 L \quad \pi_2 L \cdots \quad \pi_S L]e < \\ [0 \quad \pi_1 M_1 \quad \pi_2 M \cdots \quad \pi_S M \quad 0]e\end{array}$$

Simplifying, $\sum_{i=1}^{S}\sum_{j=1}^{k}\pi_{ij}\lambda_j < \sum_{i=1}^{S}\sum_{j=1}^{k}\pi_{ij}\mu_j$

Stationary Distribution
Let $x = (x_0, x_1, x_2, \cdots)$ be the steady state probability vector of the Makov process $X(t) = (N(t), I(t), S(t)), t \geq 0$. Then, $x_i = x_{i-1}R^i$, where R is the minimal non negative solution of the matrix equation,

$$R^2A_2 + RA_1 + A_0 = 0$$

Here, $x_i = x_i(j, m), 1 \leq j \leq S, 1 \leq m \leq k$.

4.4 PERFORMANCE MEASURES

1. Expected number of customers in the queue, $E_c = \sum_{i=0}^{\infty} ix_i(0) + \sum_{i=0}^{\infty}\sum_{j=1}^{S}\sum_{m=0}^{k} ix_i(j, m)$.
2. Expected number of items in the in the stage m, $E_m = \sum_{i=0}^{\infty}\sum_{j=1}^{S} jx_i(j, m)$.
3. Expected number of items scrapped, $E_s = \sum_{i=0}^{\infty}\sum_{j=1}^{S}\alpha jx_i(j, k)$.
4. Probability that all items are sold in the first stage, $E_s = \sum_{i=1}^{\infty}\mu_1 x_i(j, 1)$.
5. Probability that all items are scrapped, $x_0(S, k)$.

4.5 DISTRIBUTION OF THE DURATION OF A CYCLE

The expected cycle length is the duration of time elapsed between two consecutive replenishment orders. To compute this we proceed as follows.

Choose an ϵ and a $H(\epsilon)$ such that the probability of the number of customers in system exceeding H has probability $< \epsilon$. Take ϵ to be quite small (of the order 10^{-5}). Accordingly H will be large. Yet we have a finite system. We have the corresponding system state probability vector; the sum of these probabilities is very close to 1. Starting from any system state, with the number of items in inventory S

(an order has been placed and got materialised), we look at the evolution until the next replenishment takes place, either through realization of CLT or by the sales of the last item left in the inventory in that cycle.
Consider the Markov chain $X = \{X(t) : t \geq 0\} = \{(N(t), I(t), S(t)) : t \geq 0\}$

with the state space $\{(i, j, l) : 0 \leq i \leq H, 1 \leq j \leq S, 1 \leq l \leq k\}U\{\Delta\}$, where$\{\Delta\}$ is the absorbing state of the Markov chain which denotes the realization of CLT.

Let $q_i = Pr\{N(t) = i, I(t) = S, S(t) = 1\}$
and $\eta_i = q_i / \sum_{i=0}^{H} q_i$ Then $\eta = (\eta_0, \eta_1, \ldots\ldots.\eta_H)$ is the initial probability vector of the distribution. The infinitesimal generator of the process is

$$\hat{U} = \begin{bmatrix} \mathbf{V} & V^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where,

$$V = \begin{bmatrix} V_{S,S} & V_{S,S-1} & \mathbf{0} & \cdots & \mathbf{0} & V_{S,0} \\ \mathbf{0} & V_{S-1,S-1} & V_{S-1,S-2} & \mathbf{0} & \cdots & V_{S-1,0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & V_{s,s} & V_{s,s-1} & \ddots & V_{s,0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \ddots & V_{1,1} & V_{1,0} \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & V_{0,0} \end{bmatrix}_{(S+1)\times(S+1)}$$

Here,

for $s + 1 \leq i \leq S$

$$V_{i,i} = \begin{bmatrix} B_{S,S} & L & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_{S,S} & L & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & A_{S,S} & L \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_{S,S} \end{bmatrix}_{(H+1)\times(H+1)}$$

and for $1 \leq i \leq s$

$$V_{i,i} = \begin{bmatrix} B_{1,1} & L & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_{1,1} & L & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & A_{1,1} & L \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_{1,1} \end{bmatrix}_{(H+1)\times(H+1)}$$

For $2 \leq i \leq S$

$$V_{i,i-1} = \begin{bmatrix} \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ M & \ddots & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & M & \mathbf{0} \end{bmatrix}_{(H+1)\times(H+1)}$$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

$$V_{1,0} = \begin{bmatrix} A_{1,0} & \mathbf{0} & \cdots & \mathbf{0} \\ M_1 & A_{1,0} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & M_1 & A_{1,0} \end{bmatrix}_{(H+1)\times(H+1)}$$

For $2 \leq i \leq S$

$$V_{i,0} = \begin{bmatrix} A_{1,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_{1,0} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & A_{1,0} \end{bmatrix}_{(H+1)\times(H+1)}$$

$V_{0,0} = -\beta I_{H+1}$ and

$$V^0 = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ B \\ \vdots \\ B \end{bmatrix}_{(S+1)\times 1}$$

$$B = \begin{bmatrix} \beta \\ \vdots \\ \beta \end{bmatrix}_{k\times 1}$$

The probability density function of the distribution is, $f(x) = \eta e^{Vx} V^0$

So the mean of the distribution is the first raw moment, $\mu_1^{'} = -\eta V^{-1} e$
Therefore expected cycle length$= -\eta V^{-1} e$

5 OPTIMIZATION PROBLEM

Based on the above performance measures we construct a cost function to find the profit of the retailer. Let C_r be the price per unit of the item in the rth stage for$r = 1, \cdots, k$. Assume that the holding cost increases with the age of the stocked item and let H_r be the holding cost per unit of the item in the$r^t h$ stage. Let K be the fixed ordering cost while placing a replenishment order and c be the variable procurement cost (per unit) of the item to the store.
The total revenue $= \sum_{r=1}^{k} (C_r - H_r) E_r + C_s crap E_s crap - (K + cS)/cycle\ length$, where $C_s crap$ is the tagged price for scrapped item and $E_s crap$ is the expected number of scrapped items.

6 NUMERICAL ILLUSTRATION

In this section we provide numerical illustration of the profit function with variation in the values of s and S in both Phase Type and Erlang cases.
Table 1: **Effect of S and s on Profit in Phase-Type Case** We fixthe parameters as $k = 4, m =$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

$3, \lambda_1 = 4, \lambda_2 = 4.5, \lambda_3 = 5, \lambda_4 = 6, \mu_1 = 7, \mu_2 = 7.5, \mu_3 = 8, \mu_4 = 8.5, \beta = 5, \alpha = [0.4, 0.4, 0.2]$

$$T = \begin{bmatrix} -15 & 4 & 6 \\ 1 & -13 & 5 \\ 3 & 4 & -17 \end{bmatrix}$$

and costs as $C_1 = 10, C_2 = 8, C - 3 = 6, C_4 = 2, C_s crap = 1.5, H_1 = 2, H_2 = 1.5, H_3 = 1, H_4 = 1, c = 5, K = 2$

$S \downarrow s \rightarrow$	1	2	3	4	5	6	7
6	105.91	104.67	103.04	99.02	96.26		
7	112.82	112.40	111.67	110.60	107.32	105.32	
8	121.60	121.38	121.01	120.32	119.20	116.07	113.18
9	131.24	131.09	130.86	130.50	129.80	128.57	125.47
10	141.34	141.22	141.06	140.84	140.47	139.77	138.50
11	151.73	151.62	151.49	151.33	151.10	150.72	149.98
12	162.32	162.21	162.01	161.97	161.80	161.57	161.17
13	173.06	172.94	172.95	172.71	172.58	172.42	172.19
14	183.89	183.78	183.67	183.55	183.44	183.31	183.15
15	194.81	194.69	194.58	194.46	194.35	194.23	194.11

Table 2: **Effect of S and s on Profit in Erlang Case** We fix the parameters as $k = 4, m = 3, \lambda_1 = 4, \lambda_2 = 4.5, \lambda_3 = 5, \lambda_4 = 6, \mu_1 = 7, \mu_2 = 7.5, \mu_3 = 8, \mu_4 = 8.5, \beta = 5, \alpha = 5$
and the costs as $C_1 = 10, C_2 = 8, C - 3 = 6, C_4 = 2, C_s crap = 1.5, H_1 = 2, H_2 = 1.5, H_3 = 1, H_4 = 1, c = 5, K = 2$

$S \downarrow s \rightarrow$	1	2	3	4	5	6	7
6	75.09	64.82	53.67	41.60	29.13		
7	87.14	77.71	67.49	56.25	43.83	30.73	
8	98.51	89.98	80.77	70.57	59.13	46.30	32.52
9	109.19	101.58	93.35	84.23	73.92	62.19	48.88
10	119.28	112.55	105.31	97.24	88.08	77.58	65.51
11	128.74	122.86	116.52	109.42	101.34	92.04	81.27
12	137.66	132.57	127.05	120.86	113.78	105.60	96.10
13	146.07	141.69	136.94	131.58	125.42	118.27	109.93
14	154.08	150.34	146.27	141.66	146.35	130.14	122.88
15	161.66	158.49	155.03	151.10	146.53	141.17	134.89

From the table it is clear that the profit increses as S increases but it decreases as s increases. The change of profit when s is changed is very sensitive in Erlang case compared to Phase Type case.

REFERENCES

[1] **Lian, Z.** and **Neuts, M.F.** (2005).A Discrete-Time Model for Common Lifetime Inventory Systems.*Mathematics of Operations Research, 30*(3), 718–732.
https://doi.org/10.1287/moor.1040.0133.

[2] **Berk, E.** and **Gürler, L.** (2008).Analysis of the (Q, r) Inventory Model for Perishables with Positive Lead Times and Lost Sales.*Operations Research, 56*(5), 1238–1246.
https://doi.org/10.1287/opre.1080.0582.

[3] **Krishnamoorthy, A.** ,**Shajin, D.** and **Lakshmy, B.** (2016).On a queueing-inventory with reservation, cancellation, common life time and retrial.*Ann Oper Res 247*, 365–389.
https://doi.org/10.1007/s10479-015-1849-x.

[4] **Broekmeulen, R.** and **Donselaar, K.** (2009).A heuristic to manage perishable inventory with batch ordering, positive lead-times, and time-varying demand.*Computers and Operations Research*

36(11),3013-3018.
https://doi.org/10.1016/j.cor.2009.01.017.

[5] **Williams, C. L.** and **Patuwo, B. E.** (1999).A perishable inventory model with positive order lead times, positive lead-times, and time-varying demand.*European Journal of Operational Research, 116*(2), 352-373.
https://doi.org/10.1016/S0377-2217(98)00105-2.

[6] **Duan, L.** and **Ventura, J. A.** (2019).A dynamic supplier selection and inventory management model for a serial supply chain with a novel supplier price break scheme and flexible time periods.*European Journal of Operational Research, 272*(3), 979-998.
https://doi.org/10.1016/j.ejor.2018.07.031.

[7] **Schwarz, M.**, **Sauer, C.**, **Daduna, H.**,**Kulik, R.** and **Szekli, R.**(2006).M/M/1 queueing system with inventory. *Queueing Systems: Theory and Applications 1, 54*(1), 55-78.
https://doi.org/10.1007/s11134-006-8710-5.

[8] **Krishnamoorthy, A.**, **Dhanya Shajin.**, and **Narayanan,W.**(2021) .Inventory with Positive Service Time: a Survey. *Edited volume with V. Anisimov and N. Limnious, Wiley,.*
https://doi.org/10.1002/9781119755234.ch6.

[9] **Kirubhashankar, C. K.**, **uniappan, P.** and **Ismail, A. M.** (2021). An production inventory model for deteriorating product with price breaks. *In Journal of Physics: Conference Series (Vol. 1770, No. 1, p. 012102),.*
https://dx.doi.org/10.1088/1742-6596/1770/1/012102.

[10] **Mehrez,A.** and **Ben-Arieh,D.**(1991).All-unit discounts, multi-item inventory model with stochastic demand, service level constraints and finite horizon.*The International Journal of Production Research 29*(8), 1615-1628.
https://doi.org/10.1080/00207549108948035.

[11] **Amirthakodi, M.**, **Radhamami, V.**, and **Sivakumar, B.** (2015). A perishable inventory system with service facility and feedback customers. *Annals of Operations Research, 233*, 25-55, https://doi.org/10.1007/s10479-015-1821-9.

[12] **Wee, H. M.** (1999). Deteriorating inventory model with quantity discount, pricing and partial backordering. *International Journal of Production Economics, 59*(1-3), 511-518, https://doi.org/10.1016/S0925-5273(98)00113-3.

[13] **Janssen, L.**,**Claus, T.** and **Sauer, J.** (2016). Literature review of deteriorating inventory models by key topics from 2012 to 2015 *International Journal of Production Economics*, 182, 86-112, http://dx.doi.org/10.1016/j.ijpe.2016.08.019.