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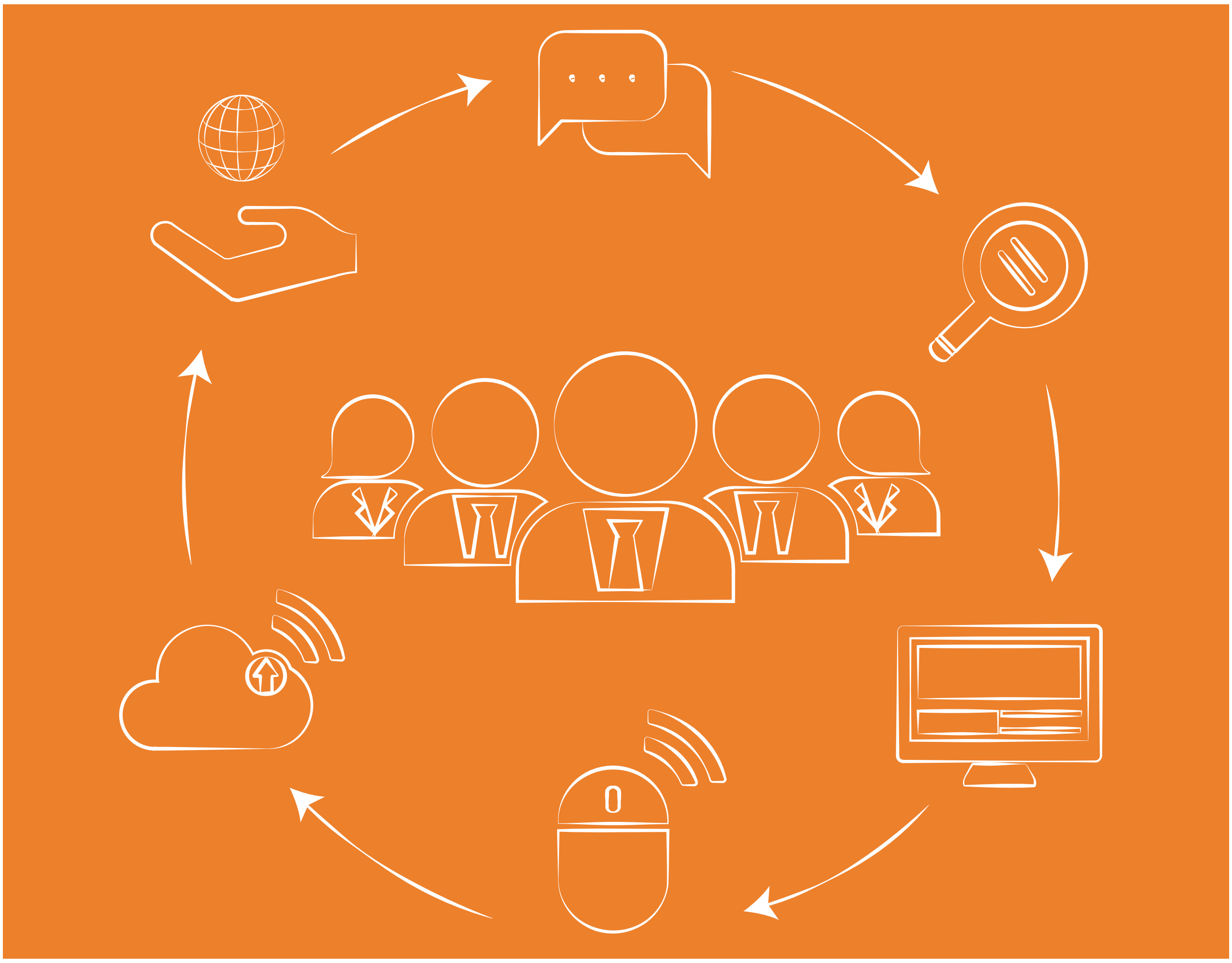

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/01/

FIXED POINT THEOREMS FOR SUZUKI NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT

In this paper, we investigate the existence of fixed points for Suzuki nonexpansive mappings in the setting of Banach spaces using the asymptotic center technique. We also establish the convergence of regular approximate fixed point sequence to the fixed points of Suzuki nonexpansive mappings. Examples are also given to illustrate the results. Our theorems generalize several results in the literature.

KEYWORDS

Banach space, Nonexpansive mapping, Suzuki nonexpansive mapping, Fixed point, Approximate fixed point.

1 INTRODUCTION

Fixed point results for nonexpansive mappings in Banach spaces are of great importance in the development of fixed point theory and are widely used to solve problems in diverse fields such as differential equations, game theory, engineering, medicine and many more (see [3, 14, 16]. The possibility of using the theory in a wide range of applications has attracted many researchers and has consequently resulted in a rapid growth of research in this field. Several authors have introduced extensions of nonexpansive mappings such as generalized nonexpansive mappings, relatively nonexpansive mappings, α- nonexpansive mappings, etc. (see [1, 6, 15]) and proved fixed point results in Banach spaces and various other spaces as well. In 2008, Suzuki introduced a new condition called condition (C) [17]. The mapping which satisfies condition (C) is now known as Suzuki nonexpansive mapping. Suzuki proved that all nonexpansive mappings satisfy the condition (C). Unlike nonexpansive mappings, the Suzuki nonexpansive mappings need not be always continuous. We can find a couple of examples for mappings which are not continuous but satisfying condition (C) in [17].

There are several techniques for finding the fixed points of nonexpansive mappings. One of the most widely used techniques, introduced by Edelstein in 1972 [5], uses the concept of asymptotic radius and asymptotic center of a sequence relative to a set K. Many researchers have used the properties and geometric behavior of the asymptotic center of sequences under consideration, to prove several fixed point results for nonexpansive mappings (see [5, 9]). For any sequence, asymptotic center can be considered as the intersection of some closed balls [7]. Therefore, the asymptotic center is always closed. But it need not be nonempty. Researchers proved that if a set K is nonempty, weakly compact and convex, then the asymptotic center of any sequence in K has the same properties as K [7]. These results boosted the usefulness of asymptotic center technique as a tool to find fixed points for Suzuki nonexpansive mapping. Dhompongsa [4] used this technique in Suzuki nonexpansive mapping and proved that if K is a Banach space and T is a self mapping of K satisfying condition (C), then for any bounded approximate fixed point sequence in K, the asymptotic center relative to K is invariant under T. Another equally important method to find fixed points in Banach spaces is the Chebyshev center technique, which uses the concept of Chebyshev radius and Chebyshev center to analyze the geometric structure of a set. A good amount of research work is reported in literature which makes use of these two techniques to find fixed points (see [4, 5, 7–10, 13].

In this paper, we primarily focus on the asymptotic center technique and show that it is possible to derive an interesting relation between the asymptotic radius and Chebyshev radius under certain conditions. In [8], Kirk proved a fixed point result for nonexpansive mapping in reflexive Banach spaces having normal structure. We extended this result for Suzuki nonexpansive mapping in weakly compact Banach spaces. Also, we investigated some sufficient conditions for the existence of fixed points for Suzuki nonexpansive mapping in a closed, bounded and convex subset of a Banach space. Apart from these, we also developed certain sufficient conditions for the convergence of regular approximate fixed point sequences to the fixed points of Suzuki nonexpansive mappings.

1.1 PRELIMINARIES

Definition 1. [7] *A mapping T on a subset K of a Banach space X is called a nonexpansive mapping if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in K$.*

Definition 2. [17] *Let T be a mapping on a subset K of a Banach space X. Then T is said to satisfy condition (C) if for all $x, y \in K$,*

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \implies ||Tx - Ty|| \leq ||x - y||.$$

The mapping satisfying condition (C) is called Suzuki nonexpansive mapping.

Definition 3. [10] *Let $T : K \to K$ be any mapping. A sequence $\{x_n\}$ in K is called approximate fixed point sequence if $||Tx_n - x_n|| \to 0$ as $n \to \infty$.*

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Definition 4. [4] *Let K be a nonempty closed and convex subset of a Banach space X and $\{x_n\}$, a bounded sequence in X. For $x \in X$, the asymptotic radius of $\{x_n\}$ at x is defined as*

$$r(x, \{x_n\}) = \limsup\{||x_n - x||\}.$$

The asymptotic radius and asymptotic center of $\{x_n\}$ relative to K are defined as follows:

$$r \equiv r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}$$

$$A \equiv A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r\}.$$

Definition 5. [9] *A bounded sequence is said to be regular if each of its subsequence has the same asymptotic radius.*

Definition 6. [9] *A bounded sequence is said to be uniform if each of its subsequence has the same asymptotic center.*

Definition 7. [7] *For any subset K of X, the radius of K relative to x, Chebyshev radius of K, Chebyshev center of K and the diameter of K are defined as follows:*
For any $x \in X$, $r_x(K) = \sup\{||x - y|| : y \in K\}$

$$r(K) = \inf\{r_x(K) : x \in K\}$$

$$C(K) = \{x \in K : r_x(K) = r(K)\}$$

$$diam(K) = \sup\{r_x(K) : x \in K\}.$$

Definition 8. *A nonempty, closed, convex subset D of a given set K is said to be a minimal invariant set for a mapping $T : K \to K$ if $T(D) \subseteq D$ and D has no nonempty, closed and convex proper subsets which are T-invariant.*

Definition 9. [7, 8] *A convex subset K of X is said to have normal structure if each bounded, convex subset S of K with $diam S > 0$ contains a nondiametral point.*

Definition 10. [2] *A convex set K of X is said to have asymptotic normal structure if, given any bounded convex subset S of K which contains more than one point and given any decreasing net of nonempty subsets $\{s_\alpha; \alpha \in A\}$ of S, the asymptotic center of $\{s_\alpha; \alpha \in A\}$ in S is a proper subset of S.*

Definition 11. [11] *Let X be a Banach space. X is said to have Opial property if for each weakly convergent sequence $\{x_n\}$ in X with weak limit z and for all $x \in X$ with $x \neq z$,*

$$\limsup ||x_n - z|| < \limsup ||x_n - x||.$$

Lemma 1. [17, Lemma 6] *Let T be a mapping on a bounded convex subset K of a Banach space X. Assume that T satisfies condition (C). Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and*

$$x_{n+1} = \lambda T x_n + (1 - \lambda) x_n$$

for $n \in \mathbb{N}$, where λ is a real number belonging to $[\frac{1}{2}, 1)$. Then

$$\lim_{n \to \infty} ||Tx_n - x_n|| = 0$$

holds.

Lemma 2. [7, Lemma 9.1] *Let $\{x_n\}$ be a sequence in a Banach space X and K a nonempty subset of X.*
(a) If K is weakly compact, then $A(K, \{x_n\}) \neq \phi$.
(b) If K is convex, then $A(K, \{x_n\})$ is convex.

Lemma 3. [4, Lemma 3.1] *Let K be a subset of a Banach space X, and $T : K \to K$ be a mapping satisfying condition (C). Suppose $\{x_n\}$ is a bounded approximate fixed point sequence for T. Then $A(K, \{x_n\})$ is invariant under T.*

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Proposition 1. [9, Preposition 1] *Every bounded sequence has a regular subsequence.*

Theorem 1. [7, Theorem 3.2] *Suppose K is a nonempty, weakly compact, convex subset of a Banach space. Then for any mapping $T : K \to K$ there exists a closed convex subset of K which is minimal T-invariant.*

Theorem 2. [12, Theorem 1] *A convex subset K of X has normal structure if and only if K has asymptotic normal structure.*

Remark 1. [7] *It is clear that if for any sequence $\{x_n\}$ in K and $x \in X$, $r(x, \{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$.*

Remark 2. [9] *If $\{x_{n_k}\}$ is a subset of $\{x_n\}$, then $r(K, \{x_{n_k}\}) \leq r(K, \{x_n\})$ and if $r(K, \{x_{n_k}\}) = r(K, \{x_n\})$, then $A(K, \{x_n\}) \subseteq A(K, \{x_{n_k}\})$.*

2 RESULTS

Theorem 3. *Let K be a weakly compact convex subset of a Banach space X and $T : K \to K$ satisfies condition (C). Assume that K is minimal T-invariant and $\{x_n\}$ is an approximate fixed point sequence in K. Then*
(i) $A(K, \{x_n\}) = K$
(ii) $r(K, \{x_n\}) = r(K)$.

Proof. Let K be a weakly compact and convex subset of a Banach space X.
Suppose $diam(K) = 0$. Then there is nothing to prove.
Now, suppose $diam(K) > 0$.
Let $\{x_n\}$ be any bounded approximate fixed point sequence in K. Then $A(K, \{x_n\})$ is closed.
Also by Lemma 2, $A(K, \{x_n\})$ is nonempty and convex. Thus $A(K, \{x_n\})$ is weakly compact.
By Lemma 3, $A(K, \{x_n\})$ is T-invariant and by the minimality of K, we have $A(K, \{x_n\}) = K$.
Since K is weakly compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in K$ such that $x_{n_k} \to z$ weakly.
Clearly $\{x_{n_k}\}$ is an approximate fixed point sequence in K.
Since $A(K, \{x_{n_k}\}) = K$, $\limsup ||x_{n_k} - x|| = r(K, \{x_{n_k}\})$ for all $x \in K$.
Also, we have for any $x \in K$, $\limsup ||x_{n_k} - x|| \leq \sup\{||x - y|| : y \in K\}$.
Therefore,

$$r(K, \{x_{n_k}\}) \leq r_x(K) \implies r(K, \{x_{n_k}\}) \leq r(K). \tag{1}$$

Now, for any $y \in K$, $x_{n_k} - y \to z - y$ weakly and hence we have

$$||z - y|| \leq \limsup ||x_{n_k} - y|| = r(K, \{x_{n_k}\}).$$

Hence for all $y \in K$,

$$\begin{aligned} ||z - y|| \leq r(K, \{x_{n_k}\}) &\implies r_z(K) \leq r(K, \{x_{n_k}\}) \\ &\implies r(K) \leq r(K, \{x_{n_k}\}). \end{aligned} \tag{2}$$

Thus from (1) and (2) we get

$$r(K, \{x_{n_k}\}) = r(K). \tag{3}$$

We know that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$,

$$r(K, \{x_{n_k}\}) \leq r(K, \{x_n\}). \tag{4}$$

Since $r(K, \{x_{n_k}\}) = r(K)$, if $x \in C(K)$, then for all $y \in K$, $||x - y|| \leq r(K, \{x_{n_k}\})$.
Therefore, for all x_n,

$$||x_n - x|| \leq r(K, \{x_{n_k}\}) \implies r(K, \{x_n\}) \leq r(K, \{x_{n_k}\}). \tag{5}$$

Hence from (4) and (5), $r(K, \{x_n\}) = r(K, \{x_{n_k}\})$. Thus from (3) we get
$r(K, \{x_n\}) = r(K)$.

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Corollary 1. *Let K be a weakly compact convex subset of a Banach space X and $T : K \to K$ satisfies condition (C). If K is minimal T-invariant, then every approximate fixed point sequence in K are regular and uniform.*

Proof. Let $\{x_n\}$ be an approximate fixed point sequence in K. Then every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is also an approximate fixed point sequence.
Hence by (ii) in Theorem 3, $r(K, \{x_n\}) = r(K, \{x_{n_k}\})$.
Therefore, $\{x_n\}$ is regular.
By (i) in Theorem 3, $A(K, \{x_n\}) = A(K, \{x_{n_k}\}) = K$.
Hence $\{x_n\}$ is uniform.

Corollary 2. [10, Preposition 6.3] *Let K be a weakly compact convex subset of a Banach space X, and $T : K \to K$ be a nonexpansive mapping. Assume that K is minimal for T, that is, no closed convex bounded proper subset of K is invariant for T. If $\{x_n\}$ is an approximate fixed point sequence in K, then $A(K, \{x_n\}) = K$.*

Proof. Since every nonexpansive mapping satisfies condition (C), by above theorem, $A(K, \{x_n\}) = K$.

Theorem 4. *Let K be a nonempty, weakly compact and convex subset of a Banach space X and suppose K has normal structure. Then every mapping $T : K \to K$ satisfying condition (C) has a fixed point.*

Proof. By Theorem 1, we can consider K as closed, convex minimal T- invariant subset.
Suppose $diamK > 0$.
Consider an approximate fixed point sequence $\{x_n\} \subseteq K$.
By (i) in Theorem 3, $A(K, \{x_n\}) = K$.
Now, define $W_n := \{x_m : m \geq n\},\ n \in \mathbb{N}$.
Clearly, $\{W_n, n \in \mathbb{N}\}$ is a decreasing chain of nonempty bounded subsets of K.
We can easily prove that Asymptotic center of $\{W_n, n \in \mathbb{N}\} = A(K, \{x_n\}) = K$.
Since K has normal structure, by Theorem 2, K has asymptotic normal structure.
Thus $A(K, \{x_n\}) = K$, which is a contradiction.
Therefore, $diamK = 0$ and hence K has only one element x (say).
Thus $T(x) = x$.

We obtain the result of Kirk [8] and Theorem 4.1 in [7] as corollaries of our result.

Corollary 3. [8, Theorem] *Let K be a nonempty, bounded, closed and convex subset of a reflexive Banach space X, and suppose that K has normal structure. If $T : K \to K$ is nonexpansive, then T has a fixed point.*

Proof. Since a bounded, closed and convex subset of a reflexive Banach space is weakly compact and every nonexpansive mapping is Suzuki nonexpansive, by the above theorem, T has a fixed point.

Corollary 4. [7, Theorem 4.1] *Let K be a nonempty, weakly compact, convex subset of a Banach space, and suppose K has normal structure. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

Proof. Since every nonexpansive mapping is Suzuki nonexpansive, by the above theorem, T has a fixed point.

The following theorem gives sufficient conditions for the existence of fixed points for Suzuki nonexpansive mapping in Banach spaces.

Theorem 5. *Let K be a closed, bounded and convex subset of a Banach space X and $T : K \to K$ satisfies condition (C). If $T(K)$ is contained in a compact subset of K, then T has a fixed point in K.*

Proof. Let $\{x_n\}$ be an approximate fixed point sequence in K.
Therefore, $||Tx_n - x_n|| \to 0$.
Consider the sequence $\{Tx_n\}$ in $T(K)$.
Since $T(K)$ is a subset of a compact set of K, there exist a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ and $z \in K$ such that $Tx_{n_k} \to z$.
Therefore, $\lim_{k\to\infty} ||x_{n_k} - z|| = \lim_{k\to\infty} ||x_{n_k} - Tx_{n_k}|| = 0$.
Hence $x_{n_k} \to z$.
Clearly, $\{x_{n_k}\}$ is a bounded approximate fixed point sequence and $A(K, \{x_{n_k}\}) = \{z\}$.
Also by Lemma 3, $A(K, \{x_{n_k}\})$ is T-invariant and hence $T(z) = z$.

The following is an example to illustrate this theorem.

Example 1. *In the space l_2 consider the closed unit ball*
$K = \{x = (x_1, x_2, ...) \in l_2 : ||x||_2 \leq 1\}$.
Define $T : K \to K$ as

$$T(x) = \begin{cases} (\frac{1}{3}, 0, 0, ..), & if\ x = (1, 0, 0, ...) \\ 0, & otherwise. \end{cases}$$

For all $x, y \neq (1, 0, 0, ...)$ in K, $||Tx - Ty||_2 = 0 \leq ||x - y||_2$.
If $x = (1, 0, 0, ...)$ then $\frac{1}{2}||x - Tx||_2 = \frac{1}{2}||(\frac{2}{3}, 0, 0, ...)||_2 = \frac{1}{3}$.
Therefore, if $\frac{1}{2}||x - Tx||_2 \leq ||x - y||_2$ for any $y \in K$, then $\frac{1}{3} \leq ||x - y||_2$.
Thus $||Tx - Ty||_2 = \frac{1}{3} \leq ||x - y||_2$.
Hence T satisfies condition (C).
$T(K) = \{(0, 0, 0, ...), (\frac{1}{3}, 0, 0, ...)\}$ is a subset of a compact set of K.
Thus all conditions in the above theorem are satisfied.
Since $T(0) = 0$, T has a fixed point.

Theorem 6. *Let X be a Banach space with Opial property and K be a closed, bounded and convex subset of X. Let $T : K \to K$ satisfies condition (C) and if $T(K)$ is contained in a weakly compact subset of K, then T has a fixed point in K.*

Proof. Let $\{x_n\}$ be an approximate fixed point sequence in K. Then we have, $||Tx_n - x_n|| \to 0$.
Since $\frac{1}{2}||Tx_n - x_n|| \leq ||Tx_n - x_n||$ and T satisfies condition (C), we have $||T^2x_n - Tx_n|| \leq ||Tx_n - x_n||$ for all $n \in \mathbb{N}$.
Therefore, $\lim_{n\to\infty} ||T^2x_n - Tx_n|| \leq \lim_{n\to\infty} ||Tx_n - x_n|| = 0$.
Hence $\{T(x_n)\}$ is an approximate fixed point sequence.
Since $T(K)$ is a subset of a weakly compact set of K, there exist a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ and $z \in K$ such that $Tx_{n_k} \to z$ weakly as $k \to \infty$.
By Opial property, for any $x \neq z$, $\limsup ||Tx_{n_k} - z|| < \limsup ||Tx_{n_k} - x||$.
Therefore, for any $x \neq z, r(z, \{Tx_{n_k}\}) < r(x, \{Tx_{n_k}\}) \implies r(K, \{Tx_{n_k}\}) = r(z, \{Tx_{n_k}\})$ and $A(K, \{Tx_{n_k}\}) = \{z\}$.
By Lemma 3, $A(K, \{x_{n_k}\})$ is T-invariant and hence $T(z) = z$.

Corollary 5. [17, Theorem 4] *Let T be a mapping on a convex subset K of a Banach space X. Assume that T satisfies condition (C). Assume also that either of the following holds:*
(i) K is compact;
(ii) K is weakly compact and X has the Opial property.
Then T has a fixed point.

Proof. Suppose K is compact and convex. Therefore, K is closed, bounded and convex.
Also, since $T : K \to K$, we have $T(K) \subseteq K$. Thus $T(K)$ is a subset of a compact set.
Hence by Theorem 5, T has a fixed point.
Now suppose K is weakly compact, convex and X has the Opial property.
Therefore, K is closed, bounded and convex. Also, since $T : K \to K$, we have $T(K) \subseteq K$.

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Thus $T(K)$ is a subset of a weakly compact set.
Hence by Theorem 6, T has a fixed point.

Theorem 7. *Let K be a compact subset of a Banach space X. Let $T : K \to K$ satisfies condition (C). Let $\{x_n\}$ be a regular approximate fixed point sequence in K. Then $\{x_n\}$ converge strongly to a fixed point of T.*

Proof. Let $\{x_n\}$ be a regular approximate fixed point sequence in K.
Since K is compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in K$ such that $x_{n_k} \to z$.
Therefore, $A(K, \{x_{n_k}\}) = \{z\}$.
Since $\{x_{n_k}\}$ is an approximate fixed point sequence, by Lemma 3, $A(K, \{x_{n_k}\})$ is T-invariant. Hence $T(z) = z$.
Since $\{x_n\}$ is regular, $r(K, \{x_n\}) = r(K, \{x_{n_k}\}) = 0$.
Since K is compact and $\{x_n\}$ is an approximate fixed point sequence, $A(K, \{x_n\})$ is nonempty.
We know that if $\{x_n\}$ is regular, then for any subsequence $\{x_{n_k}\}$,
$A(K, \{x_n\}) \subseteq A(K, \{x_{n_k}\})$.
Therefore, $A(K, \{x_n\}) = \{z\}$. Hence $x_n \to z$.

The following example shows that even if T has fixed points, if $\{x_n\}$ is not regular, then $\{x_n\}$ need not converge to a fixed point.

Example 2. *Consider the compact set $K = [-1, 1]$ in $\mathbb{R}$ and define $T : K \to K$ as $T(x) = x$.*
Clearly T is a nonexpansive mapping and hence satisfies condition (C).
Consider $x_n = (-1)^n$ for all $n \in \mathbb{N}$
For all $n \in \mathbb{N}$, $||x_n - Tx_n|| = 0$. Thus $\{x_n\}$ is an approximate fixed point sequence.
But $\{x_n\}$ does not converge to a fixed point.
This will not contradict the above theorem because $\{x_n\}$ is not regular.
Asymptotic radius of $\{x_n\} = r(K, \{x_n\}) = 1$ and $A(K, \{x_n\}) = \{0\}$.
Consider the subsequence $\{x_{2n}\} = \{1, 1, 1, ...\}$. Clearly $x_{2n} \to 1$.
Therefore, $A(K, \{x_{2n}\}) = \{1\}$ and $r(K, \{x_{2n}\}) = 0$.
Hence $\{x_n\}$ is not regular.

Theorem 8. *Let K be a subset of a Banach space X and $T : K \to K$ satisfies condition (C). Suppose $T(K)$ is contained in a compact subset of K and let $\{x_n\}$ be a regular approximate fixed point sequence in K with nonempty asymptotic center. Then $\{x_n\}$ converge strongly to a fixed point of T.*

Proof. Let $\{x_n\}$ be a regular approximate fixed point sequence in K with nonempty asymptotic center.
Consider the sequence $\{Tx_n\}$ in $T(K)$. Since $T(K)$ is compact, there exist a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ and $z \in T(K)$ such that $Tx_{n_k} \to z$.
Therefore, $\lim_{n\to\infty} ||x_{n_k} - z|| \leq \lim_{n\to\infty} ||x_{n_k} - Tx_{n_k}|| = 0$.
Thus $x_{n_k} \to z$ and hence $A(K, \{x_{n_k}\}) = \{z\}$.
Since $\{x_{n_k}\}$ is an approximate fixed point sequence, by Lemma 3, $A(K, \{x_{n_k}\})$ is T-invariant.
Hence $T(z) = z$.
Since $\{x_n\}$ is regular, $r(K, \{x_n\}) = r(K, \{x_{n_k}\}) = 0$.
Also if $\{x_n\}$ is regular, then for any subsequence $\{x_{n_k}\}$, we have $A(K, \{x_n\}) \subseteq A(K, \{x_{n_k}\})$.
Thus we have $A(K, \{x_n\}) = \{z\}$, which implies that $x_n \to z$.

3 CONCLUSIONS

In this paper, we have used the asymptotic center technique to establish the existence of fixed points for Suzuki nonexpansive mappings in Banach spaces. We have shown that under certain condition, the asymptotic radius and Chebyshev radius are equal. Using this result, we have established that every approximate fixed point sequence is regular as well as uniform. The convergence of regular approximate fixed point sequences to a fixed points of the Suzuki nonexpansive mapping is also established. A couple of examples are given to illustrate the results.

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/02/

A SURVEY ON THE FIXED POINT THEOREMS VIA ADMISSIBLE MAPPING

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ABSTRACT

In this survey, we discuss the crucial role of the notion of admissible mapping in the metric fixed point theory. Adding admissibility conditions to the statements leads not only to generalizing the existing results but also unifying several corresponding results in different settings. In particular, a contraction via admissible mapping involves and covers contractions defined on partially ordered sets, and contractions forming cyclic structure.

KEYWORDS

Admissible mapping, fixed point

1 INTRODUCTION

The metric fixed point theory is one of the most attractive research topics that lies in the intersection of three disciplines of the mathematics; topology, applied mathematics and nonlinear functional analysis. In the literature, it was assumed that the first metric fixed point theorem was proved by Banach [18] in 1922. On the other hand, it was noted that the idea of the fixed point theorem was used before Banach's paper. Indeed, the fixed point results were used to prove the existence and uniqueness solution of the initial value problems in the some research papers of Liouville, Picard, Poincaré and so on. Despite this fact, the sole purpose of Banach's theorem is the indicate the existence and uniqueness of the fixed point.

The Banach's fixed point theorem stated perfectly and its proof is also awesome: Every contraction in a complete metric space guarantee not only the existence but also uniqueness of a fixed point. In the proof, Banach indicates how the desired fixed point should be found. Regarding that it is a core tool in the solution of the certain differential equation, the existence and uniqueness of the solution of the differential equations are induced to the existence and uniqueness of a fixed point. Indeed, this observation can be the main reason why the metric fixed point theory has been investigated heavily.

In the last three decades, thousands of new results have been announced in the framework of the metric fixed point theory. Most of them claimed that it was the generalization of the existing results. Mainly, these announced results are slight generalizations or extensions of the existing ones. Indeed, predominantly, most of the proofs are the mimic of the proof of the pioneer fixed point theorem of Banach: Construct a sequence (usually, by using the Picard operator); indicate that this sequence is convergent and finally prove that the limit of this recursive sequence is the required fixed point of the considered operator. The fact that existing so many publications on this topic has encouraged researchers to organize and unify this scattered literature. One of the best examples of this trend was given by Samet *et al.* [61] in 2012 by involving the notion of admissible mappings.

Now, we shall explain why the new notion of Samet *et al.* [61], an admissible mapping, is important. First of all, we need to underline that Banach's famous fixed point theorem has been generalized and extended in many different ways in the literature. The most classic approach for generalization and extension is to change the definitions of contraction. The other approach and method are that a given contraction function is in cyclic form. One of the other methods for generalization and extension is to add a partial order on the structure where contraction is defined. Admissible mappings make it possible to put together these three approaches in a single statement. For the clarification of the fixed point theory literature, admissible mapping plays one of the key roles.

In this manuscript, we shall revisit the literature by using the auxiliary function: admissible mapping. We shall provide examples and put several consequences to illustrate how this approach works successfully.

2 Preliminaries

In this section, we collect some necessary tools (notions, notations) that are essential to express the results.

First of all, we shall fixed some basic notations: Hereafter, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers. Furthermore, the symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of reals, positive reals and the set of nonnegative reals, respectively. Throughout the manuscript, all considered sets are non-empty.

We shall first recall the admissible mapping defined by Samet *et al.* [61]. Let $\alpha : X \times X \to [0, \infty)$ be a function. Then, the mapping $T : X \to X$ is said to be α-admissible [61] if, for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Next, we recollect the notion of triangular α-admissible that plays crucial rules in usage of triangle inequality axiom of the metric.

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Definition 1. *[41] A self-mapping $T : X \to X$ is called triangular α-admissible if*

$$(T_1)\ \ T \text{ is } \alpha - \text{admissible},$$
$$(T_2)\ \ \alpha(x,z) \geq 1, \quad \alpha(z,y) \geq 1 \Longrightarrow \alpha(x,y) \geq 1, \quad x,y,z \in X.$$

Example 1. *Suppose that $M = (0,+\infty)$. Define $T : M \to M$ and $\alpha : M \times M \to [0,\infty)$ by*

*(1) $T(x) = \ln(x+1)$, for all $x \in M$ and $\alpha(x,y) = \begin{cases} 1, & \text{if } x \geq y; \\ 0, & \text{if } x < y. \end{cases}$
Then T is α-admissible.*

*(2) $T(x) = \sqrt[3]{x}$, for all $x \in M$ and $\alpha(x,y) = \begin{cases} e^{x-y}, & \text{if } x \geq y; \\ 0, & \text{if } x < y. \end{cases}$
Then T is α-admissible.*

For more examples of such mappings are presented in [30, 40, 41, 46–48, 61] and references therein.

In 2014, Popescu [54] conclude that the notion of triangular α-admissible can be refined slightly, as follows:

Definition 2. *[54] Let $T : X \to X$ be a self-mapping and $\alpha : X \times X \to [0,\infty)$ be a function. Then T is said to be α-orbital admissible if*

$$(T3)\ \ \alpha(x,Tx) \geq 1 \Rightarrow \alpha(Tx,T^2x) \geq 1.$$

Definition 3. *[54] Let $T : X \to X$ be a self-mapping and $\alpha : X \times X \to [0,\infty)$ be a function. Then T is said to be triangular α-orbital admissible if T is α-orbital admissible and*

$$(T4)\ \ \alpha(x,y) \geq 1 \text{ and } \alpha(y,Ty) \geq 1 \Rightarrow \alpha(x,Ty) \geq 1.$$

As it is expected, each α-admissible mapping is an α-orbital admissible mapping and each triangular α-admissible mapping is a triangular α-orbital admissible mapping. Notice also that the converse is false, see e.g. ([54] Example 7).

A metric space (X,d) is said α-regular [54], if for every sequence $\{x_n\}$ in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq 1$ for all k.

Lemma 1. *[54] Let $T : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}_0$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.*

In what follows, we recall the following auxiliary family of the functions, namely, comparison and c-comparison functions. A mapping $\psi : [0,\infty) \to [0,\infty)$ is called a *comparison function* if it is increasing and $\psi^n(t) \to 0$, $n \to \infty$, for any $t \in [0,\infty)$. We denote by Φ, the class of the corporation function $\psi : [0,\infty) \to [0,\infty)$. For more details and examples , see e.g. [21, 58]. Among them, we recall the following essential result.

Lemma 2. *(Berinde [21], Rus [58]) If $\psi : [0,\infty) \to [0,\infty)$ is a comparison function, then:*

(1) each iterate ψ^k of ψ, $k \geq 1$, is also a comparison function;

(2) ψ is continuous at 0;

(3) $\psi(t) < t$, for any $t > 0$.

Later, Berinde [21] introduced the concept of (c)-*comparison function* in the following way.

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Definition 4. *(Berinde [21]) A function $\psi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if*

(c_1) ψ is increasing,

(c_2) there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\psi^{k+1}(t) \leq a\psi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

From now on, the letter Ψ is reserved to indicate the family of functions (c)-comparison function. It is evident that Lemma 2 is valid for $\psi \in \Psi$. Further, (c_2) yields

$$\sum_{n=1}^{+\infty} \psi^n(t) < \infty,$$

for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.

Samet *et al.* [61] prove the first fixed point result via admissible mapping by introducing the following concepts.

Definition 5. *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \text{ for all } x, y \in X.$$

It is obvious that, any contractive mapping forms an $\alpha - \psi$ contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, $k \in (0, 1)$.

The following is the interesting fixed point theorem of Samet et al. [61]

Theorem 1. *Let (X, d) be a complete metric space and $T : X \to X$ be an $\alpha - \psi$ contractive mapping. Suppose that*

(i) T is $\alpha-$admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous, or

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then there exists $u \in X$ such that $Tu = u$.

Note that in this setting, for the uniqueness additional condition is considered.

Theorem 2. *Adding to the hypotheses of Theorem 1 (resp. Theorem 1) the condition: For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, we obtain uniqueness of the fixed point.*

Another successful attempt to simplify and clarify the literature of the metric fixed point theory was done by Khojasteh *et al.* [44] in 2015. In this paper, the authors introduced the notion of the simulation functions to combine the several existing results. In what follows, we recall the definition of this auxiliary function.

Definition 6. *(See [44]) A function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is said to be* simulation *if it satisfies the following conditions:*

(ζ_1) $\zeta(0,0)=0$;

(ζ_2) $\zeta(t,s)<s-t$ *for all* $t,s>0$;

(ζ_3) *if* $\{t_n\},\{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(t_n,s_n)<0. \tag{1}$$

The family of all simulation functions $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ will be denoted by $\mathcal{Z}$. On account of (ζ_2), we observe that

$$\zeta(t,t)<0 \text{ for all } t>0,\ \zeta\in\mathcal{Z}. \tag{2}$$

Notice also that the condition (ζ_1) is superfluous due to (ζ_2).

Example 2. *(See e.g. [12, 15, 42–44, 56]) Let* $\zeta_i:[0,\infty)\times[0,\infty)\to\mathbb{R}$, $i\in\{1,2,3\}$, *be mappings defined by*

(i) $\zeta_1(t,s)=\psi(s)-\phi(t)$ *for all* $t,s\in[0,\infty)$, *where* $\phi,\psi:[0,\infty)\to[0,\infty)$ *are two continuous functions such that* $\psi(t)=\phi(t)=0$ *if, and only if,* $t=0$, *and* $\psi(t)<t\le\phi(t)$ *for all* $t>0$.

(ii) $\zeta_2(t,s)=s-\dfrac{f(t,s)}{g(t,s)}t$ *for all* $t,s\in[0,\infty)$, *where* $f,g:[0,\infty)\to(0,\infty)$ *are two continuous functions with respect to each variable such that* $f(t,s)>g(t,s)$ *for all* $t,s>0$.

(iii) $\zeta_3(t,s)=s-\varphi(s)-t$ *for all* $t,s\in[0,\infty)$, *where* $\varphi:[0,\infty)\to[0,\infty)$ *is a continuous function such that* $\varphi(t)=0$ *if, and only if,* $t=0$.

(iv) If $\varphi:[0,\infty)\to[0,1)$ *is a function such that* $\limsup_{t\to r^+}\varphi(t)<1$ *for all* $r>0$, *and we define*

$$\zeta_T(t,s)=s\,\varphi(s)-t \qquad \text{for all } s,t\in[0,\infty),$$

then ζ_T *is a simulation function.*

(v) If $\eta:[0,\infty)\to[0,\infty)$ *is an upper semi-continuous mapping such that* $\eta(t)<t$ *for all* $t>0$ *and* $\eta(0)=0$, *and we define*

$$\zeta_{BW}(t,s)=\eta(s)-t \qquad \text{for all } s,t\in[0,\infty),$$

then ζ_{BW} *is a simulation function.*

(vi) If $\phi:[0,\infty)\to[0,\infty)$ *is a function such that* $\int_0^\varepsilon\phi(u)du$ *exists and* $\int_0^\varepsilon\phi(u)du>\varepsilon$, *for each* $\varepsilon>0$, *and we define*

$$\zeta_K(t,s)=s-\int_0^t\phi(u)du \qquad \text{for all } s,t\in[0,\infty),$$

then ζ_K *is a simulation function.*

Suppose (X,d) is a metric space, T is a self-mapping on X and $\zeta\in\mathcal{Z}$. We say that T is a $\mathcal{Z}$-*contraction* with respect to ζ [44], if

$$\zeta(d(T(x),T(y)),d(x,y))\ge 0, \text{ for all } x,y\in X.$$

It is evident that renowned Banach contraction forms $\mathcal{Z}$-*contraction* with respect to ζ where $\zeta(t,s)=ks-t$ with $k\in[0,1)$ and $s,t\in[0,\infty)$.

Theorem 3. *On the complete metric space, every* $\mathcal{Z}$-contraction *possesses a unique fixed point.*

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3 A theorem with many consequences

In this section, we shall consider a theorem that generalizes and hence unifies a number of existing results. Consequently, we list many corollaries.

Definition 7. *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalized Suzuki type $(\alpha - \psi) - \mathcal{Z}$-contraction mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$, $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } \zeta(\psi(M(x, y)), \alpha(x, y)d(Tx, Ty)) \geq 0, \tag{3}$$

where $M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$.

Theorem 4. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalized Suzuki type $(\alpha - \psi) - \mathcal{Z}$-contraction mapping and satisfies the following conditions:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous.

Then there exists $u \in X$ such that $Tu = u$.

Proof. On account of the assumption (ii) of the theorem, there is a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Starting with this initial point, we shall built-up a recursive sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. We, first, observe that incase of $x_{n_0} = x_{n_0+1}$ for some n_0, we conclude that $u = x_{n_0}$ is a fixed point of T. Accordingly, we presume that $x_n \neq x_{n+1}$ for all n. Hence, we find that

$$0 < \frac{1}{2}d(x_n, x_{n+1}) = \frac{1}{2}d(x_n, Tx_n) \leq d(x_n, x_{n+1}),$$

for all n.

On the other hand, employing that T is $\alpha-$admissible, we derive that

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Inductively, we have

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n = 0, 1, \ldots \tag{4}$$

From (3) and (4), it follows that for all $n \geq 1$, we have

$$\frac{1}{2}d(x_n, Tx_n) \leq d(x_n, x_{n+1}),$$

implies that

$$\zeta(\psi(M(x_n, x_{n+1})), \alpha(x_n, x_{n+1}))d(Tx_n, Tx_{n+1}))) \geq 0,$$

which is equivalent to

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}) \leq \psi(M(x_n, x_{n-1})). \tag{5}$$

Now, we shall simplify the right hand side of the inequality above, as follows

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\left\{d(x_n, x_{n-1}), \frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{2}, \frac{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)}{2}\right\} \\ &= \max\left\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2}, \frac{d(x_{n-1}, x_{n+1})}{2}\right\} \\ &\leq \max\left\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2}\right\} \\ &\leq \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}. \end{aligned}$$

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Consequently, regarding (5) together with the fact that ψ is a nondecreasing function, we derive

$$d(x_{n+1}, x_n) \leq \psi\left(\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}\right), \tag{6}$$

for all $n \geq 1$. If for some $n \geq 1$, we have $d(x_n, x_{n-1}) \leq d(x_n, x_{n+1})$, from (6), we obtain that

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}),$$

a contradiction. Thus, for all $n \geq 1$, we have

$$\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} = d(x_n, x_{n-1}). \tag{7}$$

Using (6) and (7), we get that

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1})) < d(x_n, x_{n-1}), \tag{8}$$

for all $n \geq 1$. By induction, we get

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0)), \text{ for all } n \geq 1. \tag{9}$$

From (9) and using the triangular inequality, for all $k \geq 1$, we have

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\ &\leq \sum_{p=n}^{n+k-1} \psi^n(d(x_1, x_0)) \\ &\leq \sum_{p=n}^{+\infty} \psi^n(d(x_1, x_0)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{10}$$

Since T is continuous, we obtain from (10) that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0. \tag{11}$$

From (10), (11) and the uniqueness of the limit, we get immediately that u is a fixed point of T, that is, $Tu = u$.

In what follows, the continuity of the contraction in Theorem 4 is refined.

Theorem 5. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalized Suzuki type $(\alpha - \psi) - \mathcal{Z}$-*contraction *mapping and satisfies the following conditions:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then there exists $u \in X$ such that $Tu = u$.

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Proof. Following the proof of Theorem 4 line by line, we deduce that the recursive sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. From (4) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k.

Now, we use (3) to deduce $u \in X$ is the required fixed point. For this purpose, we need to show that $\frac{1}{2}d(x_{n(k)}, x_{n(k)+1}) = \frac{1}{2}d(x_{n(k)}, Tx_{n(k)}) \leq d(x_{n(k)}, u)$ or $\frac{1}{2}d(x_{n(k)+1}, x_{n(k)+2}) = \frac{1}{2}d(x_{n(k)+1}, Tx_{n(k)+1}) \leq d(x_{n(k)+1}, u)$. Suppose, on the contrary, that $\frac{1}{2}d(x_{n(k)}, x_{n(k)+1}) > d(x_{n(k)}, u)$ and $\frac{1}{2}d(x_{n(k)+1}, x_{n(k)+2}) > d(x_{n(k)+1}, u)$. On account of the triangle inequality, we have

$$\begin{aligned} d(x_{n(k)}, x_{n(k)+1}) &\leq d(x_{n(k)}, u) + d(u, x_{n(k)+1}) \\ &< \tfrac{1}{2}d(x_{n(k)}, x_{n(k)+1}) + \tfrac{1}{2}d(x_{n(k)+1}, x_{n(k)+2}) \\ \text{on account of (8)} & \\ &< \tfrac{1}{2}d(x_{n(k)}, x_{n(k)+1}) + \tfrac{1}{2}d(x_{n(k)}, x_{n(k)+1}) = d(x_{n(k)}, x_{n(k)+1}), \end{aligned} \tag{12}$$

a contradiction. Thus, we have

$$\frac{1}{2}d(x_{n(k)}, Tx_{n(k)}) \leq d(x_{n(k)}, u) \text{ or } \frac{1}{2}d(x_{n(k)+1}, Tx_{n(k)+1}) \leq d(x_{n(k)+1}, u).$$

After this observation, by applying (3), for all k, we find

$\frac{1}{2}d(x_{n(k)}, Tx_{n(k)}) \leq d(x_{n(k)}, u)$ implies $\zeta(\psi(M(x_{n(k)}, u)), \alpha(x_{n(k)}, u)d(Tx_{n(k)}, Tu)) \geq 0$ which yields that

$$d(x_{n(k)+1}, Tu) = d(Tx_{n(k)}, Tu) \leq \alpha(x_{n(k)}, u)d(Tx_{n(k)}, Tu) \leq \psi(M(x_{n(k)}, u)). \tag{13}$$

On the other hand, we have

$$M(x_{n(k)}, u) = \max\left\{d(x_{n(k)}, u), \frac{d(x_{n(k)}, x_{n(k)+1}) + d(u, Tu)}{2}, \frac{d(x_{n(k)}, Tu) + d(u, x_{n(k)+1})}{2}\right\}.$$

Letting $k \to \infty$ in the above equality, we obtain that

$$\lim_{k\to\infty} M(x_{n(k)}, u) = \frac{d(u, Tu)}{2}. \tag{14}$$

Suppose that $d(u, Tu) > 0$. From (14), for k large enough, we have $M(x_{n(k)}, u) > 0$, which yields that $\psi(M(x_{n(k)}, u)) < M(x_{n(k)}, u)$. Hence, from (13), we find

$$d(x_{n(k)+1}, Tu) < M(x_{n(k)}, u).$$

Letting $k \to \infty$ in the above inequality, using (14), we obtain that

$$d(u, Tu) \leq \frac{d(u, Tu)}{2},$$

which is a contradiction. Thus we have $d(u, Tu) = 0$, that is, $u = Tu$.

Notice that the proved theorems above are valid when we replace Instead of $M(x, y)$ with $N(x, y)$, where

$$N(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}\right\}.$$

The following example indicate that the hypotheses in Theorem 4 and Theorem 5 do not guarantee uniqueness of the fixed point.

Example 3. *Consider the set* $X = \{(2, 1), (1, 2)\} \subset \mathbb{R}^2$ *endowed with the standard Euclidean distance*

$$d((x, y), (u, v)) = |x - u| + |y - v|,$$

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for all $(x,y),(u,v)\in X$. It is evident that (X,d) is a complete metric space. It is clear that $T(x,y)=(x,y)$ that is trivially continuous. On account of the inequality $0=d(T(x,y),T(u,v))\le d((x,y),(u,v))$ we have

$$\zeta(\psi(M((x,y),(u,v))),\alpha((x,y),(u,v))d(T(x,y),T(u,v)))\ge 0,$$

for any $\psi\in\Psi$. Consequently, we have

$$\alpha((x,y),(u,v))d(T(x,y),T(u,v))\le\psi(M((x,y),(u,v))),$$

for all $(x,y),(u,v)\in X$, where

$$\alpha((x,y),(u,v))=\begin{cases}1 & \text{if} \quad (x,y)=(u,v),\\ 0 & \text{if} \quad (x,y)\neq(u,v).\end{cases}$$

Consequently, T is a generalized Suzuki type $(\alpha-\psi)-\mathcal{Z}$-contraction *mapping. On the other hand, for all $(x,y),(u,v)\in X$, we have*

$$\alpha((x,y),(u,v))\ge 1\to(x,y)=(u,v)\to T(x,y)=T(u,v)\to\alpha(T(x,y),T(u,v))\ge 1.$$

Thus, the mapping T is $\alpha-$admissible. Furthermore, for all $(x,y)\in X$, we have $\alpha((x,y),T(x,y))\ge 1$. Hence, the assumptions of Theorem 4 are fulfilled. Note that the assumptions of Theorem 5 are also satisfied, indeed if $\{(x_n,y_n)\}$ is a sequence in X that converges to some point $(x,y)\in X$ with $\alpha((x_n,y_n),(x_{n+1},y_{n+1}))\ge 1$ for all n, from the definition of α, we have $(x_n,y_n)=(x,y)$ for all n, that yields $\alpha((x_n,y_n),(x,y))=1$ for all n. Notice that, in this case, T has two fixed points in X.

For the uniqueness of a fixed point of such mappings, we need to the following additional condition:

(H) For all $x,y\in \mathrm{Fix}(T)$, there exists $z\in X$ such that $\alpha(x,z)\ge 1$ and $\alpha(y,z)\ge 1$.

Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 6. *Adding condition (H) to the hypotheses of Theorem 4 (resp. Theorem 5), we obtain that u is the unique fixed point of T.*

Proof. Suppose, on the contrary, that v is another fixed point of T. From (H), there exists $z\in X$ such that

$$\alpha(u,z)\ge 1 \text{ and } \alpha(v,z)\ge 1. \tag{15}$$

Due to the fact that T is $\alpha-$admissible together with (15), we find

$$\alpha(u,T^nz)\ge 1 \text{ and } \alpha(v,T^nz)\ge 1,\ \text{ for all } n. \tag{16}$$

Construct a sequence $\{z_n\}$ in X by $z_{n+1}=Tz_n$ for all $n\ge 0$ and $z_0=z$.

Taking (16) into account, for all n, we have

$$0=\frac{1}{2}d(u,Tu)\le d(u,z_n) \text{ implies } \zeta(\psi(M(u,z_n)),\alpha(u,z_n)d(Tu,Tz_n))\ge 0, \tag{17}$$

which is equivalent to

$$d(u,z_{n+1})=d(Tu,Tz_n)\le\alpha(u,z_n)d(Tu,Tz_n)\le\psi(M(u,z_n)). \tag{18}$$

On the other hand, we find

$$\begin{aligned} M(u,z_n) &= \max\left\{d(u,z_n),\frac{d(z_n,z_{n+1})}{2},\frac{d(u,z_{n+1})+d(z_n,u)}{2}\right\}\\ &\le \max\left\{d(u,z_n),\frac{d(z_n,u)+d(u,z_{n+1})}{2}\right\}\\ &\le \max\{d(u,z_n),d(u,z_{n+1})\}.\end{aligned}$$

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On account of the inequality above, the expression (18) and the monotone property of ψ, we derive that

$$d(u, z_{n+1}) \leq \psi\left(\max\{d(u, z_n), d(u, z_{n+1})\}\right), \tag{19}$$

for all n. Without restriction to the generality, we can suppose that $d(u, z_n) > 0$ for all n. If $\max\{d(u, z_n), d(u, z_{n+1})\} = d(u, z_{n+1})$, we get from (19) that

$$d(u, z_{n+1}) \leq \psi(d(u, z_{n+1})) < d(u, z_{n+1}),$$

which is a contradiction. Thus we have $\max\{d(u, z_n), d(u, z_{n+1})\} = d(u, z_n)$, and

$$d(u, z_{n+1}) \leq \psi(d(u, z_n)),$$

for all n. This implies that

$$d(u, z_n) \leq \psi^n(d(u, z_0)), \text{ for all } n \geq 1.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(z_n, u) = 0. \tag{20}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(z_n, v) = 0. \tag{21}$$

From (20) and (21), it follows that $u = v$. Thus we proved that u is the unique fixed point of T.

3.1 Immediate consequences

. The first immediate consequence is obtained by removing the Suzuki condition, as in the following definition.

Definition 8. *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalized type $(\alpha - \psi) - \mathcal{Z}$-*contraction *mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$, $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have*

$$\zeta(\psi(M(x, y)), \alpha(x, y)d(Tx, Ty)) \geq 0, \tag{22}$$

where $M(x, y) = \max\left\{d(x, y), \dfrac{d(x, Tx) + d(y, Ty)}{2}, \dfrac{d(x, Ty) + d(y, Tx)}{2}\right\}$.

Theorem 7. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalized Suzuki type $(\alpha - \psi) - \mathcal{Z}$-*contraction *mapping and satisfies the following conditions:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous*
or

*(iii)** if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k,*

(iv) the condition (H) holds.

Then there exists $u \in X$ such that $Tu = u$.

We skipped the proof. Indeed, it is verbatim of the combinations of the proofs of Theorem 4, Theorem 5 and Theorem 6, by removing the related lines about the Suzuki condition.

Taking Example 2 into account, both Theorem 6 and Theorem 7 yields several consequences. In this direction, one of the first example, by considering the case (i) Example 2 is the following theorem

Definition 9. *Let (X,d) be a metric space and $T:X\to X$ be a given mapping. We say that T is a generalized $\alpha-\psi$ contractive mapping if there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that for all $x,y\in X$, we have*

$$\alpha(x,y)d(Tx,Ty)\leq\psi(M(x,y)), \tag{23}$$

where $M(x,y)=\max\left\{d(x,y),\dfrac{d(x,Tx)+d(y,Ty)}{2},\dfrac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Theorem 8. *Let (X,d) be a complete metric space. Suppose that $T:X\to X$ is a generalized $\alpha-\psi$ contractive mapping and satisfies the following conditions:*

(i) *T is $\alpha-$admissible ;*

(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;*

*(iii)** *T is continuous*
or

*(iii)*** *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\geq 1$ for all k,*

(iv) *the condition (H) holds.*

Then there exists $u\in X$ such that $Tu=u$.

It is evident that by taking, the other cases of Example 2, into account, one can further consequences. We prefer to skip these consequences by the sake of the length of the manuscript.

4 Further Consequences

In this section, we shall indicate that several existing results in the literature can be deduced easily from our Theorem 6.

4.1 Standard fixed point theorems

Taking Theorem 6 into account, employing $\alpha(x,y)=1$ for all $x,y\in X$, we obtain immediately the following fixed point theorem.

Corollary 1. *Let (X,d) be a complete metric space and $T:X\to X$ be a given mapping. Suppose that there exists a function $\psi\in\Psi$ such that*

$$\frac{1}{2}d(x,Tx)\leq d(x,y) \text{ implies } \zeta(\psi(M(x,y)),d(Tx,Ty))\geq 0,$$

for all $x,y\in X$, where $M(x,y)=\max\left\{d(x,y),\dfrac{d(x,Tx)+d(y,Ty)}{2},\dfrac{d(x,Ty)+d(y,Tx)}{2}\right\}$. *Then T has a unique fixed point.*

In the same way, by letting $\alpha(x,y)=1$, for all $x,y\in X$, in Theorem 7, we find the following result:

Corollary 2. *Let (X,d) be a complete metric space and $T:X\to X$ be a given mapping. Suppose that there exists a function $\psi\in\Psi$ such that*

$$\zeta(\psi(M(x,y)),d(Tx,Ty))\geq 0,$$

for all $x,y\in X$, where $M(x,y)=\max\left\{d(x,y),\dfrac{d(x,Tx)+d(y,Ty)}{2},\dfrac{d(x,Ty)+d(y,Tx)}{2}\right\}$. *Then T has a unique fixed point.*

Analogously, by letting $\alpha(x,y) = 1$, for all $x, y \in X$, in Theorem 8, we find the following result:

Corollary 3. *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \psi(M(x,y)),$$

for all $x, y \in X$. Then T has a unique fixed point.

The following fixed point theorems follow immediately from Corollary 3.

Corollary 4 (see Berinde [22])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \psi(d(x,y)),$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 5 (see Ćirić [26])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \leq \lambda \, max \left\{ d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(x,Ty) + d(y,Tx)}{2} \right\},$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 6 (see Hardy and Rogers [33])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist constants $A, B, C \geq 0$ with $(A + 2B + 2C) \in (0,1)$ such that*

$$d(Tx,Ty) \leq Ad(x,y) + B[d(x,Tx) + d(y,Ty)] + C[d(x,Ty) + d(y,Tx)],$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 7 (Banach Contraction Principle [18])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \leq \lambda \, d(x,y),$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 8 (see Kannan [36])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0,1/2)$ such that*

$$d(Tx,Ty) \leq \lambda \, [d(x,Tx) + d(y,Ty)],$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 9 (see Chatterjea [24])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0,1/2)$ such that*

$$d(Tx,Ty) \leq \lambda \, [d(x,Ty) + d(y,Tx)],$$

for all $x, y \in X$. Then T has a unique fixed point.

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Corollary 10 (Dass-Gupta Theorem [29]). *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu < 1$ such that*

$$d(Tx,Ty) \leq \mu d(y,Ty)\,\frac{1+d(x,Tx)}{1+d(x,y)} + \lambda d(x,y), \quad \text{for all } x,y \in X. \tag{24}$$

Then T has a unique fixed point.

*S*ketch of the Proof. Consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \;\; t \geq 0 \tag{25}$$

and

$$\alpha(x,y) = \begin{cases} 1 - \mu\dfrac{d(y,Ty)(1+d(x,Tx))}{(1+d(x,y))d(Tx,Ty)}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

The rest is simple evaluation. For the detailed proof, we refer to Samet [60].

Corollary 11 (Jaggi theorem [35]). *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. Suppose there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu < 1$ such that*

$$d(Tx,Ty) \leq \mu\frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \lambda d(x,y), \quad \text{for all } x,y \in X,\; x \neq y. \tag{27}$$

Then there exist $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{R}$ such that T is an α-ψ contraction. Then T has a unique fixed point.

*S*ketch of the Proof.

Consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \;\; t \geq 0 \tag{28}$$

and

$$\alpha(x,y) = \begin{cases} 1 - \mu\dfrac{d(x,Tx)d(y,Ty)}{d(x,y)d(Tx,Ty)}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

The rest is a simple evaluation. For the detailed proof, we refer to Samet [60].

Theorem 9 (Berinde Theorem [23]). *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exists $\lambda \in (0,1)$ and $L \geq 0$ such that*

$$d(Tx,Ty) \leq \lambda\, d(x,y) + L\, d(y,Tx), \quad \text{for all } x,y \in X. \tag{30}$$

Then T has a fixed point.

*S*ketch of the Proof.

Consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \;\; t \geq 0$$

and

$$\alpha(x,y) = \begin{cases} 1 - L\dfrac{d(y,Tx)}{d(Tx,Ty)}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

The rest is straightforward. For more details for the proof, we refer to Samet [60].

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Theorem 10 (Ćirić's non-unique fixed point theorem [27].). *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. there exists $\lambda \in (0,1)$ such that for all $x,y \in X$, we have*

$$\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} - \min\{d(x,Ty), d(y,Tx)\} \leq \lambda\, d(x,y). \tag{32}$$

Then T has a fixed point.

*S*ketch of the Proof. Consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t,\ \ t \geq 0 \tag{33}$$

and

$$\alpha(x,y) = \begin{cases} \min\left\{1, \frac{d(x,Tx)}{d(Tx,Ty)}, \frac{d(y,Ty)}{d(Tx,Ty)}\right\} - \min\left\{\frac{d(x,Ty)}{d(Tx,Ty)}, \frac{d(y,Tx)}{d(Tx,Ty)}\right\}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

The rest is simple evaluation. For the detailed proof, we refer to Samet [60].

Theorem 11 (Suzuki Theorem [63]). *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exists $r \in (0,1)$ such that*

$$(1+r)^{-1} d(x,Tx) \leq d(x,y) \Longrightarrow d(Tx,Ty) \leq r\, d(x,y), \quad \textit{for all } x,y \in X. \tag{35}$$

Then T has a unique fixed point.

*S*ketch of the Proof. Consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = r\, t,\ \ t \geq 0$$

and

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (1+r)^{-1} d(x,Tx) \leq d(x,y), \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

From (35), we have

$$\alpha(x,y) d(Tx,Ty) \leq \psi(d(x,y)), \quad \text{for all } x,y \in X.$$

Then T is an α-ψ contraction.

4.2 Fixed point theorems on metric spaces endowed with a partial order

In the last two decades, one of the trend in fixed point theory is the revisit the well-known fixed point theorem on metric spaces endowed with partial orders [65]. Among all, Ran and Reurings in [55] revisited the Banach contraction principle in partially ordered sets with some applications to matrix equations. Another version of the generalization of Banach contraction principle in partially ordered sets was proposed by Nieto and Rodríguez-López in [49]. Later, this trend was supported by several authors, see e.g. [2, 13, 28, 34, 53, 59] and the references cited therein . In this section, we shall show that Theorem 6 implies easily various fixed point results on a metric space endowed with a partial order. At first, we need to recall some concepts.

Definition 10. *Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if*

$$x,y \in X,\ \ x \preceq y \Longrightarrow Tx \preceq Ty.$$

Definition 11. *Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.*

Definition 12. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preceq, d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

We have the following result.

Corollary 12. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } \zeta(\psi(M(x, y)), d(Tx, Ty)) \geq 0,$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Proof. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } x \succeq y, \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, T is a generalized $\alpha - \psi$ contractive mapping, that is,

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)),$$

for all $x, y \in X$. From condition (i), we have $\alpha(x_0, Tx_0) \geq 1$. Moreover, for all $x, y \in X$, from the monotone property of T, we have

$$\alpha(x, y) \geq 1 \Longrightarrow x \succeq y \text{ or } x \preceq y \Longrightarrow Tx \succeq Ty \text{ or } Tx \preceq Ty \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Thus T is $\alpha-$admissible. Now, if T is continuous, the existence of a fixed point follows from Theorem 4. Suppose now that $(X, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. From the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. This implies from the definition of α that $\alpha(x_{n(k)}, x) \geq 1$ for all k. In this case, the existence of a fixed point follows from Theorem 5. To show the uniqueness, let $x, y \in X$. By hypothesis, there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, which implies from the definition of α that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Thus we deduce the uniqueness of the fixed point by Theorem 6.

Corollary 13. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$\zeta(\psi(M(x, y)), d(Tx, Ty)) \geq 0,$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

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Corollary 14. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(M(x, y)),$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

The following results are immediate consequences of Corollary14.

Corollary 15. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Corollary 16. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda \, max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\},$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Corollary 17. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist constants $A, B, C \geq 0$ with $(A + 2B + 2C) \in (0, 1)$ such that*

$$d(Tx, Ty) \leq Ad(x, y) + B[d(x, Tx) + d(y, Ty)] + C[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

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Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Corollary 18 (see Ran and Reurings [55], Nieto and López [49])**.** *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda\, d(x, y),$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Corollary 19. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq \lambda\, [d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Corollary 20. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq \lambda\, [d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

4.3 Fixed point theorems for cyclic contractive mappings

The notion of "cyclic contraction mapping"to find a fixed point was proposed by Kirk, Srinivasan and Veeramani [45]. In this paper, they revisited the famous Banach Contraction Mapping Principle was proved by Kirk, Srinivasan and Veeramani [45] via cyclic contraction. Following this pioneer work [45], several fixed point theorems in the framework of cyclic contractive mappings have appeared (see, for instant, [1, 37, 39, 51, 52, 57]). In this section, we shall proved that Theorem 6 implies several fixed point theorems in the context of cyclic contractive mappings.

We have the following result.

Corollary 21. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$ and $\zeta \in \mathcal{Z}$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;

(II) there exists a function $\psi \in \Psi$ such that

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \text{ implies } \zeta(\psi(M(x,y)), d(Tx,Ty)) \geq 0;$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Proof. Since A_1 and A_2 are closed subsets of the complete metric space (X,d), then (Y,d) is complete. Define the mapping $\alpha : Y \times Y \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } (x,y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 \text{ otherwise.} \end{cases}$$

From (II) and the definition of α, we can write

$$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y)),$$

for all $x, y \in Y$. Thus T is a generalized $\alpha - \psi$ contractive mapping.

Let $(x,y) \in Y \times Y$ such that $\alpha(x,y) \geq 1$. If $(x,y) \in A_1 \times A_2$, from (I), $(Tx,Ty) \in A_2 \times A_1$, which implies that $\alpha(Tx,Ty) \geq 1$. If $(x,y) \in A_2 \times A_1$, from (I), $(Tx,Ty) \in A_1 \times A_2$, which implies that $\alpha(Tx,Ty) \geq 1$. Thus in all cases, we have $\alpha(Tx,Ty) \geq 1$. This implies that T is $\alpha-$admissible.

Also, from (I), for any $a \in A_1$, we have $(a,Ta) \in A_1 \times A_2$, which implies that $\alpha(a,Ta) \geq 1$.

Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. This implies from the definition of α that

$$(x_n, x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \text{ for all } n.$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the Euclidean metric, we get that

$$(x,x) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which implies that $x \in A_1 \cap A_2$. Thus we get immediately from the definition of α that $\alpha(x_n, x) \geq 1$ for all n.

Finally, let $x, y \in \text{Fix}(T)$. From (I), this implies that $x, y \in A_1 \cap A_2$. So, for any $z \in Y$, we have $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$. Thus condition (H) is satisfied.

Now, all the hypotheses of Theorem 6 are satisfied, we deduce that T has a unique fixed point that belongs to $A_1 \cap A_2$ (from (I)).

Corollary 22. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$ and $\zeta \in \mathcal{Z}$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;

(II) there exists a function $\psi \in \Psi$ such that

$$\zeta(\psi(M(x,y)), d(Tx,Ty)) \geq 0;$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Corollary 23. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) there exists a function $\psi \in \Psi$ *such that*

$$d(Tx, Ty) \leq \psi(M(x, y)), \text{ for all } (x, y) \in A_1 \times A_2.$$

Then T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

The following results are immediate consequences of Corollary 23.

Corollary 24 (see Pacurar and Rus [51]). *Let* $\{A_i\}_{i=1}^{2}$ *be nonempty closed subsets of a complete metric space* (X, d) *and* $T : Y \to Y$ *be a given mapping, where* $Y = A_1 \cup A_2$*. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) there exists a function $\psi \in \Psi$ *such that*

$$d(Tx, Ty) \leq \psi(d(x, y)), \text{ for all } (x, y) \in A_1 \times A_2.$$

Then T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

Corollary 25. *Let* $\{A_i\}_{i=1}^{2}$ *be nonempty closed subsets of a complete metric space* (X, d) *and* $T : Y \to Y$ *be a given mapping, where* $Y = A_1 \cup A_2$*. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) there exists a constant $\lambda \in (0, 1)$ *such that*

$$d(Tx, Ty) \leq \lambda \, max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\}, \text{ for all } (x, y) \in A_1 \times A_2.$$

Then T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

Corollary 26. *Let* $\{A_i\}_{i=1}^{2}$ *be nonempty closed subsets of a complete metric space* (X, d) *and* $T : Y \to Y$ *be a given mapping, where* $Y = A_1 \cup A_2$*. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) there exist constants $A, B, C \geq 0$ *with* $(A + 2B + 2C) \in (0, 1)$ *such that*

$$d(Tx, Ty) \leq Ad(x, y) + B[d(x, Tx) + d(y, Ty)] + C[d(x, Ty) + d(y, Tx)], \text{ for all } (x, y) \in A_1 \times A_2.$$

Then T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

Corollary 27 (see Kirk et al. [45]). *Let* $\{A_i\}_{i=1}^{2}$ *be nonempty closed subsets of a complete metric space* (X, d) *and* $T : Y \to Y$ *be a given mapping, where* $Y = A_1 \cup A_2$*. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) there exists a constant $\lambda \in (0, 1)$ *such that*

$$d(Tx, Ty) \leq \lambda \, d(x, y), \text{ for all } (x, y) \in A_1 \times A_2.$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Corollary 28. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X, d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;

(II) there exists a constant $\lambda \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq \lambda \left[d(x, Tx) + d(y, Ty)\right], \text{ for all } (x, y) \in A_1 \times A_2.$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Corollary 29. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X, d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;

(II) there exists a constant $\lambda \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq \lambda \left[d(x, Ty) + d(y, Tx)\right], \text{ for all } (x, y) \in A_1 \times A_2.$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

5 Conclusion

The fixed point theorem is one of the most actively studied research fields of recent times. Naturally, there are many publications on this subject and several new results are announced. This causes the literature to become rather disorganized and dysfunctional. The more troublesome situation is that the existing theorems are rediscovered again and again due to this messiness. More accurately, the results have been repeated. It is therefore essential to organize the fixed-point theory literature, weeding out false and/or repetitive results, and, if possible, combining and unifying existing results into a more general framework. In this work, we show that using admissible mapping, many existing fixed point theorems can be written as a consequence of the main theorem we have given.

Note that the consequences of the main result of the paper, Theorem 6 is not complete. It is possible to add several corollaries. On the other hand, we prefer to skip these possible consequences, since it is clear how the possible result can be concluded from our main theorem and how can be proved. Further, we underline that the main theorem can be derived in the distinct abstract spaces, such as, partial metric space, b-metric space, quasi-metric space, and so on.

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ESSENTIAL SPECTRUM OF DISCRETE LAPLACIAN - REVISITED

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ABSTRACT

Consider the discrete Laplacian operator A *acting on* $l^2(\mathbb{Z})$. *It is well known from the classical literature that the essential spectrum of* A *is a compact interval. In this article, we give an elementary proof for this result, using the finite-dimensional truncations* A_n *of* A. *We do not rely on symbol analysis or any infinite-dimensional arguments. Instead, we consider the eigenvalue-sequences of the truncations* A_n *and make use of the filtration techniques due to Arveson. Usage of such techniques to the discrete Schrödinger operator and to the multi-dimensional settings will be interesting future problems.*

KEYWORDS

Essential Spectrum, Discrete Laplacian

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1 INTRODUCTION

In this short article, we consider the discrete Laplacian operator A defined on $l^2(\mathbb{Z})$, as follows:

$$A(x(n)) = (x(n-1) + x(n+1)); x = (x(n)) \in l^2(\mathbb{Z}), n \in \mathbb{Z}.$$

This operator arises naturally in many physical situations. For example, when we approximate a partial differential equation by finite differences, such bounded operators come into the picture. This operator is widely used in image processing, particularly in edge detection problems. There are extensions of the discrete Laplacian to various settings, such as multi-dimensional operator (on $\mathbb{Z}^n$) and Laplacian on graphs, etc. An operator close to this example is the discrete Schrödinger operator. This operator can be considered as a perturbation of the discrete Laplacian, defined as follows;

$$H(x(n)) = (x(n-1) + x(n+1) + v(n)x(n)); x = (x(n)) \in l^2(\mathbb{Z}), n \in \mathbb{Z}.$$

Here the sequence $v = v(n)$ is a bounded sequence called the potential.

It is well-known from the classical theory that the spectrum of A is the compact interval $[-2, 2]$. In this article, we use the filtration techniques developed by W. B. Arveson in [1] and some elementary method to give a simple proof of this result. We plan to use such techniques in the computation of the spectrum of the discrete Schrödinger operator. However, the spectrum of the discrete Schrödinger operator can be very complicated, depending on the potential function. For example, if you choose the almost Mathieu potential, the spectrum will be a Cantor-like set (The Ten-Martini Conjecture, see [2] for, eg.).

The article is organized as follows. In the next section, we describe some essential results from [1,3] in connection with the spectral approximation of an infinite-dimensional bounded self-adjoint operator. In the third section, we use these techniques to give an elementary proof of the connectedness of the essential spectrum of A. A possible application to the spectral computation of some special class of discrete Schrödinger operators is mentioned at the end of this article.

2 OPERATORS IN THE ARVESON'S CLASS

"How to approximate spectra of linear operators on separable Hilbert spaces?"is a fundamental question and was considered by many mathematicians. One of the successful methods is to use the finite-dimensional theory in the computation of the spectrum of bounded operators in an infinite dimensional space through an asymptotic way. In 1994, W.B. Arveson identified a class of operators for which the finite-dimensional truncations are helpful in the spectral approximation [1]. We introduce this class of operators here.

Let A be a bounded self-adjoint operator defined on a complex separable Hilbert space $\mathcal{H}$ and $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $\mathcal{H}$. Consider the finite dimensional truncations of A, that is $A_n = P_n A P_n$, where P_n is the projection of $\mathcal{H}$ onto the span of first n elements $\{e_1, e_2, \ldots, e_n\}$ of the basis. We recall the notion of essential points and transient points introduced in [1].

Definition 1. *Essential point: A real number λ is an essential point of A, if for every open set U containing λ, $\lim_{n\to\infty} N_n(U) = \infty$, where $N_n(U)$ is the number of eigenvalues of A_n in U.*

Definition 2. *Transient point: A real number λ is a transient point of A if there is an open set U containing λ, such that* $\sup N_n(U)$ *with n varying on the set of all natural number is finite.*

Remark 3. *Note that a number can be neither transient nor essential.*

Denote $\Lambda = \{\lambda \in R; \lambda = \lim \lambda_n, \lambda_n \in \sigma(A_n)\}$ and Λ_e as the set of all essential points. The following spectral inclusion result for a bounded self-adjoint operator A is of high importance. Let $\sigma(A), \sigma_{ess}(A)$ denote the spectrum and essential spectrum of A respectively.

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Theorem 4. *[1] The spectrum of a bounded self-adjoint operator is a subset of the set of all limit points of the eigenvalue sequences of its truncations. Also, the essential spectrum is a subset of the set of all essential points. That is,*

$$\sigma(A) \subseteq \Lambda \subseteq [m, M] \quad \textit{and} \quad \sigma_{ess}(A) \subseteq \Lambda_e.$$

W.B Arveson, generalized the notion of band limited matrices in [1], and achieved some useful results in the case of some special class of operators. He used an arbitrary filtration H_n (an increasing subsequence of closed subspaces with the union dense in $\mathcal{H}$) and the sequence of orthogonal projections onto H_n to introduce his class of operators. Here we consider only a special case.

Definition 5. *The* ***degree*** *of a bounded operator A on $\mathbb{H}$ is defined by*

$$deg(A) = \sup_{n \geq 1} \ rank(P_n A - A P_n).$$

A Banach $*-$algebra of operators can be defined, which we call **A**rveson's class, as follows.

Definition 6. *A is an operator in the* ***Arveson's class*** *if*

$$A = \sum_{n=1}^{\infty} A_n, \quad \textit{where } deg(A_n) < \infty \textit{ for every } n \textit{ and convergence is in the}$$

operator norm, in such a way that

$$\sum_{n=1}^{\infty} (1 + \ deg(A_n)^{\frac{1}{2}}) \|A_n\| < \infty$$

The following gives a concrete description of operators in Arveson's class.

Theorem 7. *[1] Let $(a_{i,j})$ be the matrix representation of a bounded operator A, with respect to $\{e_n\}$, and for every $k \in \mathbb{Z}$ let*

$$d_k = \sup_{i \in \mathbb{Z}} |a_{i+k,i}|$$

be the sup norm of the k^{th} diagonal of $(a_{i,j})$. Then A will be in Arveson's class whenever the series $\sum_k |k|^{1/2} d_k$ converges.

Remark 8. *In particular, any operator whose matrix representation $(a_{i,j})$ is band-limited, in the sense that $a_{i,j} = 0$ whenever $|i - j|$ is sufficiently large, must be in Arveson's class. Therefore, the operator under our consideration is in Arveson's class, as we see that its matrix representation is tridiagonal.*

The following result allows us to confine our attention to essential points while looking for essential spectral values for certain classes of operators.

Theorem 9. *[1] If A is a bounded self-adjoint operator in the Arveson's class, then $\sigma_{ess}(A) = \Lambda_e$ and every point in Λ is either transient or essential.*

3 SPECTRUM OF DISCRETE LAPLACIAN

Consider the discrete Laplacian operator A defined on $l^2(\mathbb{Z})$, as follows:

$$A(x(n)) = (x(n-1) + x(n+1)); x = (x(n)) \in l^2(\mathbb{Z}), n \in \mathbb{Z}.$$

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If we use the standard orthonormal basis on $l^2(\mathbb{Z})$, the truncations A_n will have the following matrix representations:

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0 & & . & & & & \\ 1 & 0 & 1 & 0 & 0 & & . & & & \\ 0 & 1 & 0 & 1 & 0 & 0 & & . & & \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & & . & \\ & 0 & 0 & 1 & 0 & 1 & 0 & 0 & & . \\ . & & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \\ & . & & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ & & . & & 0 & 0 & 1 & 0 & 1 & 0 \\ & & & . & & 0 & 0 & 1 & 0 & 1 \\ & & & & . & & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now we recollect some properties of the discrete Laplacian operator below.

Lemma 1. *$\lambda \in \sigma_{ess}(A)$ if and only if $-\lambda \in \sigma_{ess}(A)$.*

Proof. Notice that this operator A is in Arveson's class, introduced in the last section. Therefore, we have $\lambda \in \sigma_{ess}(A)$ if and only if λ is an essential point. That is exactly when $N_n(U) \to \infty$ for every neighborhood U of λ. The characteristic polynomials of A_n are $P_n(z) = z^n - a_{n-2}z^{n-2} + \ldots \pm 1$, when n is even and $P_n(z) = -z^n + a_{n-2}z^{n-2} - \ldots \pm a_1 z$, when n is odd. Here the coefficients can be computed as follows. $a_k = \frac{(n-k+2)(n-k+4)\ldots(n+k)}{2^k.k!}$

Therefore the eigenvalues of A_n are distributed symmetrically on both sides of 0 in the interval $[-2,2]$. Hence the number of truncated eigenvalues in any neighbourhood of $-\lambda$ and λ are the same if the neighbourhoods are of the same length. We can conclude that $\lambda \in \sigma_{ess}(A)$ if and only if $-\lambda \in \sigma_{ess}(A)$.

Lemma 2. *The operator norm $\|A\| = 2$ and $\pm 2 \in \sigma_{ess}(A)$.*

Proof. For every $x \in l^2(\mathbb{Z})$, we have

$$\|Ax\|^2 = \sum_{-\infty}^{\infty} (x(n-1) + x(n+1))^2 = \sum_{-\infty}^{\infty} \left[(x(n-1))^2 + (x(n+1))^2 + 2x(n-1)x(n+1)\right] \leq 4\|x\|^2.$$

Therefore, $\|A\| \leq 2$.

To prove the equality, consider the sequence $x_n = \left(\ldots 0,0,0,\ldots\frac{1}{n},\frac{1}{n},\frac{1}{n}\ldots\frac{1}{n},0,0,0\ldots\right)$, where $\frac{1}{n}$ repeats n^2 times and all other entries are 0. Then x_n has norm 1 and $\|Ax_n\|$ increases to 2. Hence $\|A\| = 2$. Since A is a bounded self-adjoint operator, either $\|A\|$ or $-\|A\|$ is always a spectral value. That is 2 or -2 is in the spectrum, $\sigma(A)$. However, they are not eigenvalues of A, as we see below.

If ± 2 is an eigenvalue of A, then 4 is an eigenvalue of $A^2 = B + 2I$ where B is defined by $B(e_n) = e_{n-2} + e_{n+2}$. (Observe that A^2 is defined as $A^2(e_n) = e_{n-2} + 2e_n + e_{n+2}$). This will imply that 2 is eigenvalue of B. If $Bx = 2x$, for some nonzero x, then $x(n+2) + x(n-2) = 2x(n)$, for all n. Let $x(N) = p \neq 0$, for some N and $x(N-2) = q$. Then $x(n+2k) = (k+1)p - kq$ for every $k\mathbb{Z}$. Such an element x will not be in $l^2(Z)$.

Hence ± 2 is an essential spectral value. By Lemma 1, both ± 2 are in the essential spectrum, $\sigma_{ess}(A)$. Therefore both 2 and -2 are in $\sigma_{ess}(A)$.

Theorem 10. *The spectral gaps of A, if they exist, will appear symmetrically with respect to the origin. That means corresponding to each spectral gap on the positive real axis, there exists a spectral gap on the negative real axis. In particular, A cannot have an odd number of spectral gaps.*

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Proof. We noticed that $A^2 = B + 2I$ where B is defined by $B(e_n) = e_{n-2} + e_{n+2}$. It is worthwhile to notice further that $\|B\| = 2$. This follows easily from the following arguments.

$$\|Bx\|^2 = \sum_{-\infty}^{\infty} (x(n-2) + x(n+2))^2 = \sum_{-\infty}^{\infty} \left[(x(n-2))^2 + (x(n+2))^2 + 2x(n-2)x(n+2)\right] \leq 4\|x\|^2.$$

Therefore we have $\|B\| \leq 2$. Now consider the sequence $x_n = \left(...0, 0, 0, ...\frac{1}{n}, \frac{1}{n}, \frac{1}{n}...\frac{1}{n}, 0, 0, 0...\right)$ where $\frac{1}{n}$ repeats n^2 times and all other entries 0, has norm 1 and $\|Bx_n\|$ increases to 2. Hence $\|B\| = 2$. Also as in the case of A, the truncated eigenvalues are distributed symmetrically on both sides of 0, so that -2 and 2 are in the essential spectrum. Also, since $A^2 = B + 2I$, this implies that 0 is an essential spectral value of A^2, and hence of A. Hence any spectral gap of A can occur either to the right or left side of 0. By Lemma 1, each spectral gap on the right side of 0 will also give a spectral gap on the left side. Hence the proof.

Theorem 11. *The essential spectrum of A is connected. The spectrum and the essential spectrum coincide with the compact interval* $[-2, 2]$.

Proof. First, we show that A has no eigenvalues. This will imply that the spectrum and essential spectrum coincide, as the essential spectrum consists of discrete eigenvalues of finite multiplicity. If $Ax = \lambda x$, for some nonzero x, then

$$x(n+1) + x(n-1) = \lambda x(n), \text{ for all } n.$$

Let $x(N) = p \neq 0$, for some N and $x(N-1) = q$. Then a recursive argument similar to that in the proof of Lemma 2 will show that such a vector will not lie in $l^2(Z)$.

By Theorem 10, spectral gaps can occur symmetrically to the origin. Hence it suffices to show that there is no spectral gap to the right side of the origin. We consider each possible case for the existence of a spectral gap. We rule them out one by one. First consider the case when the spectral gap is of the form $(a, 2)$, with $0 \leq a \leq 1$. In this case, since the interval $(a, 2)$ contains no essential points (as the essential spectrum coincides with the set of all essential points), it will contain at most K eigenvalues of truncations for infinitely many n. Let $\lambda_{n1}, \lambda_{n2}, \lambda_{n3}, \ldots \lambda_{nK}$ be those eigenvalues. From the expression of characteristic polynomials, it is evident that the determinant of $A_n' s$ is either 0 or ± 1. Since the eigenvalues are distributed symmetrically to both sides of 0, we have the product of positive eigenvalues equal 1 for n even. Let $s_K := \prod_{i=1}^{K} \lambda_{ni}$. Then $\frac{1}{s_K} > \frac{1}{2^K}$, since $\lambda_{ni} < 2$ for $i = 1, 2, \ldots K$.

But since 0 is in the essential spectrum, it is an essential point, and we can find an N such that the interval $\left(0, \frac{1}{2}\right)$ contains at least $K + 1$ eigenvalues of A_n for every $n \geq N$. For such an $n \geq N$, let $\alpha_{n1}, \alpha_{n2}, \alpha_{n3}, \ldots \alpha_{nN-K}$ be the eigenvalues of A_n, in $(0, a)$. Therefore we have,

$$\frac{1}{2^K} < \frac{1}{s_K} = \prod_{i=1}^{N-K} \alpha_{ni} < \prod_{i=1}^{K+1} \alpha_{ni} < \frac{1}{2^{K+1}} < \frac{1}{2^K}.$$

The first equality holds since the product of eigenvalues is 1, and the consequent inequality is because $a \leq 1$ (each additional α_{ni} we multiply will be a positive number below 1. Hence, the product will satisfy this inequality). This contradiction leads to the fact that $(a, 2)$, with $0 \leq a \leq 1$ is not a spectral gap.

Now we see that $(a, 2)$, with $a > 1$ cannot be a spectral gap. For if $(a, 2), a > 1$ is a gap, then it will contain at most K eigenvalues of truncations for infinitely many n. Let $\lambda_{n1}, \lambda_{n2}, \lambda_{n3}, \ldots \lambda_{nK}$ be those eigenvalues. As in the above case, let $s_K := \prod_{i=1}^{K} \lambda_{ni}$. Then $\frac{1}{s_K} > \frac{1}{2^K}$ Find an N such that the interval $\left(0, \frac{1}{2a^N}\right)$ contains at least $K + 1$ eigenvalues of A_N. Now let $\alpha_{n1}, \alpha_{n2}, \alpha_{n3}, \ldots \alpha_{nN-K}$ be the eigenvalues of A_N, in $(0, a)$. Therefore we have

$$\frac{1}{s_k} = \prod_{i=1}^{N-K} \alpha_{ni} < \left(\frac{1}{2a^N}\right)^{K+1} a^{N-(2K+1)} < \left(\frac{1}{2}\right)^{K+1} < \frac{1}{2^K}.$$

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The inequality is a consequence of $a > 1$. This contradiction leads to the fact that $(a, 2), a > 1$ is not a gap.

Hence we have seen that there cannot have a spectral gap of the form $(a, 2)$, with a being a non-negative real number. The number 2 does not play any role in the proof. We can easily imitate the proof techniques for intervals of the form (a, b), with $0 \leq a < b \leq 2$.

Remark 12. *We can have a different and straight forward argument to show that $(a, 2)$ cannot be a spectral gap. However, this method cannot be extended for arbitrary intervals (a, b), with $0 \leq a < b \leq 2$. For if $(a, 2)$ is a gap, then 2 will be an isolated point in the essential spectrum. Since the interval $(a, 2)$ contains at most K eigenvalues of truncations, but 2 is an essential point, we need 2 is an eigenvalue of A_n for large values of n with multiplicity increases to infinity. Nevertheless, 2 is not an eigenvalue of A_n for any n. For if*

$$\begin{bmatrix} 0 & 1 & & & & & & & & \\ 1 & 0 & 1 & & & & & & & \\ & 1 & 0 & 1 & & & & & & \\ & & 1 & 0 & 1 & & & & & \\ & & & 1 & 0 & 1 & & & & \\ & & & & . & . & . & & & \\ & & & & & . & . & . & & \\ & & & & & & . & . & . & \\ & & & & & & & 1 & 0 & 1 \\ & & & & & & & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \\ . \\ \\ x_n \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \\ . \\ \\ x_n \end{bmatrix}$$

then
$x_2 = 2x_1$
$x_3 = 3x_1$
$x_4 = 4x_1, \ldots x_{n-1} = (n-1)x_1, x_n = nx_1$
Also $x_{n-1} = 2x_n = 2nx_1$
But this will hold only when $x_1 = 0$ which makes $x = 0$ and hence 2 is not an eigenvalue. Hence we conclude that $(a, 2)$ is not a gap.

Remark 13. *The eigenvalues of the matrices A_n are explicitly calculated to be $2cos(\pi k/n + 1), k = 1, 2, 3 \ldots n$. We may arrive at some conclusions from that information also.*

Remark 14. *An important question is whether we can approximate the eigenvalues of A using the eigenvalues of truncation. Since the operator we considered has no eigenvalues, it is interesting to see from the truncations itself whether the operator has an eigenvalue or not. In general, we observe the following; If $\lambda = \lim \lambda_n, \lambda_n \in \sigma(A_n)$ and if the sequence of eigenvectors x_n corresponding to λ_n, is Cauchy in $\mathcal{H}$, then λ is an eigenvalue of A. This can easily be seen as follows.*

Let x_n converges to some x in $\mathcal{H}$. $\lambda = \lim \lambda_n, x = \lim x_n$ together imply $\lambda x = \lim \lambda_n x_n$ Also $\lim A_n x_n = Ax$. Hence for any $\epsilon > 0$, there is an N such that $\|\lambda - \lambda_N x_N\| < \frac{\varepsilon}{2}, \|Ax - A_N x_N\| < \frac{\varepsilon}{2}$ Hence for any $\epsilon > 0, \|Ax - \lambda x\| < \varepsilon$. That is λ is an eigenvalue of A.

4 CONCLUDING REMARKS

We used only elementary tools and the filtration techniques due to Arveson to prove the connectedness of the essential spectrum. When we consider the Discrete Schrödinger operator H defined by

$$H(x(n)) = (x(n-1) + x(n+1) + v(n)x(n)); x = (x(n)) \in l^2(\mathbb{Z}), n \in \mathbb{Z},$$

with the potential sequence $v = v(n)$ being periodic, there will be spectral gaps unless when v is constant (see [4, 5] for example). However, if we write the matrix representation with respect to the standard orthonormal basis, it is tridiagonal; hence, Arveson's techniques are available. Here the spectral analysis depends on the nature of the potential. The spectral gap issues of such operators were studied with the linear algebraic techniques in [6]. The spectral gap issues of arbitrary bounded self-adjoint operators can be found in the literature (see [7, 8] for example).

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Another interesting point is to carry over such techniques to the multi-dimensional case by replacing $\mathbb{Z}$ by $\mathbb{Z}^n$.

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REIDEMEISTER NUMBER IN LEFSCHETZ FIXED POINT THEORY

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ABSTRACT

Several interesting numbers such as the homotopy invariant numbers the Lefschets number $L(f)$, the Nielsen number $N(f)$, fixed point index $i(X, f, U)$ and the Reidemeister number $R(f)$ play important roles in the study of fixed point theorems. The Nielsen number gives more geometric information about fixed points than other numbers. However the Nielsen number is hard to compute in general. To compute the Nielsen number, Jiang related it to the Reidemeister number $R(f_\pi)$ of the induced homomorphism $f_\pi : \pi_1(X) \to \pi_1(X)$ when X is a lens space or an H-space (Jian type space). For such spaces, either $N(f) = 0$ or $N(f) = R(f)$ the Reidemeister number of f_π and if $R(f) = \infty$ then $N(f) = 0$ which implies that f is homotopic to a fixed point free map. This is a review article to discuss how these numbers are related in fixed point theory.

KEYWORDS

Twisted conjugacy, Reidemiester number, Lefschetz number, Nielsen number, Jiang space

1 INTRODUCTION

Let $\phi : G \to G$ be an endomorphism of an infinite group G. One has an equivalence relation $\sim_\phi$ on G defined as $x \sim_\phi y$ if there exists a $g \in G$ such that $y = gx\phi(g)^{-1}$. The equivalence classes are called the Reidemeister classes of ϕ or ϕ-conjugacy classes. When ϕ is the identity, the Reidemeister classes of ϕ are the usual conjugacy classes. The Reidemeister classes of ϕ are the orbits of the action of G on itself defined as $g.x = gx\phi(g^{-1})$. The Reidemeister classes of ϕ containing $x \in G$ is denoted $[x]_\phi$ or simply $[x]$ when ϕ is clear from the context. The set of all Reidemeister classes of ϕ is denoted by $\mathcal{R}(\phi)$. We denote by $R(\phi)$ the cardinality of $\mathcal{R}(\phi)$ if it is finite and if it is infinite we set $R(\phi) := \infty$ and $R(\phi)$ is called the Reidemeister number of ϕ on G. We say that G has the R_∞-property if the Reidemeister number of ϕ is infinite for every automorphism ϕ of G. If G has the R_∞-property, we call G an R_∞-group.

The notion of Reidemeister number originated in the Nielsen–Reidemeister fixed point theory. See [**?**] and the references therein. The problem of determining which classes of groups have R_∞-property is an area of active research. Many mathematicians have been trying to determine which class of groups have the R_∞-property using the internal structure of the class, such as Lie group structure, C^*-algebra structure or purely algebraic properties of the class. There is no particular way to solve this problem, which makes it more difficult and interesting. If X is an H-space or a lens space, their fundamental groups are abelian. The Reidemeister number of an endomorphism of an abelian group is easily computable in many cases. In fact, if G is an abelian group, $\mathcal{R}(\phi)$ is an abelian group under the well defined operation $[x][y] := [xy], x, y \in G$.

The R_∞-property does not behave well with respect to finite index subgroups and quotients as the D_∞ and any free group of rank $n > 1$ has the R_∞-property although the infinite cyclic group and finitely generated free abelian groups, which are quotients of free groups, do not (ref. [5], [4]). Thus the R_∞-property is not invariant under quasi isometry, that is it is not *geometric* among the class of all finitely generated groups. The works of Levitt and Lustig [5] and Felshtyn [2] show that this property is geometric in the class of non-elementary hyperbolic groups. It is been proved in [7] that the R_∞-property is geometric for the class of all finitely generated groups that are quasi-isometric to irreducible lattices in real semisimple Lie groups with finite centre and finitely many connected components. The R_∞-property for irreducible lattices was proved in [6].

We have stated some results without proofs. For proofs and further readings, we refer the reader [1].

2 THE LEFSCHETZ NUMBER

Let X be a connected compact ANR and $f : X \to X$ a continuous map. We have seen fixed point theorems like Brouwer fixed point theorem that states "Any map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point"where $\mathbb{D}^n$ is the closed disk in $\mathbb{R}^n$ and the traditional Lefschetz fixed point theorem that states "If $L(f) \neq 0$ then f has a fixed point", where $L(f)$ is the Lefschetz number with respect to the rational homology. Our statement of the Lefschetz fixed point theorem differs from the traditional one. We will prove the theorem for $L(f, \mathbb{F})$, where $\mathbb{F}$ is any field, because it is often easier to compute $L(f, \mathbb{F})$ if the field is chosen properly than it is to compute $L(f)$, and the conclusion is for all maps homotopic to f rather than just for the map f. An important reason, however, was that the converse of the traditional statement is -"If $L(f) = 0$, then f is fixed point free"- and this is trivially false (we will see an example). On the other hand, the converse of our statement is -"If $L(f, \mathbb{F}) = 0$ for all fields $\mathbb{F}$, then there is a fixed point free map g homotopic to f". This is true. A proof can be seen in [1].

To define the Lefschetz number we need the following definitions. A subset A of a space X is called a *neighbourhood retract* of X if there exists an open subset U of X containing A and a retraction of U onto A, i.e., a map $r : U \to A$ such that the restriction of r to A is the identity map. A space X is an *absolute neighbourhood retract* (**ANR**) if it has the following property: If X imbeds into a separable metric space Y, then X is a neighbourhood retract of Y. The ANR property is a topological invariant. A compact space X is a *compact* $\mathbf{A}NR$ if and only if there exists an imbedding $i : X \to I^\infty$ such that

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$i(X)$ is a neighbourhood retract of I^∞, where

$$I^\infty = \sqcup_{n\in\mathbb{N}}[\frac{-1}{n}, \frac{1}{n}]$$

is the infinite Hilbert cube with the metric $d(x_n, y_n) = (\sum |x_n{}^2 - y_n{}^2|)^{1/2}$. I^∞ itself is an ANR by definition. Since $i : \mathbb{D}^n \to I^\infty$ defined by $(x_1, ...x_n) \mapsto (x_1, x_2/2, ...x_n/n, 0, 0....)$ is an imbedding such that $i(\mathbb{D}^n)$ is a retraction of I^∞, the n-cell $\mathbb{D}^n$ is an ANR. Any locally finite polyhedron is an ANR. Open subsets of an ANR and a neighbourhood retract of an ANR are also an ANR.

Throughout this note we will assume X is a connected compact ANR. Note that any compact ANR has only countably many connected components with each is open and an ANR. For a compact ANR, the homology $H_*(X, \mathbb{F})$ is a finitary graded $\mathbb{F}$-vector space and any map $f : X \to X$ induces a morphism $f_* : H_*(X, \mathbb{F}) \to H_*(X, \mathbb{F})$ between the homology groups which is a morphism of finitary graded vector space over $\mathbb{F}$.

Definition 1. *Let $f : X \to X$ be a map on X, $\mathbb{F}$ be a field. The Lefschetz number of f over $\mathbb{F}$ is defined to be the number:*

$$L(f, \mathbb{F}) := \sum(-1)^q Tr(f_q)$$

We denote $L(f) = L(f, \mathbb{Q})$.

We state the Lefschets fixed point theorem without proof.

Theorem 1 (Lefschetz Fixed Point Theorem ([1])). *If X is a compact ANR and $f : X \to X$ is a map such that $L(f, \mathbb{F}) \neq 0$ for some field $\mathbb{F}$, then every map homotopic to f has a fixed point.*

For any field $\mathbb{F}$ the homology and cohomology are isomorphic and the induced morphism is the transpose of f_* so we can define the Lefschetz number using the cohomolgy too.

Observe that: (1) for the identity map 1_X of X, the Lefschetz number

$$L(1_X) = \sum_q(-1)^q Tr(1_q) = \sum(-1)^q dim(H_q(X, \mathbf{q})) = \sum(-1)^q b_q = \chi(X)$$

where $\chi(X)$ is the Euler characteristic and b_q is the q^{th} Betti number of X. (2) Since homotopic maps induce the same homomorphism on the homology groups, if $f : X \to X$ any continuous map and if g is homotopic to f then $L(g, \mathbb{F}) = L(f, \mathbb{F})$ for all field $\mathbb{F}$.

A space X is said to have the *fixed point property* if every continuous self map on X has a fixed point. Thus a contractible compact ANR X has fixed point property since $H_q(X, \mathbb{Q}) = 0, \forall q \neq 0$ and $H_0 = \mathbb{Q}$, thus $L(f) = 1$ (since every map on a path connected space induces the identity morphism on H_0) implies f has a fixed point by the Lefschetz fixed point theorem. For $X = \mathbb{S}^n$, the $n-$sphere, $\chi(X) = 0$ whenever n is odd. Thus the converse of the traditional fixed point theorem is false. Brouwer Fixed point theorem is an immediate consequence of the Lefschetz fixed point theorme, for; let $f : \mathbb{D}^n \to \mathbb{D}^n$ is any map. Since I^∞ is a contractible compact ANR and since retract of a space with fixed point property has the fixed point property, f has a fixed point.

3 Index for ANRs

For X, a compact ANR, a map $f : X \to X$, and an open set U of X without fixed points of f on its boundary it is possible to associate a number $i(X, f, U)$, the index of f on U. We define the index for such triples.

3.1 The axioms for an index

Let $\mathcal{C}_A$ denote the collection of all connected compact ANR spaces X where the Lefschetz number is defined since $H_*(X, \mathbf{q})$ is finitary. We define index for triples $i(X, f, U)$ with the following propeties:

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1. $X \in \mathcal{C}_A$,
2. $f : X \to X$ is a map,
3. U is open in X,
4. there are no fixed points of f on the boundary of U.

The collection of such triples (X, f, U) is denoted by $\mathcal{C}$. Observe that $(X, f, X), (X, f, \phi)$ satisfy these properties.
A (*fixed point*) *index* on $\mathcal{C}_A$ is a function $i : \mathcal{C} \to \mathbb{Q}$ which satisfies the following axioms:

1. (Localization). If $(X, f, U) \in \mathcal{C}$ and $g : X \to X$ is a map such that $g(x) = f(x)$ for all $x \in \bar{U}$ (the closure of U), then
$$i(X, f, U) = i(X, g, U).$$
2. (Homotopy). For $X \in \mathcal{C}_A$ and $H : X \times I \to X$ a homotopy, define $f_t : X \to X$ by $f_t(x) = H(x, t)$. If $(X, f_t, U) \in \mathcal{C}$ for all $t \in I$, then
$$i(X, f_0, U) = i(X, f_1, U).$$
3. (Additivity). If $(X, f, U) \in \mathcal{C}$ and $U_1, ...U_n$ is a set of mutually disjoint open subsets of U such that $f(x) \neq x$ for all
$$x \in U - \bigcup_{j=1}^{n} U_j,$$
then
$$i(X, f, U) = \sum_{j=1}^{n} i(X, f, U_j).$$
4. (Normalization). If $X \in \mathcal{C}_A$ and $f : X \to X$ is a map, then
$$i(X, f, X) = L(f).$$
5. (Commutativity) If $X, Y \in \mathcal{C}_A$ and $f : X \to Y, g : Y \to X$ are maps such that $(X, gf, U) \in \mathcal{C}$, then
$$i(X, gf, U) = i(Y, fg, g^{-1}(U)).$$

The localization axiom 1 obviously makes the definition of the index "local" in the sense that $i(X, f, U)$ is not affected by the behaviuor of f outside of $\bar{U}$. The normalization axiom 4 connects the index to Lefschetz theory. The homotopy and commutativity axioms are generalizations of properties of the Lefschetz number.

Lemma 1. *If there is an index i on $\mathcal{C}_A$ and if $(X, f, U) \in \mathcal{C}$, such that $i(X, f, U) \neq 0$, then f has a fixed point in U.*

Proof. Note that $i(X, f, \phi) = i(X, f, \phi) + i(X, f, \phi)$ by additivity 3 ($U = U_1 = U_2 = \phi$). Thus $i(X, f, \phi) = 0$ since it is rational. Suppose $f(x) \neq x$ on U. Then we can apply additivity 3 for the given open set U and $U_1 = \phi$, and we get $i(X, f, U) = i(X, f, \phi) = 0$. Which is a contradiction. Hence f has a fixed point in U.

Lemma 2. *Assume there is an index on $\mathcal{C}$ and if $X \in \mathcal{C}_A, f : X \to X$ a map such that $L(f) \neq 0$ then every map homotopic to f has a fixed point.*

Proof. Let g be any map homotopic to f; then $L(g) = L(f) \neq 0$. By the normalization axiom, $i(X, g, X) = L(g) \neq 0$, so g has a fixed point in X by 1.

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The last Lemma 1 makes the important point that Index Theory is more powerful than Lefschetz Theory in the sense that the existance of a function on $\mathcal{C}_A$ satisfying just two of the axioms of an index, namely additivity 3 and normalization 4, is enough to imply that the Lefschetz Fixed Point Theorem 1 is true for all maps on spaces in a collection $\mathcal{C}'$ on which an index is defined.

Example 1. *Let X be a compact connected ANR and $(X, f, U) \in \mathcal{C}$ where f is a constant map say $f(x) = x_0,\ \forall x \in X$. Then*

$$i(X, f, U) = \begin{cases} 0 & \text{if } x_0 \notin U \\ 1 & \text{if } x_0 \in U \end{cases}$$

For, if $x_0 \notin U$. Then by additivity 3 for the given U and $U_1 = \phi$, $i(X, f, U) = i(X, f, \phi) = 0$. Now suppose $x_0 \in U$. Let $Y = \{x_0\}$ the singleton space and $g : X \to Y, h : Y \to X$ be the maps $x \mapsto x_0, h = 1_Y$ respectively. Then $i(X, f, U) = i(X, hg, U) = i(Y, gh, Y) = L(gh) = L(1_Y)$ by the commutativity 5 and normalization 4 axioms. Since any map f on a path connected space induces the identity on the homology group H_0 and since Y is path connected and higher homology groups are trivial, $L(1_Y) = 1$. Hence $i(X, f, U) = 1$ in this case. The following theorem tells us that such an index exists on $\mathcal{C}_A$. Details can be seen in chapters IV and V of [1].

Theorem 2. *For the collection $\mathcal{C}_A$ of all connected compact ANR, there is a unique index defined on it satisfying all the five axioms.*

Now we are ready to define an index on the Nielsen classes of a map $f : X \to X$.

4 The Nielsen Number

For X, a compact ANR, and a map $f : X \to X$, we shall define a non-negative integer $N(f)$, called the *Nielsen number* of f. The Nielsen number is a lower bound for the number of fixed points of f.

4.1 Nielsen Classes

Assume that the set $Fixf$ of all fixed points of f is non-empty. Two points $x_0, x_1 \in Fixf$ are f-equivalent if there is a path $c : I \to X$ from x_0 to x_1 such that c and $f \circ c$ are homotopic with respect to the end points. This relation defines an equivalence relation on $Fixf$. The equivalence classes are called *Nielsen classes* or *fixed point classes* of f. It is known that the set of Nielsen classes of a map f on a connected, compact, ANR X is finite.

Theorem 3. *A map $f : X \to X$ on a compact ANR has only finitely many Nielsen classes.*

Hence we will denote the set of fixed point classes by $\mathcal{N}(f) := \{F_1, F_2, ..., F_n\}$.

4.2 Nielsen Number

Let $f : X \to X$ be a map with fixed point classes $F_1, ..., F_n$. Then for each $j = 1, .., n$, there is an open set $U_j \subset X$ such that $F_j \subset U_j$ and $\bar{U}_j \cap Fixf = F_j$. Let i be the index on $\mathcal{C}_A$. Note that $(X, f, U_j) \in \mathcal{C}$. Define the *index* $i(F_j)$ of the fixed point class F_j by $i(F_j) = i(X, f, U_j)$. This definition is independent of the choice of the open set $U_j \subseteq X$ such that $F_j \subseteq U_j$ and $\bar{U}_j \cap Fixf = F_j$ because, suppose U, V are two such open sets. If $x \in U - U \cap V$ then since x belongs to $U, x \notin F_k$ for $k \neq j$, while $x \notin V$, so $x \notin F_j$. Thus $x \notin Fixf$. By the additivity axiom, $i(X, f, U) = i(X, f, U \cap V)$. The same reasoning implies $i(X, f, V) = i(X, f, U \cap V)$. A fixed point class F of f is said to be *essential* if $i(F) \neq 0$ and *inessential* otherwise. The *Nielsen number* $N(f)$ of the map f is defined to be the number of fixed point classes of f that are essential.
A fixed point theorem with this number is that:

Theorem 4. *Any continuous map f on a connected compact ANR has at least $N(f)$ fixed points.*

Theorem 5. *Let $f,\ g : X \to X$ be homotopic maps; then $N(f) = N(g)$.*

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Thus any continuous function g homotopic to f has at least $N(f)$ fixed points.
The example 1 is an example with only one fixed point class and it is essential, i.e, $N(f) = 1$.

Computing $N(f)$ is difficult in general, in some cases it can be computed via the Reidemeister number $R(f)$ by knowing $L(f)$ and f_π , the induced homomorphism on the fundamental group of X. The main tool to compute the Nielsen number is the Jiang subgroup of the fundamental group. Before going to the computation method of Nielsen number we will see how it is related to the Reidemeister number $R(f)$.

5 The Reidemeister Number

Let $f : X \to X$ be a map on a connected compact ANR and let $Fixf = \{x \in X : f(x) = x\}$. Let $p : \widetilde{X} \to X$ be the universal covering of X and $\tilde{f} : \widetilde{X} \to \widetilde{X}$ be a lifting of f, ie. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called conjugate if there is a $\gamma \in \Gamma = \pi_1(X)$ such that $\tilde{f}' = \gamma\tilde{f}\gamma^{-1}$. Note that if $\tilde{f}$ is a lift of f and $\gamma \in \Gamma$, then $p(Fix\tilde{f}) = p(Fix\gamma\tilde{f}\gamma^{-1})$ and $p(Fix\tilde{f}) = p(Fix\tilde{f}')$, then $\tilde{f}' = \gamma\tilde{f}\gamma^{-1}$ for some $\gamma \in \Gamma$. This is an equivalence relation on the set $Fixf = \bigsqcup_{[\tilde{f}]} p(Fix\tilde{f})$, ie. $Fixf$ is a disjoint union of projections of fixed points of lifts from distinct lifting classes. The subset $p(Fix\tilde{f}) \subseteq Fixf$ is called the *fixed point class* of f determined by the lifting class $[\tilde{f}]$. Note that for any point $y \in \widetilde{X}$, since $(f \circ p)_\pi(\pi_1(\widetilde{X})) \subset p_\pi(\pi_1(X))$, there is a unique map $\tilde{f}$ such that $p \circ \tilde{f} = f \circ p$ and $\tilde{f}(y) = y$. In particular, any fixed point of f is a projection of a fixed point of some lift $\tilde{f}$ of f.

Now we define the Reidemeister number of a group homomorphism $\varphi : G \to G$. Let $\varphi : G \to G$ be a group homomorphism on a group G. One has an action of G on itself given by $g.x = gx\varphi(g)^{-1}$. Two elements $x, y \in G$ are said to be $\varphi-$ twisted conjugate, $x \sim_\varphi y$, if they are in the same orbit of this action. The orbits are called the φ-twisted conjugacy classes or the *Reidemeister classes* and $R(\varphi)$, the number of Reidemeister classes is called the *Reidemeister number* of φ. If $\varphi = Id$ then $R(\varphi)$ coincides with the number of conjugacy classes of G.

By fixing a lift $\tilde{f}$ of f and $\alpha \in \Gamma \simeq \pi_1(X)$, $\tilde{x} \in \widetilde{X}$, we obtain a unique element $\tilde{\alpha} \in \Gamma$ such that $\tilde{\alpha}\tilde{f}(\tilde{x}) = \tilde{f}(\alpha(\tilde{x}))$. Thus we obtain a homomorphism $\phi : \pi_1(X) \to \pi_1(X)$ such that $\tilde{f}\alpha = \phi(\alpha)\tilde{f}, \forall\alpha \in \Gamma$. Also once we fix a lift $\tilde{f}$ of f, then every lift is of the form $\alpha\tilde{f}$ for some $\alpha \in \Gamma$. Let $\alpha, \beta \in \Gamma$. Then

$$[\alpha\tilde{f}] = [\beta\tilde{f}] \iff \beta\tilde{f} = \gamma\alpha\tilde{f}\gamma^{-1}$$

for some $\gamma \in \Gamma$. ie.

$$\iff \beta\tilde{f} = \gamma\alpha\phi(\gamma^{-1})\tilde{f}$$

By the uniqueness of lifts, we have $[\alpha\tilde{f}] = [\beta\tilde{f}] \iff \beta = \gamma\alpha\phi(\gamma^{-1})$ for some $\gamma \in \Gamma$. Furthermore, by choosing appropriate base point, ϕ can be identified with the induced homomorphism f_π on Γ. From now on, we write $R(f)$ for $R(f_\pi)$. It follows from the above definition that there is a one-one correspondence between the lifting classes of f and the Reidemeister classes of f_π.

Remark 1. *If we choose a different lifting $\tilde{f}'$ and thus a different homomorphism ϕ', we get a bijection between the ϕ-Reidemeister classes and the ϕ'-Reidemeister classes so that the cardianality of such sets is a constant.*

5.1 Relationship with Nielsen number

Suppose $x_1, x_2 \in Fixf$ are in the same Nielsen class, ie. there exists a path $c : I \to X$ from x_1 to x_2 such that $f \circ c$ and c are homotopic relative to the end points. Let $\tilde{f}$ be a lift of f and $\tilde{x}_1 \in Fix\tilde{f}$ such that $p(\tilde{x}_1) = x_1$. Lift c to a path $\tilde{c}$ starting from $\tilde{x}_1$ and ending at some $\tilde{x}_2$ in $\tilde{X}$. Then $\tilde{f} \circ \tilde{c}$ projects onto $f \circ c$ which is homotopic to c. Thus $\tilde{f} \circ \tilde{c}$ also ends at $\tilde{x}_2$. Hence $\tilde{f}(\tilde{x}_2) = \tilde{x}_2$. In otherwords, they belong to the same lifting class. Conversely, let $\tilde{x}_1, \tilde{x}_2 \in Fix\tilde{f}$ such that $p(\tilde{x}_1) = x_1 \neq x_2 = p(\tilde{x}_2)$. Let $\tilde{c} : I \to \widetilde{X}$ be a path from $\tilde{x}_1$ to $\tilde{x}_2$. Then $c = p \circ \tilde{c}$ is a path from x_1 to x_2 in X and $p(\tilde{f} \circ \tilde{c}) = f \circ p \circ \tilde{c} = f \circ c$, ie. $\tilde{f} \circ \tilde{c}$ projects onto $f \circ c$. In fact, the loop $\tilde{c}(\tilde{f} \circ \tilde{c})^{-1}$ projects to the loop $c(f \circ c)^{-1}$. Since $\widetilde{X}$ is simply-connected, the former loop is trivial in $\pi_1(\widetilde{X})$ and thus the later loop is homotopic to the trivial loop, ie. $c \sim f \circ c$. That is x_1 and x_2 are in the same Nielsen class. This shows that there is a one-to-one

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map, say ψ, from the set of Nielsen classes to the set of Reidemeister classes and which implies that $N(f) \leq R(f)$. Note that a lifting class $p(Fix\tilde{f})$ might be empty, but Nielsen classes are non-empty. Also $R(f)$ need not be finite while $N(f)$ is always finite. For example, if $f = 1_X$ then any two points are Nielsen equivalent, thus $N(f) \leq 1$ while $R(f)$ is the number of conjugacy classes in $\pi_1(X)$. In particular, if $\pi_1(X)$ is abelian then $R(f) = |\pi_1(X)|$.

6 Computing Nielsen Number

First, let us consider a simple example: For a simply connected space X, there is only one Nielsen class for any self map f of X, so $N(f) \leq 1$. In this case $L(f) = 0 \Rightarrow N(f) = 0$ or $L(f) \neq 0 \Rightarrow N(f) = 1$. $N(f)$ does not give more information than $L(f)$.

The main tool to calculate $N(f)$ is the Jiang subgroup $T(f) \leq \pi_1(X)$ introduced by B. Jiang(1963).

6.1 The Jiang Subgroup

Fix a point x_0 in a compact connected ANR X and a self map f on X. We denote by $Map(X)$ the set of all maps from X to itself with the supremum metric $d(f,g) = sup\{d(f(x), g(x)) \mid x \in X\}$, then it is a complete metric space. Let $p : Map(X) \to X$ be the map given by $p(g) = g(x_0)$. Then p induces a homomorphism $p_\pi : \pi_1(Map(X), f) \to \pi_1(X, f(x_0))$. The Jiang subgroup $T(f, x_0)$ is the image of the homomorphism p_π. Equivalently, an element $\alpha \in \pi_1(X, f(x_0))$ is said to be in the Jiang subgroup $T(f, x_0)$ of f if there is a loop H in $Map(X)$ based at f such that the loop c in X defined by $c(t) = H(t)(x_0)$ is homotopic to α.

Lemma 3. *The Jiang subgroup is independent of the base point, ie.* $T(f, x_0) \simeq T(f, x_1)$ *for any* $x_0, x_1 \in X$.

Theorem 6. *If f is such that* $T(f, x_0) \simeq \pi_1(X, x_0)$*. Then all the fixed point classes have the same index. If* $f : X \to X$ *is such that* $T(f, x_0) = \pi_1(X, x_0)$*, then* $L(f) = 0$ *implies* $N(f) = 0$. Proof.*If* $Fixf = \phi$*, then certainly* $N(f) = 0$*. Otherwise, let* $\{F_1, F_2, ..., F_n\}$ *be the different fixed point classes of f, and assume* $x_0 \in F_1$ *(Lemma 3). By Theorem 6* $i(F_j) = i(F_1)$ *for every j; so, by additivity (3) and normalization (4) axioms,*

$$0 = L(f) = \sum_j i(F_j) = n\ i(F_1)$$

Thus $i(F_1) = 0 \Rightarrow i(F_j) = 0$ *for every j, which implies* $N(f) = 0$. □

Lemma 4. *If f and g are homotopic, then* $T(f, x_0) \simeq T(g, x_0)$.

Lemma 5. $f : X \to X, x_0, x_1 \in X$*. Then there exists a map* $g : X \to X$ *such that both* $f^{-1}(x_0), x_1 \in g^{-1}(x_0)$.

This lemma implies that, given f, x_0 *as above, there is a map* $g \sim f$ *such that* $g(x_0) = x_0$*. Hence we can choose* $x_0 \in Fixf$ *(Lemmas 3, 4, 5). We will drop the base point from the fundamental group and the Jiang subgroup. The Jiang subgroup of the identity map on X is denoted by* $T(X)$ *and* $T(f) = T(f, x_0)$.

Theorem 7. *For any map* $f : X \to X$, $T(X) \subseteq T(f)$. Proof.*Let* $\alpha \in T(X) \leq \pi_1(X)$*. Then there is a loop* $[H] \in \pi_1(Map(X), 1_X)$ *based at the identity map such that* $[pH] = \alpha$*. Define a loop H' in $Map(X)$ (based at f) by* $H'(t)(x) = H(t)(f(x))$*. Then, since* $f(x_0) = x_0$*, it follows that* $H'(t)(x_0) = H(t)(x_0)$*, which proves that* $\alpha = [pH] = [pH'] \in T(f)$. □*An ANR is an* H-space *if there is an element* $e \in X$ *and a map* $\mu : X \times X \to X$ *such that* $\mu(x, e) = \mu(e, x) = x, \forall x \in X$*. (The fundamental group of an H-spaces is abelian), (*$\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3, \mathbb{S}^7$ *are the only spheres which are H-spaces). An important property of an H-space is:*

Theorem 8. *If X is a H-space, then* $T(X) = \pi_1(X)$. Proof.*We use e as the base point. Let c be any loop in X at e and define* $H : [0,1]^a Map(X) by H(t)(x)=\mu(c(t), x)$*. Thus* $[c] \in T(X)$. □

Note that for any H-space, $L(f) = 0$ *implies* $N(f) = 0$.

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Now on we will work with X a connected polyhedron and will fix a triangulation (K, τ) on X. A space X is aspherical *if $\pi_n(X) = 1$, for all $n \geq 2$.*

Theorem 9. *Let X be a connected aspherical polyhedron and $f : X^a X.ThenZ(f_\pi(\pi_1(X))) \subseteq T(f)$.*

Note that, if $f_\pi(\pi_1(X)) \subseteq Z(\pi_1(X))$, then $Z(f_\pi(\pi_1(X))) \subseteq \pi_1(X)$. If $f_\pi(\pi_1(X)) \subseteq Z(\pi_1(X))$, then $T(f) = \pi_1(X)$. Proof.*$f_\pi(\pi_1(X)) \subseteq Z(\pi_1(X))Z(f_\pi(\pi_1(X))) \subseteq T(f)T(f) = \pi_1(X)$. □Now on we assume $L(f) \neq 0$ (then there is at least one essential fixed point class, ie. $L(f) \neq 0N(f) \geq 1$ (by additivity 3), and X a compact ANR. If we apply the equivalence relation of f_π-equivalence (twisted action) to the Jiang subgroup $T(f) \subseteq \pi_1(X)$, then the set of equivalence classes is denoted by $T'(f)$. Let $J(f)$ be the cardinality of $T'(f)$. In other words, $J(f)$ is the number of f_π- twisted classes in $\pi_1(X)$ which contain elements of $T(f)$.*

Theorem 10. *If $\alpha \in T(f)$, then there is an essential fixed point class F of f such that $\psi(F) = [\alpha]$, the Reidemeister class containing α, where ψ is the map from the set of all Nielsen classes to the set of all Reidemeister classes of f discussed in section 5.1. It follows that $J(f) \leq N(f)$. If $T(f) = \pi_1(X)$, then $N(f) = R(f)$.* Proof.*$T(f) = \pi_1(X)$ implies that $J(f) = R(f)$ by definition. We know that $N(f) \leq R(f)$. Now the result follows from theorem 10, it states $J(f) \leq N(f)$.* □

Example 2. *Let $X = \mathbb{S}^1$, the circle, an aspherical H-space with $\pi_1(X) =$. Then $T(X) = T(1_X) = \pi_1(\mathbb{S}^1) =$. If $f : \mathbb{S}^{1a} S^1$ be any map, then $T(X) \subseteq T(f)T(f) = \pi_1(\mathbb{S}^1)$. Now $L(f) = 0 \iff f_\pi$ is the identity isomorphism (since $H_0(\mathbb{S}^1) == H_1$). Thus $L(f) = 0N(f) = 0$. If f_π is not the identity isomorphism, say $f_\pi(1) = q \neq 1$, then $T(f) = \pi_1(\mathbb{S}^1)N(f) = R(f) = \#Coker(1 - f_\pi) = |1 - q|$ since for an abelan group G and a given homomorphism $\phi : G^a G, theReidemeisternumber R(\phi) = \#Coker(1-\phi)$.*

Example 3. *Let $f : \mathbb{S}^{2a} S^2$ be a rotation by an angle θ. Let $p, n \in \mathbb{S}^2$ be the south and north poles and are the only fixed points of f. Since $\mathbb{S}^2$ is simply connected, there is exactly one Nielsen class F and hence $N(f) \leq 1$. Note that f is homotopic to the identity map. Thus $L(f) = L(1) = \chi(\mathbb{S}^2) = 2 \neq 0N(f) \geq 1$. Hence $N(f) = 1$ and $i(F) = 1$.*

Example 4. *Let $f : \mathbb{S}^{na} S^n$ be the map $f(x) = -x$ for all $x \in \mathbb{S}^n$. Then $L(f) = 1 - deg(f)$, where degree of f is $deg(f) = (-1)^{n+1}$.*

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APPLICATIONS OF FIXED POINT THEOREMS TO SOLUTIONS OF OPERATOR EQUATIONS IN BANACH SPACES

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ABSTRACT

In this paper we use Browder's and Gohde's fixed point theorem, Kirk's fixed point theorem and the Sadovskii fixed point theorem to obtain solutions of operator equations in Banach spaces.

KEYWORDS

fixed point, Banach spaces

1 INTRODUCTION

Perhaps the most famous fixed-point theorem is the Banach's contraction principle which has several applications. Motivated by this we have considered in this review article, applications of other well-known fixed-point theorems in various kinds of Banach spaces. This article should be of interest to mathematicians working in the fields of fixed-point theory and functional analysis.
In Section 1 we apply the Browder's and Göhde's fixed point theorem for the existence of solutions of operator equations involving asymptotically nonexpansive mappings in uniformly convex Banach spaces. In Section 2 we apply Kirk's fixed point theorem for the existence of solutions of the operator equation $x - Tx = f$ in reflexive Banach spaces and in Section 3 we apply the Sadovskii fixed point theorem for existence of solutions of the operator equation $x - Tx = f$ in arbitrary Banach spaces.

2 Application of Browder's and Göhde's fixed point theorem

Definition 1. *[1] A mapping T from a metric space (X, d) into another metric space (Y, ρ) is said to satisfy Lipschitz condition on X if there exists a constant $L > 0$ such that*

$$\rho(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. If L is the least number for which Lipschitz condition holds, then L is called Lipschitz constant. If $L = 1$, the mapping is said to be nonexpansive.

Definition 2. *[2] Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if for each $n \in \mathbb{N}$ there exists a positive constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$||T^n x - T^n y|| \leq k_n ||x - y||$$

for all $x, y \in K$.

The Browder's and Göhde's fixed point theorem is as follows:

Theorem 1. *[3] Let X be a uniformly convex Banach space and C a nonempty, closed, convex and bounded subset of X. Then every nonexpansive mapping $T : C \to C$ has a fixed point in C.*

We now state the main theorem of Section 1.

Theorem 2. *Let X be a uniformly convex Banach space and K a nonempty subset of X. Let $T : K \to K$ be an asymptotically nonexpansive mapping and $f_n \in K$, then the operator equation*

$$k_n x = T^n x + f_n$$

where $n \in \mathbb{N}$ and k_n is the Lipschitz constant of the iterates T^n, has a solution if and only if, for any $x_1 \in K$, the sequence of iterates $\{x_n\}$ in K defined by

$$k_n x_{n+1} = T^n x_n + f_n$$

$n \in \mathbb{N}$ is bounded.

Proof. For every $n \in \mathbb{N}$, let T_{f_n} be defined to be a mapping from K into K by

$$T_{f_n}(u) = \frac{1}{k_n}[T^n u + f_n].$$

Then $u_n \in K$ is a solution of

$$x = \frac{1}{k_n}[T^n x + f_n]$$

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if and only if u_n is a fixed point of T_{f_n}. Since T is asymptotically nonexpansive it follows that T_{f_n} is nonexpansive for all $n \in \mathbb{N}$.

$$||T_{f_n}(x) - T_{f_n}(y)|| = \frac{1}{k_n}||T^n(x) - T^n(y)|| \leq ||x - y||.$$

Suppose T_{f_n} has a fixed point $u_n \in K$. Then

$$||x_{n+1} - u_n|| = ||\frac{1}{k_n}[T^n x_n + f_n] - u_n|| = ||T_{f_n}(x_n) - T_{f_n}(u_n)|| \leq ||x_n - u_n||,$$

T_{f_n} being nonexpasive. Since $\{||x_n - u_n||\}$ is non-increasing, hence $\{x_n\}$ is bounded. Conversely, suppose that$\{x_n\}$ is bounded. Let $d = diam(\{x_n\})$ and

$$B_d[x] = \{y \in K : ||x - y|| \leq d\}$$

for each $x \in K$. Set

$$C_n = \bigcap_{i \geq n} B_d[x_i] \subset K.$$

Hence C_n is a nonempty, convex set for each $n \in \mathbb{N}$. Now we claim that $T_{f_n}(C_n) \subset C_{n+1}$. Let $y \in B_d[x_n]$ which implies $||y - x_n|| \leq d$. Since T_{f_n} is nonexpansive, we get

$$||T_{f_n}(y) - T_{f_n}(x_n)|| \leq d$$

$$||\frac{1}{k_n}[T^n(y) + f_n] - \frac{1}{k_n}[T^n(x_n) + f_n]|| \leq d$$

or

$$||\frac{1}{k_n}[T^n(y) + f_n] - x_{n+1}|| \leq d$$

or

$$\frac{1}{k_n}[T^n(y) + f_n] \in B_d[x_{n+1}]$$

giving

$$T_{f_n}(y) \in B_d[x_{n+1}]$$

proving that $T_{f_n}(C_n) \subset C_{n+1}$.

Let $C = \overline{\bigcup_{n \in \mathbb{N}} C_n}$. Since C_n increases with n, C is a closed, convex and bounded subset of K. We can easily see that T_{f_n} maps C into C.

$$T_{f_n}(C) = T_{f_n}(\overline{\bigcup_{n \in \mathbb{N}} C_n}) \subseteq \overline{T_{f_n}(\bigcup_{n \in \mathbb{N}} C_n)} = \overline{\bigcup_{n \in \mathbb{N}} T_{f_n}(C_n)} \subseteq \overline{\bigcup_{n \in \mathbb{N}} C_{n+1}} = C.$$

Applying the Browder's and Göhde's theorem to T_{f_n} and C we get a fixed point of T_{f_n} in C. Since $C \subset K$, we obtain a fixed point of T_{f_n} in K.

3 Application of Kirk's Fixed Point Theorem

Let us recall some definitions and results that we shall require for the proof of the Main Theorem of Section 2.

Definition 3. *[2] Let (X, ρ) and (M, d) be metric spaces. A mapping $f : X \to M$ is said to be nonexpansive if for each $x, y \in X$,*

$$d(f(x), f(y)) \leq \rho(x, y).$$

Definition 4. *[1] A convex subset K of a Banach space X is said to have normal structure if each bounded, convex subset S of K with diam $S > 0$ contains a nondiametral point.*

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The following theorem gives application of the Browder-Göhde-Kirk's theorem for the existence of solutions of the operator equation

$$x - Tx = f.$$

It is known that every uniformly convex Banach space is reflexive. We generalize the theorem below to reflexive Banach spaces using Kirk's fixed point theorem

Theorem 3. *[3] Let X be a uniformly convex Banach space, f an element in X and $T : X \to X$ a nonexpansive mapping, then the operator equation*

$$x - Tx = f$$

has a solution x if and only if for any $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ in X defined by $x_{n+1} = Tx_n + f$, $n \in \mathbb{N}_0$ is bounded.

Definition 5. *[1] A Banach space X is said to satisfy the Opial condition if whenever a sequence $\{x_n\}$ in X converges weakly to $x_0 \in X$, then*

$$\lim_{n\to\infty} \inf \|x_n - x_0\| < \lim_{n\to\infty} \inf \|x_n - x\|$$

for all $x \in X$, $x \neq x_0$.

Lemma 1. *[3] Let X be a reflexive Banach space with the Opial condition. Then X has normal structure.*

Lemma 2. *[3] A closed subspace of a reflexive Banach space is reflexive.*

Now we state the Kirk's fixed point theorem.

Theorem 4. *[3] Let X be a Banach space and C a nonempty weakly compact, convex subset of X with normal structure, then every nonexpansive mapping $T : C \to C$ has a fixed point.*

We state the main theorem of Section 2.

Theorem 5. *Let X be a reflexive Banach space satisfying Opial condition. Let $f \in X$ and $T : X \to X$ be a nonexpansive mapping. Then the operator equation*

$$x - Tx = f$$

has a solution x if and only if for any $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ in X defined by $x_{n+1} = Tx_n + f$, $n \in \mathbb{N}_0$ is bounded.

Proof. Let T_f be the mapping from X into X given by

$$T_f(u) = Tu + f.$$

Then u is a solution of

$$x - Tx = f$$

if and only if u is a fixed point of T_f. Clearly T_f is nonexpansive. Suppose T_f has a fixed point $u \in X$. Then for all $n \in \mathbb{N}$,

$$\|x_{n+1} - u\| \leq \|x_n - u\|.$$

Hence $\{x_n\}$ is bounded.

Conversely, suppose that $\{x_n\}$ is bounded. Let $d = diam(\{x_n\})$ and

$$B_d[x] = \{y \in X : \ \|x - y\| \leq d\}$$

for each $x \in X$. Set $C_n = \bigcap_{i\geq n} B_d[x_i]$. Then C_n is a nonempty convex set for each n, and

$$T_f(C_n) \subset C_{n+1}.$$

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Let C be the closure of the union of C_n for $n \in \mathbb{N}$,

$$C = \overline{\bigcup_{n\in\mathbb{N}} C_n}.$$

Since C_n increases with n, C is a closed, convex and bounded subset of X. It is known that [1] bounded, closed and convex subsets of reflexive Banach spaces are weakly compact, hence we get that C is weakly compact.

Now since

$$T_f(C) = T_f(\overline{\bigcup C_n}) \subseteq \overline{T_f(\bigcup C_n)} = \overline{\bigcup T_f(C_n)} \subseteq \overline{\bigcup C_{n+1}} = C,$$

we get that T_f maps C into itself. By Lemma 1.3.6, C is a reflexive Banach space. Now X satisfies Opial condition and C being a closed subset of X, will also satisfy Opial condition. Hence by Lemma 1.3.5, C has normal structure. Finally, applying Kirk's fixed point theorem we get that T_f has a fixed point in C which proves the theorem.

4 Application of Sadovskii Fixed Point Theorem

We recall some definitions

Definition 6. *[1] Let (M, ρ) denote a complete metric space and let $\mathfrak{B}$ denote the collection of nonempty and bounded subsets of M. Define the Kuratowski measure of noncompactness $\alpha : \mathfrak{B} \to \mathbb{R}^+$ by taking for $A \in \mathfrak{B}$,*
$\alpha(A)$=inf$\{\epsilon > 0$ A is contained in the union of a finite number of sets in $\mathfrak{B}$ each having diameter less than $\epsilon\}$.
If M is a Banach space the function α has the following properties for $A, B \in \mathfrak{B}$
1. $\alpha(A) = 0 \Leftrightarrow \overline{A}$ is compact,
2. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

Definition 7. *[2] Let K be a subset of a metric space M. A mapping $T : K \to M$ is said to be condensing if T is bounded and continuous and if*

$$\alpha(T(D)) < \alpha(D)$$

for all bounded subsets D of M for which $\alpha(D) > 0$.

We state the Sadovskii fixed point theorem.

Theorem 6. *[2] Let K be a nonempty, bounded closed and convex subset of a Banach space and let $T : K \to K$ be a condensing mapping, then T has a fixed point.*

The main result of section 3 is the following:

Theorem 7. *Let X be an arbitrary Banach space, let $f \in X$ and $T : X \to X$ be a condensing mapping, then the operator equation*

$$x - Tx = f$$

has a solution if and only if for any $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ in X, defined by $x_{n+1} = Tx_n + f$, $n \in \mathbb{N}_0$ is bounded.

Proof. Let the mapping $T_f : X \to X$ be defined by

$$T_f(u) = Tu + f.$$

Then u is a solution of the operator equation

$$x - Tx = f$$

if and only if u is a fixed point of T_f.
Since T is bounded and continuous, T_f is also bounded and continuous. Using the properties of the Kuratowski measure of noncompactness, for all bounded subsets D of X, we have

$$\alpha(T_f(D)) = \alpha(T(D) + \{f\}) \leq \alpha(T(D)) + \alpha(\{f\}).$$

Since $\{f\}$ is compact, $\overline{\{f\}}$ is compact, implying $\alpha(\{f\}) = 0$, giving

$$\alpha(T_f(D)) \leq \alpha(T(D)) < \alpha(T(D)).$$

Since T is condensing mapping and it follows that T_f is a condensing mapping.
Suppose T_f has a fixed point u in X. Then for all $n \in \mathbb{N}$, since T_f is a continuous mapping being condensing, we get

$$\|x_{n+1} - u\| = \|Tx_n + f - u\| = \|T_f(x_n) - T_f(u)\| \leq \|x_n - u\|.$$

Hence $\{x_n\}$ is bounded.

Conversely, suppose that $\{x_n\}$ is bounded. Let $d = diam(\{x_n\})$ and for each $x \in X$

$$B_d[x] = \{y \in X : \|x - y\| \leq d\}.$$

Set $C_n = \bigcap_{i \geq n} B_d[x_i]$, then C_n is a nonempty convex set for each n. Using that T is a continuous mapping and the given Picard iteration, we have

$$\begin{aligned} y \in B_d[x_n] &\Rightarrow \|y - x_n\| \leq d \\ &\Rightarrow \|Ty - Tx_n\| \leq d \\ &\Rightarrow \|Ty - [x_{n+1} - f]\| \leq d \\ &\Rightarrow \|(Ty + f) - x_{n+1}\| \leq d \\ &\Rightarrow (Ty + f) \in B_d[x_{n+1}]. \end{aligned}$$

Applying this, we get the following

$$\begin{aligned} T_f(C_n) &= T_f(\bigcap_{i \geq n} B_d[x_i]) \\ &\subseteq \bigcap_{i \geq n} T_f(B_d[x_i]) \\ &= \bigcap_{i \geq n} \{T_f(y) : \|y - x_i\| \leq d\} \\ &= \bigcap_{i \geq n} \{(Ty + f) : \|y - x_i\| \leq d\} \\ &\subseteq \bigcap_{i \geq n+1} B_d[x_i] = C_{n+1}. \end{aligned}$$

Let us define

$$C = \overline{\bigcup_{n \in \mathbb{N}} C_n}.$$

Since C_n increases with n,

$$C_n \subset C_{n+1} \subset C_{n+2} \subset ....,$$

it follows that C is a closed, convex and bounded subset of X. Now we have

$$T_f(C) = T_f\left(\overline{\bigcup_{n \in \mathbb{N}} C_n}\right) \subseteq \overline{T_f\left(\bigcup_{n \in \mathbb{N}} C_n\right)} = \overline{\bigcup_{n \in \mathbb{N}} T_f(C_n)} \subseteq \overline{\bigcup_{n \in \mathbb{N}} C_{n+1}} = C$$

giving $T_f : C \to C$ since T_f is continuous mapping.

Finally, applying the Sadovskii fixed point theorem to T_f and C, we obtain that T_f has a fixed point in C which proves the theorem.

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FG- COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED S^* METRIC SPACES

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ABSTRACT

This is a review paper based on a recent article on FG- coupled fixed points [17], in which the authors established FG- coupled fixed point theorems in partially ordered complete S^* metric space. The results were illustrated by suitable examples, too. An S^* metric is an n-tuple metric from n-product of a set to the non negative reals. The theorems in [17] generalizes the main results of Gnana Bhaskar and Lakshmikantham [5].

KEYWORDS

FG- Coupled Fixed Point, Mixed Monotone Property, Partially Ordered Set, S^* Metric

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1 INTRODUCTION

In 1906, Maurice Frechet introduced the concept of metric as a generalization of distance. He defined a metric on a set as a function from the bi-product of the set to the non-negative reals that satisfy certain axioms. Later, several authors generalized the concept of metrics by either changing the domain or co-domain of the metric function or by varying the properties of the metric function [3,4,8,11,15,16]. An n-tuple metric called S^* metric is the latest development in this direction. Since the existence of fixed points is depending on the function as well as on its domain, studies started on fixed point theory by considering those generalized metric spaces. Now a lot of fixed point and coupled fixed point results are available under different types of metric spaces [2,6,7,9,13,14]. In [1] Abdellaoui, M.A. and Dahmani, Z. introduced S^* metric and they have proved fixed point results in S^* metric spaces. But the same concept can be seen in [10], under a different name. In [10] Mujahid Abbas, Bashir Ali, and Yusuf I Suleiman coined the name A- metric for this concept, and they have proved common coupled fixed point theorems with an illustrative example.
Recently, the concept of FG- coupled fixed points was introduced as a generalization of the concept of coupled fixed points in [12]. Some of the famous coupled fixed point theorems are generalized to FG- coupled fixed point theorems in [12,18,19].
In [17], the authors established FG- coupled fixed point theorems in the setting of partially ordered complete S^* metric spaces. This is a review paper of [17].

Some useful definitions and results are as follows:

Definition 1. *[1, 10] An* ***S^* metric*** *on a nonempty set X is a function $S^* : X^n \to [0,\infty)$ satisfying:*

(i) $S^*(x_1, x_2, \cdots, x_n) \geq 0$,

(ii) $S^*(x_1, x_2, \cdots, x_n) = 0$ *if and only if* $x_1 = x_2 = \cdots = x_n$,

(iii) $S^*(x_1, x_2, \cdots, x_n) \leq S^*(x_1, \cdots, x_1, a) + S^*(x_2, \cdots, x_2, a) + \cdots + S^*(x_n, \cdots, x_n, a)$

for any $x_1, x_2, \cdots, x_n, a \in X$. The pair (X, S^*) is called ***S^* metric space****.*

Lemma 1. *[1, 10] Suppose that (X, S^*) is an S^* metric space. Then for all $x_1, x_2 \in X$, we have $S^*(x_1, x_1, \cdots, x_1, x_2) = S^*(x_2, x_2, \cdots, x_2, x_1)$*

Definition 2. *[1, 10] We say that the sequence $\{x_p\}_{p\in\mathbb{N}}$ of the space X is* ***convergent*** *to x if $S^*(x_p, x_p, \cdots, x_p, x) \to 0$ as $p \to \infty$. We write $\lim_{p\to\infty} x_p = x$*

Definition 3. *[1, 10] We say that the sequence $\{x_p\}_{p\in\mathbb{N}}$ of the space X is of* ***Cauchy*** *if for each $\epsilon > 0$, there exist $p_0 \in \mathbb{N}$ such that for any $p, q \geq p_0$,*
$S^*(x_p, \cdots, x_p, x_q) < \epsilon$

The space (X, S^*) is **complete** if all its Cauchy sequences are convergent.

Lemma 2. *[1, 10] Let (X, S^*) be an S^* metric space. If $\{x_p\}_{p\in\mathbb{N}}$ in X converges to x, then x is unique.*

Definition 4. *[12] Let X and Y be any two non-empty sets and $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two mappings. An element $(x, y) \in X \times Y$ is said to be an FG- coupled fixed point if $F(x, y) = x$ and $G(y, x) = y$.*

Definition 5. *[12] Let $(X, \preceq_{P_1})$ and $(Y, \preceq_{P_2})$ be two partially ordered sets and $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two mappings. We say that F and G have mixed monotone property if F and G are increasing in first variable and monotone decreasing second variable, i.e., if for all $(x, y) \in X \times Y$,*
$x_1, x_2 \in X, x_1 \preceq_{P_1} x_2$ implies $F(x_1, y) \preceq_{P_1} F(x_2, y)$ and $G(y, x_2) \preceq_{P_2} G(y, x_1)$ and
$y_1, y_2 \in Y, y_1 \preceq_{P_2} y_2$ implies $F(x, y_2) \preceq_{P_1} F(x, y_1)$ and $G(y_1, x) \preceq_{P_2} G(y_2, x)$.

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Note 1. *[12] Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two mappings, then for $n \geq 1$, $F^n(x, y) = F(F^{n-1}(x, y), G^{n-1}(y, x))$ and $G^n(y, x) = G(G^{n-1}(y, x), F^{n-1}(x, y))$, and $F^0(x, y) = x$ and $G^0(y, x) = y$ for all $x \in X$ and $y \in Y$.*

Note 2. Let $(X, \preceq_{P_1})$ and $(Y, \preceq_{P_2})$ be two partially ordered sets, then we define the partial order $\leq_{12}$ on $X \times Y$ and the partial order $\leq_{21}$ on $Y \times X$ as follows:
For all $x, u \in X$ and $y, v \in Y$

$$\begin{aligned} (x, y) \leq_{12} (u, v) \quad &\Leftrightarrow \quad x \preceq_{P_1} u \text{ and } v \preceq_{P_2} y \\ (y, x) \leq_{21} (v, u) \quad &\Leftrightarrow \quad y \preceq_{P_2} v \text{ and } u \preceq_{P_1} x \end{aligned}$$

2 Main Results in [17]

Mainly two FG-coupled fixed point theorems are discussed in [17], first one deals with the existence of FG-coupled fixed point and the second deals with both the existence and uniqueness of FG-coupled fixed point. They are as follow:

Theorem 1. *[17] Let $(X, S_x^*, \preceq_{P_1})$ and $(Y, S_y^*, \preceq_{P_2})$ be two partially ordered complete S^* metric spaces and $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two mappings with mixed monotone property and satisfy the following:*

$$\begin{aligned} &S_x^*\big(F(x, y), \cdots, F(x, y), F(u, v)\big) \\ &\quad \leq a_1 S_x^*(x, \cdots, x, u) + a_2 S_x^*\big(x, \cdots, x, F(x, y)\big) + a_3 S_x^*\big(x, \cdots, x, F(u, v)\big) \\ &\qquad + a_4 S_x^*\big(u, \cdots, u, F(x, y)\big) + a_5 S_x^*\big(u, \cdots, u, F(u, v)\big), \ \forall (x, y) \leq_{12} (u, v) \end{aligned} \tag{1}$$

and

$$\begin{aligned} &S_y^*\big(G(y, x), \cdots, G(y, x), G(v, u)\big) \\ &\quad \leq b_1 S_y^*(y, \cdots, y, v) + b_2 S_y^*\big(y, \cdots, y, G(y, x)\big) + b_3 S_y^*\big(y, \cdots, y, G(v, u)\big) \\ &\qquad + b_4 S_y^*\big(v, \cdots, v, G(y, x)\big) + b_5 S_y^*\big(v, \cdots, v, G(v, u)\big), \ \forall (y, x) \leq_{21} (v, u) \end{aligned} \tag{2}$$

for the non negative $a_i, b_i, i = 1, 2, 3, 4, 5$ with
$a_1 + a_2 + na_3 + a_5 < 1$, $b_1 + b_2 + nb_4 + b_5 < 1$, $a_3 + a_5 < 1$, $b_2 + b_4 < 1$.
Also suppose that either

(I) F and G are continuous or

(II) X and Y have the following properties:

(i) if $\{z_k\}$ is an increasing sequence in X with $z_k \to z$, then $z_k \preceq_{P_1} z$ for all $k \in \mathbb{N}$

(ii) if $\{w_k\}$ is a decreasing sequence in Y with $w_k \to w$, then $w \preceq_{P_2} w_k$ for all $k \in \mathbb{N}$.

If there exist $x_0 \in X$ and $y_0 \in Y$ with $(x_0, y_0) \leq_{12} \big(F(x_0, y_0), G(y_0, x_0)\big)$, then there exist an FG-coupled fixed point.

Proof. Given $x_0 \in X$ and $y_0 \in Y$ such that $(x_0, y_0) \leq_{12} \big(F(x_0, y_0), G(y_0, x_0)\big)$. If $(x_0, y_0) = \big(F(x_0, y_0), G(y_0, x_0)\big)$ then (x_0, y_0) is an FG- coupled fixed point.
Otherwise we have

$$(x_0, y_0) <_{12} \big(F(x_0, y_0), G(y_0, x_0)\big)$$

Then by the definition of the partial order on $X \times Y$ we have either
$x_0 \preceq_{P_1} F(x_0, y_0)$ and $G(y_0, x_0) \prec_{P_2} y_0$ **or** $x_0 \prec_{P_1} F(x_0, y_0)$ and $G(y_0, x_0) \preceq_{P_2} y_0$.
Without loss of generality we assume that

$$x_0 \preceq_{P_1} F(x_0, y_0) \ \text{ and } \ G(y_0, x_0) \prec_{P_2} y_0 \tag{3}$$

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Let $x_1 = F(x_0, y_0)$ and $y_1 = G(y_0, x_0)$.
By (3) we have,

$$x_0 \preceq_{P_1} x_1 \text{ and } y_1 \prec_{P_2} y_0$$

By the mixed monotone property of F and G we have

$$\begin{aligned} F(x_0, y_0) &\preceq_{P_1} F(x_1, y_0) \\ &\preceq_{P_1} F(x_1, y_1) \end{aligned} \tag{4}$$

and

$$\begin{aligned} G(y_1, x_1) &\preceq_{P_2} G(y_0, x_1) \\ &\preceq_{P_2} G(y_0, x_0) \end{aligned} \tag{5}$$

Let $x_2 = F(x_1, y_1)$ and $y_2 = G(y_1, x_1)$.
By (4) and (5) we have,

$$x_1 \preceq_{P_1} x_2 \text{ and } y_2 \preceq_{P_2} y_1$$

Continuing this process by using the mixed monotone property of F and G and by using the definition of partial order on $X \times Y$ we get sequences $\{x_m\}$ and $\{y_m\}$ in X and Y respectively as: for all $m \in \mathbb{N} \cup \{0\}$

$$x_{m+1} = F(x_m, y_m) \text{ and } y_{m+1} = G(y_m, x_m) \tag{6}$$

with the property that for all $m \in \mathbb{N} \cup \{0\}$

$$x_m \preceq_{P_1} x_{m+1} \text{ and } y_{m+1} \preceq_{P_2} y_m \tag{7}$$

That is by the definition of partial order on $X \times Y$ and $Y \times X$ we have,

$$(x_m, y_m) \leq_{12} (x_{m+1}, y_{m+1})$$

and

$$(y_m, x_m) \geq_{21} (y_{m+1}, x_{m+1})$$

Claim: For all $k \in \mathbb{N}$

$$S_x^*(x_k, \cdots, x_k, x_{k+1}) \leq \alpha^k \; S_x^*(x_0, \cdots, x_0, x_1) \tag{8}$$

and

$$S_y^*(y_k, \cdots, y_k, y_{k+1}) \leq \beta^k \; S_y^*(y_0, \cdots, y_0, y_1) \tag{9}$$

where

$$\alpha = \frac{a_1 + a_2 + (n-1)a_3}{1 - a_3 - a_5} < 1 \text{ and } \beta = \frac{b_1 + b_5 + (n-1)b_4}{1 - b_2 - b_4} < 1 \tag{10}$$

Now, we prove the claim by the method of mathematical induction.
When $k = 1$ we have,

$$\begin{aligned} &S_x^*(x_1, \cdots, x_1, x_2) \\ &= S_x^*\big(F(x_0, y_0), \cdots, F(x_0, y_0), F(x_1, y_1)\big) \\ &\leq a_1 S_x^*(x_0, \cdots, x_0, x_1) + a_2 S_x^*\big(x_0, \cdots, x_0, F(x_0, y_0)\big) + a_3 S_x^*\big(x_0, \cdots, x_0, F(x_1, y_1)\big) \\ &\qquad + a_4 S_x^*\big(x_1, \cdots, x_1, F(x_0, y_0)\big) + a_5 S_x^*\big(x_1, \cdots, x_1, F(x_1, y_1)\big) \\ &= a_1 S_x^*(x_0, \cdots, x_0, x_1) + a_2 S_x^*(x_0, \cdots, x_0, x_1) + a_3 S_x^*(x_0, \cdots, x_0, x_2) \\ &\qquad + a_5 S_x^*(x_1, \cdots, x_1, x_2) \\ &= (a_1 + a_2) S_x^*(x_0, \cdots, x_0, x_1) + a_3 S_x^*(x_0, \cdots, x_0, x_2) + a_5 S_x^*(x_1, \cdots, x_1, x_2) \\ &\leq (a_1 + a_2) S_x^*(x_0, \cdots, x_0, x_1) + a_3 \big[(n-1) S_x^*(x_0, \cdots, x_0, x_1) \\ &\qquad + S_x^*(x_2, \cdots, x_2, x_1)\big] + a_5 S_x^*(x_1, \cdots, x_1, x_2) \\ &= (a_1 + a_2) S_x^*(x_0, \cdots, x_0, x_1) + a_3 \big[(n-1) S_x^*(x_0, \cdots, x_0, x_1) \\ &\qquad + S_x^*(x_1, \cdots, x_1, x_2)\big] + a_5 S_x^*(x_1, \cdots, x_1, x_2) \\ &= (a_1 + a_2 + (n-1)a_3) \; S_x^*(x_0, \cdots, x_0, x_1) + (a_3 + a_5) S_x^*(x_1, \cdots, x_1, x_2) \end{aligned}$$

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which implies that

$$(1-a_3-a_5)S_x^*(x_1,\cdots,x_1,x_2) \;\leq\; (a_1+a_2+(n-1)a_3)\;S_x^*(x_0,\cdots,x_0,x_1)$$

That is,

$$S_x^*(x_1,\cdots,x_1,x_2) \;\leq\; \frac{a_1+a_2+(n-1)a_3}{1-a_3-a_5}\;S_x^*(x_0,\cdots,x_0,x_1)$$

Similarly we have,

$$S_y^*(y_1,\cdots,y_1,y_2) \;\leq\; (b_1+b_5+(n-1)b_4)\;S_y^*(y_0,\cdots,y_0,y_1)+(b_2+b_4)S_y^*(y_1,\cdots,y_1,y_2)$$

which implies that

$$(1-b_2-b_4)\;S_y^*(y_1,\cdots,y_1,y_2) \;\leq\; (b_1+b_5+(n-1)b_4)\;S_y^*(y_0,\cdots,y_0,y_1)$$

That is,

$$S_y^*(y_1,\cdots,y_1,y_2) \;\leq\; \frac{b_1+b_5+(n-1)b_4}{1-b_2-b_4}\;S_y^*(y_0,\cdots,y_0,y_1)$$

Thus the claim is true for $k=1$.
Now assume the claim for $k\leq m$ and check for $k=m+1$.
Consider,

$$\begin{aligned}
&S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&=S_x^*\big(F(x_m,y_m),\cdots,F(x_m,y_m),F(x_{m+1},y_{m+1})\big)\\
&\leq a_1S_x^*(x_m,\cdots,x_m,x_{m+1})+a_2S_x^*\big(x_m,\cdots,x_m,F(x_m,y_m)\big)\\
&\qquad+a_3S_x^*\big(x_m,\cdots,x_m,F(x_{m+1},y_{m+1})\big)+a_4S_x^*\big(x_{m+1},\cdots,x_{m+1},F(x_m,y_m)\big)\\
&\qquad\qquad+a_5S_x^*\big(x_{m+1},\cdots,x_{m+1},F(x_{m+1},y_{m+1})\big)\\
&=a_1S_x^*(x_m,\cdots,x_m,x_{m+1})+a_2S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\qquad+a_3S_x^*(x_m,\cdots,x_m,x_{m+2})+a_5S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&=(a_1+a_2)S_x^*(x_m,\cdots,x_m,x_{m+1})+a_3S_x^*(x_m,\cdots,x_m,x_{m+2})+a_5S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&\leq(a_1+a_2)S_x^*(x_m,\cdots,x_m,x_{m+1})+a_3\big[(n-1)S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\qquad+S_x^*(x_{m+2},\cdots,x_{m+2},x_{m+1})\big]+a_5S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&=(a_1+a_2)S_x^*(x_m,\cdots,x_m,x_{m+1})+a_3\big[(n-1)S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\qquad+S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\big]+a_5S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&=(a_1+a_2+(n-1)a_3)\;S_x^*(x_m,\cdots,x_m,x_{m+1})+(a_3+a_5)S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-a_3-a_5)S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\\
&\leq(a_1+a_2+(n-1)a_3)\;S_x^*(x_m,\cdots,x_m,x_{m+1})\\
&\leq(a_1+a_2+(n-1)a_3)\Big[\frac{a_1+a_2+(n-1)a_3}{1-a_3-a_5}\Big]^m S_x^*(x_0,\cdots,x_0,x_1)
\end{aligned}$$

Therefore,

$$S_x^*(x_{m+1},\cdots,x_{m+1},x_{m+2})\leq\Big[\frac{a_1+a_2+(n-1)a_3}{1-a_3-a_5}\Big]^{m+1}\;S_x^*(x_0,\cdots,x_0,x_1)$$

Similarly we have,

$$\begin{aligned}
S_y^*(y_{m+1},\cdots,y_{m+1},y_{m+2}) \;\leq\;& (b_1+b_5+b_4(n-1))S_y^*(y_m,\cdots,y_m,y_{m+1})\\
&+(b_2+b_4)S_y^*(y_{m+1},\cdots,y_{m+1},y_{m+2})
\end{aligned}$$

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which implies that

$$
\begin{aligned}
&(1-b_2-b_4)S_y^*(y_{m+1},\cdots,y_{m+1},y_{m+2})\\
&\leq (b_1+b_5+b_4(n-1))S_y^*(y_m,\cdots,y_m,y_{m+1})\\
&\leq (b_1+b_5+b_4(n-1))\Big[\frac{b_1+b_5+(n-1)b_4}{1-b_2-b_4}\Big]^m S_y^*(y_0,\cdots,y_0,y_1)
\end{aligned}
$$

Therefore,

$$
S_y^*(y_{m+1},\cdots,y_{m+1},y_{m+2}) \leq \Big[\frac{b_1+b_5+(n-1)b_4}{1-b_2-b_4}\Big]^{m+1} S_y^*(y_0,\cdots,y_0,y_1)
$$

Thus the claim is true for all $k \in \mathbb{N}$.
Next we prove that $\{x_m\}$ is a Cauchy sequence in X and $\{y_m\}$ is a Cauchy sequence in Y.
Let $p,q \in \mathbb{N}$ with $p<q$.
Consider,

$$
\begin{aligned}
&S_x^*(x_p,\cdots,x_p,x_q)\\
&\leq (n-1)S_x^*(x_p,\cdots,x_p,x_{p+1})+S_x^*(x_q,\cdots,x_q,x_{p+1})\\
&= (n-1)S_x^*(x_p,\cdots,x_p,x_{p+1})+S_x^*(x_{p+1},\cdots,x_{p+1},x_q)\\
&\leq (n-1)S_x^*(x_p,\cdots,x_p,x_{p+1})+(n-1)S_x^*(x_{p+1},\cdots,x_{p+1},x_{p+2})\\
&\qquad\qquad\qquad\qquad +S_x^*(x_q,\cdots,x_q,x_{p+2})\\
&= (n-1)\Big[S_x^*(x_p,\cdots,x_p,x_{p+1})+S_x^*(x_{p+1},\cdots,x_{p+1},x_{p+2})\Big]\\
&\qquad\qquad\qquad\qquad +S_x^*(x_{p+2},\cdots,x_{p+2},x_q)\\
&\qquad\qquad\qquad \cdot\ \cdot\ \cdot\\
&\qquad\qquad\qquad \cdot\ \cdot\ \cdot\\
&\qquad\qquad\qquad \cdot\ \cdot\ \cdot\\
&\leq (n-1)\Big[S_x^*(x_p,\cdots,x_p,x_{p+1})+S_x^*(x_{p+1},\cdots,x_{p+1},x_{p+2})\\
&\qquad +\cdots+S_x^*(x_{q-2},\cdots,x_{q-2},x_{q-1})\Big]+S_x^*(x_q,\cdots,x_q,x_{q-1})\\
&= (n-1)\sum_{i=p}^{q-2} S_x^*(x_i,\cdots,x_i,x_{i+1})+S_x^*(x_{q-1},\cdots,x_{q-1},x_q)\\
&\leq (n-1)\sum_{i=p}^{q-2} \alpha^i S_x^*(x_0,\cdots,x_0,x_1)+\alpha^{q-1}S_x^*(x_0,\cdots,x_0,x_1)
\end{aligned}
$$

$$
\begin{aligned}
&= (n-1)S_x^*(x_0,\cdots,x_0,x_1)\sum_{i=p}^{q-2} \alpha^i+\alpha^{q-1}S_x^*(x_0,\cdots,x_0,x_1)\\
&\leq (n-1)\frac{\alpha^p}{1-\alpha}S_x^*(x_0,\cdots,x_0,x_1)+\alpha^{q-1}S_x^*(x_0,\cdots,x_0,x_1)\\
&\to 0 \ as\ p,\ q\to\infty \text{ since } \alpha<1
\end{aligned}
$$

Thus, $\{x_m\}$ is a Cauchy sequence in X.

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Similarly we have,

$$\begin{aligned} S_y^*(y_p,\cdots,y_p,y_q) &\leq (n-1)\sum_{i=p}^{q-2} S_y^*(y_i,\cdots,y_i,y_{i+1}) + S_y^*(y_{q-1},\cdots,y_{q-1},y_q) \\ &\leq (n-1)\sum_{i=p}^{q-2}\beta^i\ S_y^*(y_0,\cdots,y_0,y_1)+\beta^{q-1}S_y^*(y_0,\cdots,y_0,y_1) \\ &= (n-1)\ S_y^*(y_0,\cdots,y_0,y_1)\sum_{i=p}^{q-2}\ \beta^i + \beta^{q-1}S_y^*(y_0,\cdots,y_0,y_1) \\ &\leq (n-1)\frac{\beta^p}{1-\beta}S_y^*(y_0,\cdots,y_0,y_1) + \beta^{q-1}S_y^*(y_0,\cdots,y_0,y_1) \\ &\to 0\ as\ p,\ q\to\infty \text{ since } \beta<1 \end{aligned}$$

Thus, $\{y_m\}$ is a Cauchy sequence in Y.
Since X and Y are complete S^* metric spaces, there exist $x\in X$ and $y\in Y$ such that

$$\lim_{p\to\infty} x_p = x \ \text{ and } \ \lim_{p\to\infty} y_p = y \tag{11}$$

Case (I): First assume that F and G are continuous.
Therefore by (6) and (11) we have,

$$x = \lim_{p\to\infty} x_{p+1} = \lim_{p\to\infty} F(x_p,y_p) = F(x,y)$$

and

$$y = \lim_{p\to\infty} y_{p+1} = \lim_{p\to\infty} G(y_p,x_p) = G(y,x)$$

That is $F(x,y)=x$ and $G(y,x)=y$.
Thus (x,y) is an FG- coupled fixed point.
Case (II): Suppose that X and Y have the properties (i) and (ii) respectively.
By (7) we have, $\{x_m\}$ is an increasing sequence in X and $\{y_m\}$ is a decreasing sequence in Y and by (11) we have $\lim_{m\to\infty} x_m = x$ and $\lim_{m\to\infty} y_m = y$
Therefore by the hypothesis we have for all $m\in\mathbb{N}$

$$x_m \preceq_{P_1} x \ \text{ and } \ y \preceq_{P_2} y_m$$

Therefore by the definition of partial order on $X\times Y$ and $Y\times X$ we have

$$(x_m,y_m) \leq_{12} (x,y) \ \text{ and } \ (y,x) \leq_{21} (y_m,x_m)$$

Consider,

$$\begin{aligned} &S_x^*(x,\cdots,x,F(x,y)) \\ &\leq (n-1)S_x^*(x,\cdots,x,x_{m+1})+S_x^*(F(x,y),\cdots,F(x,y),x_{m+1}) \\ &= (n-1)S_x^*(x,\cdots,x,x_{m+1}) + S_x^*(F(x,y),\cdots,F(x,y),F(x_m,y_m)) \\ &= (n-1)S_x^*(x,\cdots,x,x_{m+1})+S_x^*(F(x_m,y_m),\cdots,F(x_m,y_m),F(x,y)) \\ &\leq (n-1)S_x^*(x,\cdots,x,x_{m+1}) + a_1S_x^*(x_m,\cdots,x_m,x) + a_2S_x^*(x_m,\cdots,x_m,F(x_m,y_m)) \\ &\quad +a_3S_x^*(x_m,\cdots,x_m,F(x,y))+a_4S_x^*(x,\cdots,x,F(x_m,y_m))+a_5S_x^*(x,\cdots,x,F(x,y)) \\ &= (n-1)S_x^*(x,\cdots,x,x_{m+1}) + a_1S_x^*(x_m,\cdots,x_m,x) + a_2S_x^*(x_m,\cdots,x_m,x_{m+1}) \\ &\quad +a_3S_x^*(x_m,\cdots,x_m,F(x,y)) + a_4S_x^*(x,\cdots,x,x_{m+1}) + a_5S_x^*(x,\cdots,x,F(x,y)) \\ &\leq (n-1)S_x^*(x,\cdots,x,x_{m+1}) + a_1S_x^*(x_m,\cdots,x_m,x) + a_2S_x^*(x_m,\cdots,x_m,x_{m+1}) \\ &\quad +a_3\left[(n-1)S_x^*(x_m,\cdots,x_m,x) + S_x^*(F(x,y),\cdots,F(x,y),x)\right] \\ &\quad +a_4S_x^*(x,\cdots,x,x_{m+1}) + a_5S_x^*(x,\cdots,x,F(x,y)) \end{aligned}$$

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By taking the limit as $m \to \infty$ on both sides, and by using (11) and Lemma 1 we get

$$S^*_x\big(x, \cdots, x, F(x,y)\big) \leq (a_3 + a_5)\ S^*_x\big(x, \cdots, x, F(x,y)\big)$$

since $a_3 + a_5 < 1$ we get $S^*_x\big(x, \cdots, x, F(x,y)\big) = 0$
Thus we get

$$F(x,y) = x \tag{12}$$

Similarly,

$$\begin{aligned}
&S^*_y\big(y, \cdots, y, G(y,x)\big)\\
&\leq (n-1)S^*_y(y, \cdots, y, y_{m+1}) + S^*_y(G(y,x), \cdots, G(y,x), y_{m+1})\\
&\leq (n-1)S^*_y(y, \cdots, y, y_{m+1}) + S^*_y\big(G(y,x), \cdots, G(y,x), G(y_m, x_m)\big)\\
&\leq (n-1)S^*_y(y, \cdots, y, y_{m+1}) + b_1 S^*_y(y, \cdots, y, y_m) + b_2 S^*_y\big(y, \cdots, y, G(y,x)\big)\\
&\quad + b_3 S^*_y\big(y, \cdots, y, G(y_m, x_m)\big) + b_4 S^*_y\big(y_m, \cdots, y_m, G(y,x)\big)\\
&\qquad + b_5 S^*_y\big(y_m, \cdots, y_m, G(y_m, x_m)\big)\\
&\leq (n-1)S^*_y(y, \cdots, y, y_{m+1}) + b_1 S^*_y(y, \cdots, y, y_m) + b_2 S^*_y\big(y, \cdots, y, G(y,x)\big)\\
&\quad + b_3 S^*_y(y, \cdots, y, y_{m+1}) + b_4 S^*_y\big(y_m, \cdots, y_m, G(y,x)\big) + b_5 S^*_y(y_m, \cdots, y_m, y_{m+1})\\
&\leq (n-1)S^*_y(y, \cdots, y, y_{m+1}) + b_1 S^*_y(y, \cdots, y, y_m) + b_2 S^*_y\big(y, \cdots, y, G(y,x)\big)\\
&\quad + b_3 S^*_y(y, \cdots, y, y_{m+1}) + b_4\big[(n-1)S^*_y(y_m, \cdots, y_m, y) + S^*(G(y,x), \cdots, G(y,x), y)\big]\\
&\qquad + b_5 S^*_y(y_m, \cdots, y_m, y_{m+1})
\end{aligned}$$

By taking the limit as $m \to \infty$ on both sides, using (11) and Lemma 1 we get

$$S^*_y\big(y, \cdots, y, G(y,x)\big) \leq (b_2 + b_4)\ S^*_y\big(y, \cdots, y, G(y,x)\big)$$

since $b_2 + b_4 < 1$ we get $S^*_y\big(y, \cdots, y, G(y,x)\big) = 0$
Thus we get

$$G(y,x) = y \tag{13}$$

Therefore by (12) and (13), (x,y) is an FG- coupled fixed point.
Hence the proof.

By taking $n = 2$, $X = Y$, $F = G$, $a_2 = b_2 = k$, $a_5 = b_5 = l$ and the remaining $a_i, b_i = 0$, we get a coupled fixed fixed point theorem for Kannan type mapping. We give the result as a corollary as follows:

Corollary 1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $F : X \times X \to X$ be a mapping having the mixed monotone property on X satisfying:*

$$d\big(F(x,y), F(u,v)\big) \leq k\ d\big(x, F(x,y)\big) + l\ d\big(u, F(u,v)\big)\ \forall\ (x,y) \geq (u,v)$$

for non negative k, l with $k + l < 1$
Suppose that either

(I) F is continuous or

(II) X satisfy the following:

(i) if $\{x_k\}$ is an increasing sequence in X with $x_k \to x$, then $x_k \preceq x$ for all $k \in \mathbb{N}$

(ii) if $\{y_k\}$ is a decreasing sequence in X with $y_k \to y$, then $y \preceq y_k$ for all $k \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $(x_0, y_0) \leq \big(F(x_0, y_0), F(y_0, x_0)\big)$ then F has a coupled fixed point.

By taking $n = 2$, $X = Y$, $F = G$, $a_3 = b_3 = k$, $a_4 = b_4 = l$ and the remaining $a_i, b_i = 0$, we get a coupled fixed point theorem for Chatterjea type mapping. We give the result as a corollary as follows:

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Corollary 2. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $F : X \times X \to X$ be a mapping having the mixed monotone property on X satisfying:*

$$d\big(F(x,y), F(u,v)\big) \leq k\ d\big(x, F(u,v)\big) + l\ d\big(u, F(x,y)\big)\ \forall\ (x,y) \geq (u,v)$$

for $k, l \in [0, \frac{1}{2})$
Suppose that either

(I) F is continuous or

(II) X satisfy the following:

(i) if $\{x_k\}$ is an increasing sequence in X with $x_k \to x$, then $x_k \preceq x$ for all $k \in \mathbb{N}$

(ii) if $\{y_k\}$ is a decreasing sequence in X with $y_k \to y$, then $y \preceq y_k$ for all $k \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $(x_0, y_0) \leq \big(F(x_0, y_0), F(y_0, x_0)\big)$ then F has a coupled fixed point.

Remark 1. *By putting different values to the constants $a_i, b_i;\ i = 1,2,3,4,5$ which satisfy the conditions mentioned in Theorem 1 we get various FG- coupled fixed point theorems.*

Remark 2. *By varying the constants $a_i, b_i;\ i = 1,2,3,4,5$ which satisfy the conditions mentioned in Theorem 1 and by taking $X = Y$ and $F = G$ we get different coupled fixed point theorems.*

Example 1. *Let $X = [0,1]$ and $Y = [-1, 0]$*
Consider the metric S^* defined on both X and Y as

$$S^*(a_1, \cdots, a_n) = \sum_{i=1}^{n} \sum_{i<j} |a_i - a_j|$$

For $x, u \in X$, consider the partial order $\leq$ as $x \leq u \Leftrightarrow x = u$
and for $y, v \in Y$, define partial order $\leq$ as $y \leq v \Leftrightarrow y = v$.
Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be defined as

$$F(x,y) = \frac{x-y}{2} \quad \text{and} \quad G(y,x) = \frac{2y-x}{3}$$

As per the partial order defined on X and Y it can be easily verified that F and G are mixed monotone mappings and satisfy the conditions (1) and (2).
Here $\{(x, -x) : x \in [0,1]\}$ is the set of all FG- coupled fixed points.

Theorem 2. *[17] Let $(X, S_x^*, \preceq_{P_1})$ and $(Y, S_y^*, \preceq_{P_2})$ be two partially ordered complete S^* metric spaces and $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two mappings with mixed monotone property satisfying:*

$$S_x^*\big(F(x,y), \cdots, F(x,y), F(u,v)\big) \leq aS_x^*(x, \cdots, x, u) + bS_y^*(y, \cdots, y, v),\ \forall (x,y) \leq_{12} (u,v) \tag{14}$$

and

$$S_y^*\big(G(y,x), \cdots, G(y,x), G(v,u)\big) \leq aS_y^*(y, \cdots, y, v) + bS_x^*(x, \cdots, x, u),\ \forall (y,x) \leq_{21} (v,u) \tag{15}$$

for non negative a, b with $a + b < 1$. Also suppose that either

(I) F and G are continuous or

(II) X and Y have the following properties:

(i) if $\{z_k\}$ is an increasing sequence in X with $z_k \to z$, then $z_k \preceq_{P_1} z$ for all $k \in \mathbb{N}$

(ii) if $\{w_k\}$ is a decreasing sequence in Y with $w_k \to w$, then $w \preceq_{P_2} w_k$ for all $k \in \mathbb{N}$.

If there exist $x_0 \in X$ and $y_0 \in Y$ with $(x_0, y_0) \leq_{12} \big(F(x_0, y_0), G(y_0, x_0)\big)$, then there exist an FG- coupled fixed point in $X \times Y$.
Moreover unique FG- coupled fixed point exists if

(III) for every $(x, y), (x_1, y_1) \in X \times Y$ there exist a $(u, v) \in X \times Y$ that is comparable to both (x, y) and (x_1, y_1).

Proof. Following as in the prof of Theorem 1 we can construct an increasing sequence $\{x_m\}_{m\in\mathbb{N}}$ in X and a decreasing sequence $\{y_m\}_{m\in\mathbb{N}}$ in Y defined as:

$$x_{m+1} = F(x_m, y_m) \text{ and } y_{m+1} = G(y_m, x_m) \tag{16}$$

with the property that

$$(x_m, y_m) \leq_{12} (x_{m+1}, y_{m+1})$$

and

$$(y_m, x_m) \leq_{21} (y_{m+1}, x_{m+1})$$

By (16) we have

$$x_{m+1} = F(x_m, y_m) = F^{m+1}(x_0, y_0) \text{ and } y_{m+1} = G(y_m, x_m) = G^m(y_0, x_0) \tag{17}$$

Claim: For $p \in \mathbb{N}$,

$$S_x^*(x_p, \cdots, x_p, x_{p+1}) \leq (a+b)^p \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right] \tag{18}$$

and

$$S_y^*(y_p, \cdots, y_p, y_{p+1}) \leq (a+b)^p \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right] \tag{19}$$

Now we prove the claim by the method of mathematical induction.
When $p = 1$ we have,

$$\begin{aligned} S_x^*(x_1, \cdots, x_1, x_2) &= S_x^*\big(F(x_0, y_0), \cdots, F(x_0, y_0), F(x_1, y_1)\big) \\ &\leq a\ S_x^*(x_0, \cdots, x_0, x_1) + b\ S_y^*(y_0, \cdots, y_0, y_1) \\ &\leq (a+b)\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)] \end{aligned}$$

and

$$\begin{aligned} S_y^*(y_1, \cdots, y_1, y_2) &= S_y^*(y_2, \cdots, y_2, y_1) \\ &= S_y^*\big(G(y_1, x_1), \cdots, G(y_1, x_1), G(y_0, x_0)\big) \\ &\leq a\ S_y^*(y_1, \cdots, y_1, y_0) + b\ S_x^*(x_1, \cdots, x_1, x_0) \\ &= a\ S_y^*(y_0, \cdots, y_0, y_1) + b\ S_x^*(x_0, \cdots, x_0, x_1) \\ &\leq (a+b)\ \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right] \end{aligned}$$

Therefore the claim is true for $p = 1$.
Now assume the claim for $p \leq m$ and check for $p = m + 1$.
Consider,

$$\begin{aligned} S_x^*(x_{m+1}, \cdots, x_{m+1}, x_{m+2}) &= S_x^*\big(F(x_m, y_m), \cdots, F(x_m, y_m), F(x_{m+1}, y_{m+1})\big) \\ &\leq\ a\ S_x^*(x_m, \cdots, x_m, x_{m+1}) + b\ S_y^*(y_m, \cdots, y_m, y_{m+1}) \\ &\leq a\ (a+b)^m\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)] \\ &\quad +b\ (a+b)^m\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)] \\ &= (a+b)^{m+1}\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)] \end{aligned}$$

Similarly,

$$S_y^*(y_{m+1}, \cdots, y_{m+1}, y_{m+2}) \leq (a+b)^{m+1}\ [S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)]$$

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Thus the claim is true for all $p \in \mathbb{N}$.
Next we prove that $\{x_p\}_{p\in\mathbb{N}}$ and $\{y_p\}_{p\in\mathbb{N}}$ are Cauchy sequences in X and Y respectively.
Consider for $p, q \in \mathbb{N}$ with $p < q$,

$$\begin{aligned}
&S_x^*(x_p, \cdots, x_p, x_q)\\
&\leq (n-1)S_x^*(x_p, \cdots, x_p, x_{p+1}) + S_x^*(x_q, \cdots, x_q, x_{p+1})\\
&= (n-1)S_x^*(x_p, \cdots, x_p, x_{p+1}) + S_x^*(x_{p+1}, \cdots, x_{p+1}, x_q)\\
&\leq (n-1)S_x^*(x_p, \cdots, x_p, x_{p+1}) + (n-1)S_x^*(x_{p+1}, \cdots, x_{p+1}, x_{p+2})\\
&\qquad\qquad + S_x^*(x_q, \cdots, x_q, x_{p+2})\\
&= (n-1)\Big[S_x^*(x_p, \cdots, x_p, x_{p+1}) + S_x^*(x_{p+1}, \cdots, x_{p+1}, x_{p+2})\Big]\\
&\qquad\qquad + S_x^*(x_{p+2}, \cdots, x_{p+2}, x_q)\\
&\qquad\qquad \cdots\\
&\qquad\qquad \cdots\\
&\qquad\qquad \cdots\\
&\leq (n-1)\Big[S_x^*(x_p, \cdots, x_p, x_{p+1}) + S_x^*(x_{p+1}, \cdots, x_{p+1}, x_{p+2})\\
&\quad + \cdots + S_x^*(x_{q-2}, \cdots, x_{q-2}, x_{q-1})\Big] + S_x^*(x_q, \cdots, x_q, x_{q-1})\\
&= (n-1) \sum_{i=p}^{q-2} S_x^*(x_i, \cdots, x_i, x_{i+1}) + S_x^*(x_{q-1}, \cdots, x_{q-1}, x_q)\\
&\leq (n-1) \sum_{i=p}^{q-2} (a+b)^i \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right]\\
&\qquad\qquad + (a+b)^{q-1} \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right]
\end{aligned}$$

$$\begin{aligned}
&= (n-1) \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right] \sum_{i=p}^{q-2} (a+b)^i\\
&\quad + (a+b)^{q-1} \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right]\\
&\leq (n-1) \frac{(a+b)^p}{1-a-b} \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right]\\
&\quad + (a+b)^{q-1} \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right]\\
&\to 0 \ as \ p, q \to \infty, \quad \text{since } a+b < 1.
\end{aligned}$$

Thus, $\{x_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in X.
Similarly we have,

$$\begin{aligned}
S_y^*(y_p, \cdots, y_p, y_q) \quad &\leq \quad (n-1) \frac{(a+b)^p}{1-a-b} \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right]\\
&\quad + (a+b)^{q-1} \left[S_x^*(x_0, \cdots, x_0, x_1) + S_y^*(y_0, \cdots, y_0, y_1)\right]\\
&\to 0 \ as \ p, q \to \infty, \quad \text{since } a+b < 1.
\end{aligned}$$

Thus, $\{y_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in Y.
Since X and Y are complete S^* metric spaces, there exist $x \in X$ and $y \in Y$ such that

$$\lim_{p\to\infty} x_p = x \quad \text{and} \quad \lim_{p\to\infty} y_p = y \tag{20}$$

As in the Theorem 1, by assuming the continuity of F and G we can prove that $(x, y) \in X \times Y$ is an FG- coupled fixed point.
Now, suppose that X and Y have the properties (i) and (ii) respectively.
Since $\{x_m\}$ is increasing in X and $\{y_m\}$ is decreasing in Y and by using (20) we have $\forall\, m \in \mathbb{N}$ $x_m \preceq_{P_1} x$

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and $y_m \succeq_{P_2} y$
That is by the definition of partial order on $X \times Y$ and $Y \times X$ we have

$$(x_m, y_m) \leq_{12} (x, y) \text{ and } (y_m, x_m) \geq_{21} (y, x)$$

Now consider,

$$\begin{aligned}
&S_x^*(x, \cdots, x, F(x,y)) \\
&\leq (n-1)\ S_x^*(x, \cdots, x, F(x_p, y_p)) + S_x^*(F(x,y), \cdots, F(x,y), F(x_p, y_p)) \\
&= (n-1)\ S_x^*(x, \cdots, x, F(x_p, y_p)) + S_x^*(F(x_p, y_p), \cdots, F(x_p, y_p), F(x,y)) \\
&\leq (n-1)\ S_x^*(x, \cdots, x, x_{p+1}) + a\ S_x^*(x_p, \cdots, x_p, x) + b\ S_y^*(y_p, \cdots, y_p, y) \\
&\to 0\ as\ p \to \infty
\end{aligned}$$

Thus, $F(x, y) = x$.
Similarly,

$$\begin{aligned}
&S_y^*(y, \cdots, y, G(y,x)) \\
&\leq (n-1)\ S_y^*(y, \cdots, y, G(y_p, x_p)) + S_y^*(G(y,x), \cdots, G(y,x), G(y_p, x_p)) \\
&= (n-1)\ S_y^*(y, \cdots, y, y_{p+1}) + S_y^*(G(y,x), \cdots, G(y,x), G(y_p, x_p)) \\
&\leq (n-1)\ S_y^*(y, \cdots, y, y_{p+1}) + a\ S_y^*(y, \cdots, y, y_p) + b\ S_x^*(x, \cdots, x, x_p) \\
&\to 0\ as\ p \to \infty
\end{aligned}$$

Thus, $G(y, x) = y$.
That is, $F(x, y) = x$ and $G(y, x) = y$.
Hence (x, y) is an FG- coupled fixed point.

Next we prove the uniqueness of FG- coupled fixed point.
Claim: for any two points (x_1, y_1), $(x_2, y_2) \in X \times Y$ which are comparable and for all $k \in \mathbb{N}$

$$S_x^*(F^k(x_1, y_1), \cdots, F^k(x_1, y_1), F^k(x_2, y_2)) \leq (a+b)^k \left[S_x^*(x_1, \cdots, x_1, x_2) + S_y^*(y_1, \cdots, y_1, y_2)\right] \tag{21}$$

and

$$S_y^*(G^k(y_1, x_1), \cdots, G^k(y_1, x_1), G^k(y_2, x_2)) \leq (a+b)^k \left[S_x^*(x_1, \cdots, x_1, x_2) + S_y^*(y_1, \cdots, y_1, y_2)\right] \tag{22}$$

Without loss of generality assume that $(x_1, y_1) \leq_{12} (x_2, y_2)$.
That is by the definition of partial order we have $x_1 \preceq_{P_1} x_2$ and $y_2 \preceq_{P_2} y_1$
By the mixed monotone property of F and G we have

$$\begin{aligned}
F(x_1, y_1) &\preceq_{P_1} F(x_2, y_1) \\
&\preceq_{P_1} F(x_2, y_2)
\end{aligned}$$

and

$$\begin{aligned}
G(y_1, x_1) &\succeq_{p_2} G(y_2, x_1) \\
&\succeq_{p_2} G(y_2, x_2)
\end{aligned}$$

Again by the mixed monotone property of F and G we have

$$\begin{aligned}
F^2(x_1, y_1) &= F(F(x_1, y_1), G(y_1, x_1)) \\
&\preceq_{P_1} F(F(x_2, y_2), G(y_1, x_1)) \\
&\preceq_{P_1} F(F(x_2, y_2), G(y_2, x_2)) \\
&= F^2(x_2, y_2)
\end{aligned}$$

and

$$\begin{aligned} G^2(y_1,x_1) &= G\big(G(y_1,x_1),F(x_1,y_1)\big) \\ &\succeq_{P_2} G\big(G(y_2,x_2),F(x_1,y_1)\big) \\ &\succeq_{P_2} G\big(G(y_2,x_2),F(x_2,y_2)\big) \\ &= G^2(y_2,x_2) \end{aligned}$$

Continuing like this we get $\forall\ m \in \mathbb{N}\cup\{0\}$,

$$F^m(x_1,y_1) \preceq_{P_1} F^m(x_2,y_2) \text{ and } G^m(y_1,x_1) \succeq_{P_2} G^m(y_2,x_2)$$

That is by the definition of partial order on $X \times Y$ and $Y \times X$ we have $\forall\ m \in \mathbb{N}\cup\{0\}$

$$\big(F^m(x_1,y_1),G^m(y_1,x_1)\big) \leq_{12} \big(F^m(x_2,y_2),G^m(y_2,x_2)\big)$$

and

$$\big(G^m(y_1,x_1),F^m(x_1,y_1)\big) \geq_{21} \big(G^m(y_2,x_2),F^m(x_2,y_2)\big)$$

Now, we prove the claim by the method of mathematical induction.
When $k=1$ we have,

$$\begin{aligned} & S_x^*\big(F(x_1,y_1),\cdots,F(x_1,y_1),F(x_2,y_2)\big) \\ &\leq a\ S_x^*(x_1,\cdots,x_1,x_2)+b\ S_y^*(y_1,\cdots,y_1,y_2) \\ &\leq (a+b)[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)] \end{aligned}$$

and

$$\begin{aligned} & S_y^*\big(G(y_1,x_1),\cdots,G(y_1,x_1),G(y_2,x_2)\big) \\ &= S_y^*\big(G(y_2,x_2),\cdots,G(y_2,x_2),G(y_1,x_1)\big) \\ &\leq a\ S_y^*(y_2,\cdots,y_2,y_1)+b\ S_x^*(x_2,\cdots,x_2,x_1) \\ &\leq a\ S_y^*(y_1,\cdots,y_1,y_2)+b\ S_x^*(x_1,\cdots,x_1,x_2) \\ &\leq (a+b)[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)] \end{aligned}$$

Therefore claim is true for $k=1$.
Now assume the claim for $k \leq m$ and check for $k=m+1$.
Consider,

$$\begin{aligned} & S_x^*\big(F^{m+1}(x_1,y_1),\cdots,F^{m+1}(x_1,y_1),F^{m+1}(x_2,y_2)\big) \\ &= S_x^*\Big(F\big(F^m(x_1,y_1),G^m(y_1,x_1)\big),\cdots,F\big(F^m(x_1,y_1),G^m(y_1,x_1)\big),F\big(F^m(x_2,y_2),G^m(y_2,x_2)\big)\Big) \\ &\leq aS_x^*\big(F^m(x_1,y_1),\cdots,F^m(x_1,y_1),F^m(x_2,y_2)\big)+bS_y^*\big(G^m(y_1,x_1),\cdots,G^m(y_1,x_1),G^m(y_2,x_2)\big) \\ &\leq a\ (a+b)^m[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)] \\ &\quad +b\ (a+b)^m[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)] \\ &= (a+b)^{m+1}[S_x^*(x_1,\cdots,x_1,x_2)+S_y^*(y_1,\cdots,y_1,y_2)] \end{aligned}$$

Similarly we get,

$$\begin{aligned} S_y^*\big(G^{m+1}(y_1,x_1),\cdots,G^{m+1}(y_1,x_1),G^{m+1}(y_2,x_2)\big) &\leq (a+b)^{m+1}[S_x^*(x_1,\cdots,x_1,x_2) \\ &\quad +S_y^*(y_1,\cdots,y_1,y_2)] \end{aligned}$$

Thus the claim is true for all $k \in \mathbb{N}$.
Suppose that $(x,y),(x^*,y^*)$ be any two FG- coupled fixed points.
That is

$$F(x,y)=x \text{ and } G(y,x)=y \tag{23}$$

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and

$$F(x^*,y^*)=x^* \text{ and } G(y^*,x^*)=y^* \tag{24}$$

Case 1: If (x,y) and (x^*,y^*) are comparable, then

$$\begin{aligned} S_x^*(x,\cdots,x,x^*) &= S_x^*\big(F(x,y),\cdots,F(x,y),F(x^*,y^*)\big) \\ &\le a\ S_x^*(x,\cdots,x,x^*)+b\ S_y^*(y,\cdots,y,y^*) \end{aligned} \tag{25}$$

and

$$\begin{aligned} S_y^*(y,\cdots,y,y^*) &= S_y^*\big(G(y,x),\cdots,G(y,x),G(y^*,x^*)\big) \\ &\le a\ S_y^*(y,\cdots,y,y^*)+b\ S_x^*(x,\cdots,x,x^*) \end{aligned} \tag{26}$$

Adding (25) and (26) we get

$$S_x^*(x,\cdots,x,x^*)+S_y^*(y,\cdots,y,y^*)\le(a+b)\ [S_x^*(x,\cdots,x,x^*)+S_y^*(y,\cdots,y,y^*)]$$

which implies that $S_x^*(x,\cdots,x,x^*)+S_y^*(y,\cdots,y,y^*)=0$ since $a+b<1$.
Therefore we have, $x=x^*$ and $y=y^*$.
Case 2: Suppose (x,y) and (x^*,y^*) are not comparable.
Then by the hypothesis there exist $(u,v)\in X\times Y$ which is comparable to both (x,y) and (x^*,y^*).
Consider,

$$S_x^*(x,\cdots,x,x^*) \quad = \quad S_x^*\big(F^k(x,y),\cdots,F^k(x,y),F^k(x^*,y^*)\big)$$

$$\begin{aligned} &\le (n-1)\ S_x^*\big(F^k(x,y),\cdots,F^k(x,y),F^k(u,v)\big)+S_x^*\big(F^k(x^*,y^*),\cdots,F^k(x^*,y^*),F^k(u,v)\big) \\ &\le (n-1)\ (a+b)^k[S_x^*(x,\cdots,x,u)+S_y^*(y,\cdots,y,v)] \\ &\qquad +(a+b)^k[S_x^*(x^*,\cdots,x^*,u)+S_y^*(y^*,\cdots,y^*,v)] \\ &\to 0\ as\ k\to\infty \text{ since } a+b<1 \end{aligned}$$

Thus we have $x=x^*$.
Consider,

$$\begin{aligned} &S_y^*(y,\cdots,y,y^*) \\ &= S_y^*\big(G^k(y,x),\cdots,G^k(y,x),G^k(y^*,x^*)\big) \\ &\le (n-1)\ S_y^*\big(G^k(y,x),\cdots,G^k(y,x),G^k(v,u)\big)+S_y^*\big(G^k(y^*,x^*),\cdots,G^k(y^*,x^*),G^k(v,u)\big) \\ &\le (n-1)\ (a+b)^k[S_x^*(x,\cdots,x,u)+S_y^*(y,...,y,v)] \\ &\qquad +(a+b)^k[S_x^*(x^*,\cdots,x^*,u)+S_y^*(y^*,\cdots,y^*,v)] \\ &\to 0\ as\ k\to\infty \text{ since } a+b<1 \end{aligned}$$

Thus by Definition 1 (ii) we have $y=y^*$.
Therefore, $x=x^*$ and $y=y^*$
Hence the proof.

Remark 3. *By taking $n=2$, $a=b=\frac{k}{2}$, $X=Y$ and $F=G$ and assuming condition (I) and (II) of the above theorem we get the Theorems 2.1 and 2.2 of Bhaskar and Lakshmikantham [?] respectively as corollaries to our results.*

Remark 4. *By taking $n=2$, $a=b=\frac{k}{2}$, $X=Y$ and $F=G$ and assuming condition (I) and (III) of the above theorem we get the Theorems 2.4 of Bhaskar and Lakshmikantham [?] as a corollary to our results.*

We illustrate the above theorem with the following example.

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Example 2. *Let* $X = [0, \infty)$ *and* $Y = (-\infty, 0]$ *with the usual order in* $\mathbb{R}$
Consider the S^* *metric on both* X *and* Y *as*

$$S^*(a_1, \cdots, a_n) = \sum_{i=1}^{n} \sum_{i<j} |a_i - a_j|$$

Define $F : X \times Y \to X$ *and* $G : Y \times X \to Y$ *by*

$$F(x, y) = \frac{2x - 3y}{7n} \quad \text{and} \quad G(y, x) = \frac{2y - 3x}{7n}$$

For $x, u \in X$ *and* $y, v \in Y$ *with* $x \leq u$ *and* $y \geq v$ *we have*

$$\frac{2x - 3y}{7n} \leq \frac{2u - 3y}{7n}, \quad \frac{2y - 3x}{7n} \geq \frac{2y - 3u}{7n}$$

and

$$\frac{2x - 3y}{7n} \leq \frac{2x - 3v}{7n}, \quad \frac{2y - 3x}{7n} \geq \frac{2v - 3x}{7n}$$

That is,

$$F(x, y) \leq F(u, y), \ G(y, x) \geq G(y, u) \quad \text{and} \quad F(x, y) \leq F(x, v), \ G(y, x) \geq G(v, x)$$

Therefore F *and* G *are mixed monotone mappings.*
Next we show that F *and* G *satisfy the contractive type conditions (14) and (15)*

$$\begin{aligned} S^*(F(x, y), \cdots, F(x, y), F(u, v)) &= (n-1) \left| \frac{2x - 3y}{7n} - \frac{2u - 3v}{7n} \right| \\ &\leq \frac{2}{7n} (n-1)|x - u| + \frac{3}{7n} (n-1)|y - v| \\ &= \frac{2}{7n} S^*(x, \cdots, x, u) + \frac{3}{7n} S^*(y, \cdots, y, v) \end{aligned}$$

and

$$\begin{aligned} S^*(G(y, x), \cdots, G(y, x), G(v, u)) &= (n-1) \left| \frac{2y - 3x}{7n} - \frac{2v - 3u}{7n} \right| \\ &\leq \frac{2}{7n} (n-1)|y - v| + \frac{3}{7n} (n-1)|x - u| \\ &= \frac{2}{7n} S^*(y, \cdots, y, v) + \frac{3}{7n} S^*(x, \cdots, x, u) \end{aligned}$$

Therefore F *and* G *satisfy the contractive type conditions (14) and (15) for* $a = \frac{2}{7n}$ *and* $b = \frac{3}{7n}$.
Here $(0, 0)$ *is the unique* FG*- coupled fixed point.*

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THE LERAY-SCHAUDER PRINCIPLE IN GEODESIC SPACES

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ABSTRACT

The essential maps introduced by Granas in 1976 is a best tool for proving continuation results for compact maps. Several authors modified this idea to different scenarios. This is a review article, here, we consider the continuation results based on essential maps and the Leray-Schauder principle in the setting of uniquely geodesic spaces [20].

KEYWORDS

$\Gamma-$*uniquely geodesic spaces, Essential maps, Fixed point, Leray Schauder principle.*

1 INTRODUCTION

This paper is based on the notion of essential maps defined by Granas [9] in 1976. Many studies and extensions of this concept in variety of settings have been done by several authors [2,8,16–18]. Essential map techniques are one of the best tools to prove continuation results for compact maps [1].

Compact maps play a vital role in proving the existence and uniqueness of solutions of differential and integral equations. The classical Schauder fixed point theorem proved by Juliusz Schauder in 1930 in the setting of Banach spaces as a generalization of the celebrated Brouwer's fixed point theorem and the famous Leray-Schauder principle are milestones in the theory of fixed points and both of these incredible results are based on compact maps. For many years, researchers were trying to extend the concepts in normed spaces to more general spaces. In that direction, several authors have extended very many important fixed point theorems to various general spaces [3, 15]. Most importantly, the geodesic spaces grasped the attention of many fixed point theorists after the publication of the papers [11, 12] due to Kirk. A version of Schauder fixed point theorem has been proved by Niculescu and Roventa in $CAT(0)$ spaces [15] by assuming the compactness of convex hull of finite number of elements. Followed by this, in [3], Ariza, Li and Lopez have proved the Schauder Fixed point theorem in the setting of $\Gamma-$uniquely geodesic spaces, which includes Busemann spaces, Linear spaces, $CAT(\kappa)$ spaces with diameter less than $D_k/2$, Hyperbolic spaces [19] etc.

Granas introduced essential maps in order to prove continuation results for compact maps in Banach spaces [9]. In his paper, he has proved topological transversality principle and the Leray Schauder principle, using essential maps techniques. The Leray Schauder principle was first proved for compact mappings in Banach spaces and this has been broadly used to obtain fixed points of variety of mappings under different settings and many interesting contributions can be found in the literature [4, 6, 7, 13, 14]. In [9], Granas has also proved several fixed point results for compact maps using essential mappings and homotopical methods. Followed by the results established by Granas, Agarwal and O'Regan [2] have extended the concept of essential maps to a large class of mappings and established several fixed point theorems in 2000. The concept of essential maps has been further extended to $d-$essential maps, and $d-L$-essential maps [18].

In this review article, we consider the continuation results based on essential maps, and the Leray-Schauder principle in the setting of uniquely geodesic spaces.

2 Preliminaries

We recall some basic definitions and results used in this paper.

Definition 1. [5] *Let (X,ρ) be a metric space. A geodesic segment (or geodesic)from $x \in X$ to $y \in X$ is a map γ from a closed interval $[0,l] \subseteq \mathbb{R}$ to X such that $\gamma(0) = u$, $\gamma(l) = v$ and $\rho(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all t_1, t_2 in $[0,l]$.*
(X,ρ) is said to be a geodesic space if every two points in X are joined by a geodesic and (X,ρ) is uniquely geodesic if there is exactly one geodesic joining u to v, for all $u, v \in X$.

We denote the set of all geodesic segments in X by Ω. For $A \subseteq X$, closure and boundary of A in X is denoted by $\overline{A}$ and ∂A respectively.

Definition 2. [3] *Let (X,ρ) be a metric space and $\Gamma \subseteq \Omega$ be a family of geodesic segments. We say that (X,ρ) is a $\Gamma-$uniquely geodesic space if for every $x, y \in X$, there exists a unique geodesic in Γ passes through x and y. We denote a unique geodesic segment in Γ joining x and y by $\gamma_{x,y}$.*

Remark 1. [3] *Let (X,ρ) be a $\Gamma-$uniquely geodesic space. Then, the family Γ induces a unique mapping $\bigoplus_\Gamma : X^2 \times [0,1] \to X$ such that $\bigoplus_\Gamma(x, y, \tau) \in \gamma_{x,y}$ and the following properties hold for each $u, v \in X$:*

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1. $\rho(\bigoplus_\Gamma(x,y,\tau),\bigoplus_\Gamma(u,v,s)) = |\tau - s|\rho(u,v)$ *for all* $\tau, s \in [0,1]$
2. $\bigoplus_\Gamma(u,v,0) = u$ *and* $\bigoplus_\Gamma(u,v,1) = v$

Also, it is enough to consider $\bigoplus_\Gamma(u,v,\tau) = \gamma_{u,v}(\tau\rho(u,v))$, *i.e., it is a point on* γ *at a distance* $\tau\rho(u,v)$ *from* u *and we denote it by* $(1-\tau)u \oplus \tau v$.

Definition 3. [1] *Let* X *and* Y *be two metric spaces. A map* $\zeta : X \to Y$ *is called compact if* $\zeta(X)$ *is contained in a compact subset of* Y.

Definition 4. [1] *Let* T *be a closed convex subset of a Banach space* X *and* S *be a closed subset of* T. *Denote set of all continuous, compact maps* $\zeta : S \to T$ *by* $\mathscr{K}(S,T)$ *and set of all maps* $\zeta \in \mathscr{K}(S,T)$ *with* $x \neq \zeta(x)$ *by* $\mathscr{K}_{\partial S}(S,T)$ *for* $x \in \partial S$.
A map $\zeta \in \mathscr{K}_{\partial S}(S,T)$ *is essential in* $\mathscr{K}_{\partial S}(S,T)$ *if for every map* $\xi \in \mathscr{K}_{\partial S}(S,T)$ *with* $\zeta|_{\partial S} = \xi|_{\partial S}$ *there exists* $x \in intS$ *with* $x = \xi(x)$. *Otherwise* ζ *is inessential in* $\mathscr{K}_{\partial S}(S,T)$, *that is, there exists a fixed point free* $\xi \in \mathscr{K}_{\partial S}(S,T)$ *with* $\zeta|_{\partial S} = \xi|_{\partial S}$.

Definition 5. [1] *Two maps* $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$ *are homotopic in* $\mathscr{K}_{\partial S}(S,T)$, *written* $\zeta \simeq \xi$ *in* $\mathscr{K}_{\partial S}(S,T)$, *if there exists a continuous, compact mapping* $H : S \times [0,1] \to T$ *such that* $H_t(x) := H(\cdot,\tau) : S \to T$ *belongs to* $\mathscr{K}_{\partial S}(S,T)$ *for each* $\tau \in [0,1]$ *with* $H_0 = \zeta$ *and* $H_1 = \xi$.

Definition 6. [3] *A* $\Gamma-$*uniquely geodesic space is said to have property (Q) if*

$$\lim_{\varepsilon\to 0} \sup\{\rho((1-\tau)x \oplus \tau y, (1-\tau)x \oplus \tau z) : \tau \in [0,1], \quad x,y,z \in X, \quad \rho(y,z) \leq \varepsilon\} = 0.$$

Definition 7. [3] *Let* S *be a nonempty subset of a* $\Gamma-$*uniquely geodesic space. We say that* S *is* $\Gamma-$*convex if* $\gamma_{x,y} \subseteq S$ *for all* $x, y \in S$

Remark 2. [3] *Let* $(X, \|.\|)$ *be a normed linear space,* Γ_L *the family of linear segments and let* ρ *denote the metric induced by the norm* $\|.\|$. *Then* (X,ρ) *is a* Γ_L*-uniquely geodesic space with property (Q).*

Definition 8. [3] *A metric space* (X,ρ) *is a hyperbolic space (in the sense of Reich-Shafir* [19]*) if* X *is* $\Gamma-$ *uniquely geodesic and the following inequality holds*

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \leq \frac{1}{2}\rho(y,z).$$

Many authors have developed various extensions, and modifications to Schauder's fixed point theorem [10, 15]. In [3], Schauder-type fixed point theorem in the setting of geodesic spaces with the property (Q) is proved as follows:

Theorem 1. [3] *Let* (X,ρ) *be a* $\Gamma-$ *uniquely geodesic space with property (Q), and all balls are* Γ*-convex. Let* K *be a nonempty, closed,* $\Gamma-$*convex subset of* (X,ρ). *Then, any continuous mapping* $T : K \to K$ *with compact range* $\overline{T(K)}$ *has at least one fixed point in* K.

Throughout this paper, we denote a closed $\Gamma-$convex subset of the geodesic space X by T and interior of S by int S, where S is a closed subset of T. We denote the set of all continuous, compact maps $\zeta : S \to T$ by $\mathscr{K}(S,T)$ and set of all maps $\zeta \in \mathscr{K}(S,T)$ with $x \neq \zeta(x)$ by $\mathscr{K}_{\partial S}(S,T)$ for $x \in \partial S$.

3 RESULTS

Theorem 2. [20] *Let* (X,ρ) *be a* Γ*-uniquely geodesic space satisfying property (Q) and* $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$. *Suppose that for all* $(u,\tau) \in \partial S \times [0,1]$,

$$u \neq (1-\tau)\zeta(u) \oplus \tau\xi(u)$$

i.e., geodesic segment joining $\zeta(u)$ *and* $\xi(u)$ *does not contain* u. *Then* $\zeta \simeq \xi$ *in* $\mathscr{K}_{\partial S}(S,T)$.

Proof. Let $H(u,\tau) = H_t(u) = \bigoplus_\Gamma(\zeta(u), \xi(u), \tau) = (1-\tau)u \oplus \tau\xi u$. Clearly, H is continuous. We show that $H : S \times [0,1] \to T$ is a compact map.
Let $\{x_i\}$ be a sequence in S. Since $\zeta, \xi : S \to T$ are compact maps, $\zeta(x_i) \to u$ and $\xi(x_i) \to v$ as $i \to \infty$ for some subsequence $\mathbb{S}$ of natural numbers and $u, v \in T$. Let $\tau \in [0,1]$ be such that τ is the limit of some sequence $\tau_i \in [0,1]$. Now using Remark 1,

$$\begin{aligned} \rho\big(H(x_i,\tau_i),(1-\tau)u \oplus \tau v\big) &= \rho\big(\bigoplus_\Gamma(\zeta(x_i),\xi(x_i),\tau_i),(1-\tau)u \oplus \tau v\big) \\ &\leq \rho\big((1-\tau_i)\zeta(x_i) \oplus \tau_i\xi(x_i),(1-\tau)\zeta(x_i) \oplus \tau\xi(x_i)\big) \\ &\quad +\rho\big((1-\tau)\zeta(x_i) \oplus \tau\xi(x_i),(1-\tau)u \oplus \tau v\big) \\ &\leq |\tau_i-\tau|\rho(\zeta(x_i),\xi(x_i)) \\ &\quad +\rho((1-\tau)\zeta(x_i) \oplus \tau\xi(x_i),(1-\tau)\zeta(x_i) \oplus \tau v) \\ &\quad +\rho((1-\tau)\zeta(x_i) \oplus \tau v,(1-\tau)u \oplus \tau v). \end{aligned} \tag{1}$$

Since $\tau_i \to \tau$, we get

$$|\tau_i-\tau|\rho(\zeta(x_i),\xi(x_i)) \to 0. \tag{2}$$

Since $\xi(x_i) \to v$, we have $\rho(\xi(x_i), v) \leq \epsilon$ for every $\epsilon > 0$. Thus using property (Q), we get,

$$\rho\big((1-\tau)\zeta(x_i) \oplus \tau\xi(x_i),(1-\tau)\zeta(x_i) \oplus \tau v\big) \to 0. \tag{3}$$

Similarly,

$$\rho\big((1-\tau)\zeta(x_i) \oplus \tau v,(1-\tau)u \oplus \tau v\big) \to 0. \tag{4}$$

Using (2), (3) and (4) in (1), we have

$$\{H(x_i,\tau_i)\} \to (1-\tau)u \oplus \tau v, \quad \text{for } t_i \in [0,1]. \tag{5}$$

Since T is $\Gamma-$convex, $(1-\tau)u \oplus \tau v \in T$. Hence H is compact.
But it is given that $u \neq (1-\tau)\zeta(u) \oplus \tau\xi(u)$ for $(u,\tau) \in \partial S \times [0,1]$. Hence $H_\tau(u) \in \mathscr{K}_{\partial S}(S,T)$, $H(u,0) = \zeta(u)$ and $H(u,1) = \xi(u)$. Therefore $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$.

Following result in [1] is a corollary to our theorem.

Corollary 1. [1, Theorem 6.1] *Let X be a Banach space, T a closed, convex subset of X, S a closed subset of T and $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$. Suppose that for all $(u,\lambda) \in \partial S \times [0,1]$,*

$$u \neq (1-\lambda)\zeta(u) + \lambda\xi(u).$$

Then $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$.

Proof. Using Remark 2, X is a Γ_L-uniquely geodesic space. Hence by Theorem 2, the result follows.

Next result is a characterization of inessential maps in $\mathscr{K}_{\partial S}(S,T)$ in $\Gamma-$uniquely geodesic spaces.

Theorem 3. [20] *Let (X,ρ) be a Γ-uniquely geodesic space satisfying property (Q) and let $\zeta \in \mathscr{K}_{\partial S}(S,T)$. Then ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$ if and only if there exists $\xi \in \mathscr{K}_{\partial S}(S,T)$ with $\xi(u) \neq u$ for all $u \in S$ and $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$.*

Proof. Assume that ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$. Hence there exists a map $\xi \in \mathscr{K}_{\partial S}(S,T)$ such that $\xi(u) \neq u$ for all $u \in S$ and $\zeta|_{\partial S} = \xi|_{\partial S}$ by definition. Suppose there exists $(u,\tau) \in \partial S \times [0,1]$ such that $u = (1-\tau)\zeta(u) \oplus \tau\xi(u)$. Since $\zeta|_{\partial S} = \xi|_{\partial S}$, it follows that $u = \xi(u)$, which is a contradiction to the fact that $\xi \in \mathscr{K}_{\partial S}(S,T)$. Hence $u \neq (1-\tau)\zeta(u) \oplus \tau\xi(u)$, for each $(u,\tau) \in \partial S \times [0,1]$. Hence by using Theorem 2, we have $\zeta \simeq \xi$. Thus we have $\xi u \neq u$ and $\zeta \simeq \xi$.
Conversely assume that there exists $\xi \in \mathscr{K}_{\partial S}(S,T)$ with $\xi(u) \neq u$ for all $u \in S$ and $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$. Let $H : S \times [0,1] \to T$ with $H_t \in \mathscr{K}_{\partial S}(S,T)$ for all $\tau \in [0,1]$ be a continuous compact map with $H_0 = \zeta$ and $H_1 = \xi$. Consider

$$M = \Big\{u \in S : u = H(u,\tau) \text{ for some } \tau \in [0,1]\Big\}.$$

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There arise two cases.

Case-1: $M = \emptyset$.
If $M = \emptyset$ then $H_t(u) \neq u$ for all $u \in S$ and $\tau \in [0,1]$. In particular, $\zeta(u) = H_0(u) \neq u$ for all $u \in S$. Hence the inessentiality of ζ in $\mathscr{K}_{\partial S}(S,T)$ follows.

Case-2: $M \neq \emptyset$.
Let $\{x_i\} \in M$ such that $\{x_i\} \to u$. Then $x_i = H(x_i, \tau_i)$. Using the continuity of H, we get $u = H(u,\tau)$. Thus M is a closed subset of S.
Now, suppose $M \bigcap \partial S \neq \emptyset$. If $u \in M \bigcap \partial S$, then $u \in M$ and $u \in \partial S$, which implies $u = H_t(u)$ for $u \in \partial S$, a contradiction since $H_t \in \mathscr{K}_{\partial S}(S,T)$. Hence by the Urysohn's lemma, there exist $\xi : S \to [0,1]$ continuous, with $\xi(M) = 1$ and $\xi(\partial S) = 0$.
Define $f : S \to T$ by $f(u) = H(u, \xi(u))$. Clearly, f is a continuous compact map. If $u \in \partial S$, $f(u) = H(u,0) = \zeta(u)$. Thus $f|_{\partial S} = \zeta|_{\partial S}$.
If $u = f(u)$ for some $u \in S$ then, $u = f(u) = H(u, \xi(u))$. Thus $u \in M$ and hence $\xi(u) = 1$. Hence $u = f(u) = H(u,\xi(u)) = H(u,1) = \xi(u)$. Thus $f \in \mathscr{K}_{\partial S}(S,T)$ with $u = f(u)$ with $f|_{\partial S} = \zeta|_{\partial S}$. Therefore ξ has a fixed point, which is a contradiction to our assumption. Thus $u \neq f(u)$. Hence ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$, by Definition 4. Hence the proof.

As a consequence of Theorem 3, we obtain the following corollary.

Corollary 2. *Let X be a Hyperbolic space (in the sense of [19]), T a closed, convex subset of X, S a closed subset of T and $\zeta \in \mathscr{K}_{\partial S}(S,T)$. Then the following are equivalent:*

(i) ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$.

(ii) There exists $\xi \in \mathscr{K}_{\partial S}(S,T)$ with $\xi(u) \neq u$ for all $u \in S$ and $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$.

Proof. Every Hyperbolic space is a Γ-uniquely geodesic space with property (Q). Hence the result follows from Theorem 3.

Theorem 4. *[20] Let (X,ρ) be a Γ-uniquely geodesic space satisfying property (Q). Suppose that $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$ with $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$. Then ζ is essential in $\mathscr{K}_{\partial S}(S,T)$ if and only if ξ is essential in $\mathscr{K}_{\partial S}(S,T)$.*

Proof. Suppose ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$. Then from Theorem 3, there exists $T \in \mathscr{K}_{\partial S}(S,T)$ with $\zeta \simeq T$ in $\mathscr{K}_{\partial S}(S,T)$ such that $T(u) \neq u$ for all $u \in S$. Hence, $\xi \simeq T$ in $\mathscr{K}_{\partial S}(S,T)$. Thus $\zeta \simeq \xi$ and $\xi \simeq T$ implies $\xi \simeq T$ in $\mathscr{K}_{\partial S}(S,T)$. Therefore by Theorem 3, ξ is inessential in $\mathscr{K}_{\partial S}(S,T)$. Hence the proof.

Corollary 3. *[20] Let X be a Hyperbolic space (in the sense of [19]), S be a closed, convex subset of X, S be a closed subset of T and $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$ with $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$. Then ζ is essential in $\mathscr{K}_{\partial S}(S,T)$ if and only if ξ is essential in $\mathscr{K}_{\partial S}(S,T)$.*

The above corollary is a special case of Theorem 4.

Theorem 5. *[20] Let (X,ρ) be a Γ-uniquely geodesic space with all balls are $\Gamma-$convex and satisfies property (Q). Let $w \in \operatorname{int} S$. Then the map $\zeta(S) = w$ is essential in $\mathscr{K}_{\partial S}(S,T)$.*

Proof. Consider the continuous compact map $\xi : S \to T$ which agrees with ζ on ∂S. It is enough to show that $\xi(u) = u$ for some $u \in \operatorname{int} S$. Let $\zeta : T \to T$ be given by

$$\zeta(u) = \begin{cases} \xi(u), & u \in S; \\ w, & u \in T \setminus S. \end{cases}$$

Then ζ is continuous and compact (since ξ and w are continuous and compact and on ∂S, $\zeta(u) = \xi(u) = w$). Thus by Theorem 1, $\zeta(u) = u$ for some $u \in S$. If $u \in T \setminus S$, we get $\zeta(u) = w$. Hence

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$\zeta(w) = w$, which is a contradiction, since $w \in \text{int}S$. Clearly $u \in \text{int } S$. Hence x is a fixed point of ξ. Therefore ζ is essential in $\mathscr{K}_{\partial S}(S, S)$.

The following result in [1] is a corollary to our theorem.

Corollary 4. [1, Theorem 6.5] *Let X be a Banach space, T a closed, convex subset of X, S a closed subset of T and $u \in$ int S. Then the constant map $\zeta(S) = w$ is essential in $\mathscr{K}_{\partial S}(S, T)$.*

Proof. We know that all balls in Banach spaces are convex. Hence the result follows from theorem 5.

As a consequence of the above theorems, authors proved the Leray Schauder principle in $\Gamma-$uniquely geodesic spaces in [20], which generalizes the Leray Schauder principle in Hyperbolic spaces proved in [3].

Theorem 6. [20] *Let (X, ρ) be a Γ-uniquely geodesic space satisfying property (Q), and all balls are $\Gamma-$convex. Suppose that $\zeta : S \to T$ is a continuous compact map. Then either*

(i) ζ has a fixed point in S, or

(ii) There exists $(x_0, \tau) \in \partial S \times [0, 1]$ such that $x_0 = (1 - \tau)u \oplus \tau\zeta(x_0)$.

Proof. Suppose that (ii) does not hold and $\zeta(u) \neq u$ for all $u \in \partial S$. Define $\xi : S \to T$ by $\xi(u) = w$ for all $u \in S$. Consider the map $H : S \times [0, 1] \to T$ defined by

$$H(u, \tau) := (1 - \tau)w \oplus \tau\zeta(u),$$

which is continuous and compact. Also, for all $u \in \partial S$, $H_t(u) = (1 - \tau)w \oplus \tau\zeta(u) \neq u$ for a fixed τ (since we assumed that condition (ii) does not hold). Hence by Theorem 2, $\zeta \simeq w$ in $\mathscr{K}_{\partial S}(S, T)$. But we know that w is essential in $\mathscr{K}_{\partial S}(S, T)$ by theorem 5. Hence ζ is essential in $\mathscr{K}_{\partial S}(S, T)$ by Theorem 4. Now using Definition 4, $\zeta(u) = u$ for some $u \in$ int S.

The following Theorem is a consequence of Theorem 6.

Corollary 5. [3, Theorem 23] *Let X be a Hyperbolic space(in the sense of [19]), $x_0 \in X$ and $r > 0$. Suppose that $T : B[x_0, r] \to X$ be a continuous mapping with $\overline{T(B[x_0, r])}$ compact. Then either*

(i) T has at least one fixed point in $B[x_0, r]$, or

(ii) There exists $(u, \lambda) \in \partial B[x_0, r] \times [0, 1]$ with $u = (1 - \lambda)x_0 \oplus \lambda T(u)$.

Proof. Hyperbolic spaces are geodesic spaces with property (Q), and balls are Γ-convex. Hence the result follows from the above theorem.

4 CONCLUSIONS

In this review article, we considered continuation results in Γ-uniquely geodesic spaces and the Leray-Schauder principle in $\Gamma-$uniquely geodesic spaces proved in [20]. Geodesic spaces having the property (Q) include Busemann spaces, linear spaces, $CAT(\kappa)$ spaces with diameters smaller than $D_k/2$, hyperbolic spaces (in the sense of [19], etc., and balls are Γ-convex in these spaces. All the results in this paper are thus applicable to these spaces as well. One can attempt to develop similar results for multivalued mappings in geodesic spaces and establish similar results for d-essential maps, d-L essential maps, and other general classes of maps [2, 18].

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P-BIHARMONIC PSEUDO-PARABOLIC EQUATION WITH LOGARITHMIC NON LINEARITY

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ABSTRACT

This paper deals with the existence of solutions of a p-biharmonic pseudo parabolic partial differential equation with logarithmic nonlinearity in a bounded domain. We prove the global existence of the weak solutions using the Faedo-Galerkin method and applying the concavity approach, that the solutions blow up at a finite time. Further, we provide a maximal limit for the blow-up time.

KEYWORDS

p-Biharmonic, pseudo-parabolic, global existence, blow up

1 INTRODUCTION

Here we examine the following problem for a p-biharmonic pseudo-parabolic equation with logarithmic nonlinearity.

$$\begin{cases} u_t - \Delta u_t + \Delta(|\Delta u|^{p-2}\Delta u) - \text{div}(|\nabla u|^{q-2}\nabla u) = -\text{div}(|\nabla u|^{q-2}\nabla u \log|\nabla u|) & \text{if } (x,t) \in \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{if } (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{if } x \in \Omega. \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ represents a bounded domain whose boundary $\partial\Omega$ is smooth enough, $T \in (0,\infty)$, ν indicates the normal vector on $\partial\Omega$ pointing outward, $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}$ and the condition $2 < p < q < p(1 + \frac{2}{N+2})$ holds for p and q.

Pseudo-parabolic equations address several significant physical processes, like the evolution of two components of intergalactic material, the leakage of homogeneous fluids through a rock surface, the biomathematical modeling of a bacterial film, some thin film problems, the straight transmission of nonlinear, dispersive, long waves, the heat transfer containing two temperatures, a grouping of populations, etc. Shawalter and Ting [18], [22] first examined the pseudo parabolic equations in 1969. After their precursory results, there are many papers studied the nonlinear pseudo-parabolic equations, like semilinear pseudo-parabolic equations, quasilinear pseudo-parabolic equations, and even singular and degenerate pseudo-parabolic equations (see [1], [28], [4], [6], [15], [24], [25]). A pseudo-parabolic equation with p-Laplacian $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ and logarithmic nonlinearity were studied by Nahn, and Truong [13] in 2017. Considering the equation,

$$u_t - \Delta u_t - \Delta_p u = |u|^{p-2}u \log|u|$$

and by using the potential well method proposed by Sattinger [17] and a logarithmic Sobolev inequality, they proved the existence or nonexistence of global weak solutions. Additionally, they provided requirements for both the large time decay of weak global solutions and the finite time blow-up of weak solutions. Later, many authors [26], [27], [23] considered pseudo-parabolic equations with logarithmic nonlinearity and established results for local and global existence, uniqueness, decay estimate and asymptotic behaviour of solutions, blow-up results. Logarithmic nonlinearities in parabolic and pseudo-parabolic equations were studied by Lakshmipriya et.al [11], [10] and other researchers [29], [9], [5] and they proved the existence of weak solutions and their blow up in finite time. Lower bound of Blow-up time to a fourth order parabolic equation modelling epitaxial thin film growth

Recently, higher-order equations have gained much importance in studies. Lower bound of Blow-up time to a fourth order parabolic equation modelling epitaxial thin film growth studied by Liu et.al [3]. The p-biharmonic equation

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) + \lambda|u|^{p-2}u = 0$$

were studied by Liu and Guo [14], and by using the discrete-time method and uniform estimates, they established the existence and uniqueness of weak solutions. Hao and Zhou [7] obtained results for blow up, extinction and non-extinction of solutions for the equation

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) = |u|^q - \frac{1}{|\Omega|}\int_\Omega |u|dx.$$

Wang and Liu [8] studied the p-biharmonic parabolic equation with logarithmic nonlinearity,

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) = |u|^{q-1}u \log|u|$$

for $2 < p < q < p(1 + \frac{4}{n})$ and proved the global existence, blow up, extinction and no extinction of solutions. Then Liu and Li [2] studied,

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{q-1}u \log|u|.$$

Based on the difference and variation methods, they showed the existence of weak solutions and observed large-time behaviour and the transmission of solution perturbations for $\lambda > 0, p > q > \frac{p}{2} + 1, p > \frac{n}{2}$.

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Comert and Piskin [?] studied a p-biharmonic pseudo-parabolic equation with logarithmic nonlinearity and used the potential well method and logarithmic Sobolev inequality obtained the existence of the unique global weak solution. In addition, they also exhibited polynomial decay of solutions. Motivated by these works, we have formulated our problem (1) for a p-biharmonic pseudo-parabolic equation with logarithmic nonlinearity and studied their existence and non-existence. The problem (1) for the case $p = 2$ is already investigated and proved the existence, uniqueness and blow up of solutions (see [19], [20], [21]).

The rest of this paper is arranged to the two sections below. The preliminary notations, definitions, and results we need to support our main findings are described in Section 2. Section 3 contains the major findings of this paper explained in five theorems.

2 PRELIMINARIES

In this section, we provide some fundamental ideas and facts that are necessary for us to explain our findings. In this article, we follow the notations listed below throughout. $\|.\|_r$ denotes the $L^r(\Omega)$ norm for $1 \leq r \leq \infty$, $\|.\|_{H_0^1}$ denotes the norm in $H_0^1(\Omega)$, $(.,.)_1$ denotes the $H_0^1(\Omega)$-inner product.=, r' denotes the Holder conjugate exponent of $r > 1$ (that is, $r' = \frac{r}{r-1}$).
We define the energy functional J and the Nehari functional I as follows:
$I, J : W_0^{2,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{p}\|\Delta u\|_p^p + \frac{q+1}{q^2}\|\nabla u\|_q^q - \frac{1}{q}\int_\Omega |\nabla u|^q \log|\nabla u| dx \tag{2}$$

$$I(u) = \|\Delta u\|_p^p + \|\nabla u\|_q^q - \int_\Omega |\nabla u|^q \log|\nabla u| dx \tag{3}$$

Then we have,

$$J(u) = \frac{1}{q}I(u) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\Delta u\|_p^p + \frac{1}{q^2}\|\nabla u\|_q^q \tag{4}$$

We introduce the Nehari manifold as

$$\mathcal{N} = \{u \in W_0^{2,p}(\Omega)\backslash\{0\} : I(u) = 0\}$$

also define the potential well as

$$\mathcal{W} = \{u \in W_0^{2,p}(\Omega)\backslash\{0\} : J(u) < d, I(u) > 0\}$$

where $d = \inf_{u\in\mathcal{N}} J(u)$ is referred to as the depth of the potential well.

Definition 1. *A function $u = u(x,t)$ is considered to be a weak solution of problem (1) if $u \in L^\infty(0,T; W_0^{2,p}(\Omega))$, $u_t \in L^2(0,T; H_0^1(\Omega))$ and validates*

$$(u_t, \phi) + (\nabla u_t, \nabla\phi) + (|\Delta u|^{p-2}\Delta u, \Delta\phi) + (|\nabla u|^{q-2}\nabla u, \nabla\phi) = (|\nabla u|^{q-2}\nabla u \log|\nabla u|, \nabla\phi) \tag{5}$$

for all $\phi \in W_0^{2,p}(\Omega)$ and a.e $0 \leq t \leq T$ along with $u(x,0) = u_0(x)$ in $W_0^{2,p}(\Omega)\backslash\{0\}$. Furthermore, it also agrees the energy inequality

$$\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + J(u) \leq J(u_0) \quad ,0 < t \leq T. \tag{6}$$

Lemma 1. *[12] Let ρ be a positive number. Then we have the following inequalities:*

$$x^p \log x \leq (e\rho)^{-1} \text{ for all } x \geq 1$$

and

$$|x^p \log x| \leq (ep)^{-1} \text{ for all } 0 < x < 1.$$

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The following lemma is similar to one in [8], [16]. However, we explain the proof with some changes due to the occurrence of the non-linear logarithmic term $-\text{div}(|\nabla u|^{q-2}\nabla u \log|\nabla u|)$ and the q-Laplacian $\text{div}(|\nabla u|^{q-2}\nabla u)$.

Lemma 2. *For any $u \in W_0^{2,p}(\Omega)\backslash\{0\}$, we have the following:*

(i) $\lim_{\gamma\to 0^+} J(\gamma u) = 0$ *and* $\lim_{\gamma\to\infty} J(\gamma u) = -\infty$*;*

(ii) $\frac{d}{d\gamma}J(\gamma u) = \frac{1}{\gamma}I(\gamma u)$ *for* $\gamma > 0$*;*

(iii) there exists a unique $\gamma^* = \gamma^*(u) > 0$ *such that* $\frac{d}{d\gamma}J(\gamma u)|_{\gamma=\gamma^*} = 0$*. Also* $J(\gamma u)$ *is increasing on* $0 < \gamma \le \gamma^*$*, decreasing on* $\gamma^* \le \gamma < \infty$ *and takes the maximum at* $\gamma = \gamma^*$*;*

(iv) $I(\gamma u) > 0$ *for* $0 < \gamma < \gamma^*$*,* $I(\gamma u) < 0$ *for* $\gamma^* < \gamma < \infty$ *and* $I(\gamma^* u) = 0$*.*

Proof.

(i) Applying the definition of J we have

$$J(\gamma u) = \frac{\gamma^p}{p}\|\Delta u\|_p^p + \frac{\gamma^q(q+1)}{q^2}\|\nabla u\|_q^q - \frac{\gamma^q \log\gamma}{q}\|\nabla u\|_q^q - \frac{\gamma^q}{q}\int_\Omega |\nabla u|^q \log|\nabla u| dx$$

so it is evident that $\lim_{\gamma\to 0^+} J(\gamma u) = 0$ and $\lim_{\gamma\to\infty} J(\gamma u) = -\infty$ since $2 < p < q$.

(ii) Direct computation yields,

$$\frac{d}{d\gamma}J(\gamma u) = \gamma^{p-1}\|\Delta u\|_p^p + \gamma^{q-1}\|\nabla u\|_q^q - \gamma^{q-1}\int_\Omega |\nabla u|^q \log|\gamma\nabla u| dx = \frac{1}{\gamma}I(\gamma u)$$

(iii) We have,

$$\frac{d}{d\gamma}J(\gamma u) = \gamma^{q-1}\left(\gamma^{p-q}\|\Delta u\|_p^p + \|\nabla u\|_q^q - \log\gamma\|\nabla u\|_q^q - \int_\Omega |\nabla u|^q \log|\nabla u| dx\right)$$

Now define,

$$g(\gamma) = \gamma^{p-q}\|\Delta u\|_p^p + \|\nabla u\|_q^q - \log\gamma\|\nabla u\|_q^q - \int_\Omega |\nabla u|^q \log|\nabla u| dx$$

Then we can observe that g is decreasing since

$$g'(\gamma) = (p-q)\gamma^{p-q-1}\|\Delta u\|_p^p - \frac{1}{\gamma}\|\nabla u\|_q^q < 0$$

Also, $\lim_{\gamma\to 0^+} g(\gamma) = \infty$ and $\lim_{\gamma\to\infty} g(\gamma) = -\infty$.
Hence, a unique γ^* with $g(\gamma^*) = 0$ is guaranteed.
Also, $g(\gamma) > 0$ for $0 < \gamma < \gamma^*$ and $g(\gamma) < 0$ for $\gamma^* < \gamma < \infty$.
Now, since $\frac{d}{d\gamma}J(\gamma u) = \gamma^{q-1}g(\gamma)$ we obtain $\frac{d}{d\gamma}J(\gamma u)|_{\gamma=\gamma^*} = 0$ and also $J(\gamma u)$ is increasing on $0 < \gamma \le \gamma^*$, decreasing on $\gamma^* \le \gamma < \infty$ and takes the maximum at $\gamma = \gamma^*$.

(iv) *(iv)* is obvious since $I(\gamma u) = \gamma\frac{d}{d\gamma}J(\gamma u)$.

The above lemmas are useful to prove the main results in the following section.

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3 MAIN RESULTS

In this section, we prove the existence of weak local solutions to the problem (1). Further, we show that the weak solution exists globally using the potential well method when the initial energy of the system is subcritical and critical. We show that the solution becomes unbounded in finite time and specifies an upper limit for the blow-up time.

Theorem 1. *(The Local existence)*
Let $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}$ and $2 < p < q < p(1+\frac{2}{N+2})$. Then a $T > 0$ and a unique weak solution $u(t)$ of problem(1) agreeing the energy inequality

$$\int_0^t \|u_\tau\|^2_{H_0^1} d\tau + J(u(t)) \leq J(u_0) \quad ,0 \leq t \leq T \tag{7}$$

and $u(0) = u_0$ exists.

Proof. **Existence**
Let $\{w_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $W_0^{2,p}(\Omega)$. We use the approximation,

$$u_k(x,t) = \sum_{i=1}^k a_{k,i}(t) w_i(x), \qquad k = 1, 2, \ldots$$

where $a_{k,i}(t) : [0,T] \to \mathbb{R}$ accepts the below ODE.

$$\begin{aligned}(u_{kt}, w_i) + (\nabla u_{kt}, \nabla w_i) + (|\Delta u_k|^{p-2}\Delta u_k, \Delta w_i) + (|\nabla u_k|^{q-2}\nabla u_k, \nabla w_i) \\ = (|\nabla u_k|^{q-2}\nabla u_k \log|\nabla u_k|, \nabla w_i)\end{aligned} \tag{8}$$

$i = 1, 2, \ldots, k$ and

$$u_k(x,0) = \sum_{i=1}^k a_{k,i}(0) w_i(x) \to u_0(x) \text{ in } W_0^{2,p}(\Omega)\backslash\{0\}$$

By Peano's theorem, the above ODE has a solution $a_{k,i}$ and we can find a $T_k > 0$ with $a_{k,i} \in C^1([0,T_k])$, which implies $u_k \in C^1([0,T_k]; W_0^{2,p}(\Omega))$.
Now by multiplying (8) by $a_{k,i}(t)$, summing it for $i = 1, 2, \ldots, k$ and integrating with respect to t from 0 to t we obtain,

$$\frac{1}{2}\|u_k\|^2_{H_0^1} + \int_0^t (\|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q) dt = \frac{1}{2}\|u_k(0)\|^2_{H_0^1} + \int_0^t \int_\Omega |\nabla u_k|^q \log|\nabla u_k| dx dt$$

That is,

$$\psi_k(t) = \psi_k(0) + \int_0^t \int_\Omega |\nabla u_k|^q \log|\nabla u_k| dx dt \tag{9}$$

where

$$\psi_k(t) = \frac{1}{2}\|u_k\|^2_{H_0^1} + \int_0^t (\|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q) dt \tag{10}$$

We obtain the following by employing lemma(1), Gagliardo-Nirenberg interpolation inequality and Young's inequality.

$$\begin{aligned}\int_\Omega |\nabla u_k|^q \log|\nabla u_k| dx &\leq \int_{\{x\in\Omega:|\nabla u_k|\geq 1\}} |\nabla u_k|^q \log|\nabla u_k| dx \\ &\leq (e\rho)^{-1}\|\nabla u_k\|^{q+\rho}_{q+\rho} \\ &\leq (e\rho)^{-1} C_1^{q+\rho} \|\Delta u_k\|_p^{\theta(q+\rho)} \|u_k\|_2^{(1-\theta)(q+\rho)} \\ &\leq \epsilon\|\Delta u_k\|_p^p + C(\epsilon)\|u_k\|_2^{\frac{p(1-\theta)(q+\rho)}{p-\theta(q+\rho)}}\end{aligned} \tag{11}$$

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where $\theta = \left(\frac{1}{n} + \frac{1}{2} - \frac{1}{q+\rho}\right)\left(\frac{2}{n} + \frac{1}{2} - \frac{1}{p}\right)^{-1}, \epsilon \in (0,1)$,
$C(\epsilon) = \left(\frac{p\epsilon}{\theta(q+\rho)}\right)^{\frac{\theta(q+\rho)}{\theta(q+\rho)-p}}\left(\frac{p-\theta(q+\rho)}{p}\right)\left((e\rho)^{-1}C_1^{q+\rho}\right)^{\frac{p}{p-\theta(q+\rho)}}$ and,
ρ is chosen so that $2 < q+\rho < p(1+\frac{2}{n+2})$.
Let $\beta = \frac{p(1-\theta)(q+\rho)}{2(p-\theta(q+\rho))} = \frac{np+(p-n)(q+\rho)}{p(4+n)-(n+2)(q+\rho)}$. Then $\beta > 1$ and

$$\int_\Omega |\nabla u_k|^q \log|\nabla u_k| dx \le \epsilon \|\Delta u_k\|_p^p + C(\epsilon)\|u_k\|_2^{2\beta} \tag{12}$$

Then (9) implies that,

$$\begin{aligned}
\psi_k(t) &\le \psi_k(0) + \epsilon\int_0^t \|\Delta u_k\|_p^p dt + C(\epsilon)\int_0^t \|u_k\|_2^{2\beta} dt \\
&\le C_2 + \epsilon\psi_k(t) + C(\epsilon)2^\beta \int_0^t \left(\left(\frac{1}{2}\|u_k\|_{H_0^1}^2\right)^\beta + \left(\int_0^s (\|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q) ds\right)^\beta\right) dt \\
&\le C_2 + \epsilon\psi_k(t) + C_3\int_0^t \psi_k(t)^\beta dt
\end{aligned}$$

Hence we get,

$$\psi_k(t) \le C_4 + C_5\int_0^t \psi_k(t)^\beta dt$$

Then the Gronwall-Bellman-Bihari type integral inequality gives a T such that $0 < T < \frac{C_4^{1-\beta}}{C_5(1-\beta)}$ and

$$\psi_k(t) \le C_T \text{ for all } t \in [0,T]. \tag{13}$$

Hence the solution of (8) exists in $[0,T]$ for all k.
Now multiplying (8) by $a'_{k,i}(t)$ and summing for $i = 1,2,\ldots,k$ we get,

$$\begin{aligned}
(u_{kt}, u_{kt}) + (\nabla u_{kt}, \nabla u_{kt}) + (|\Delta u_k|^{p-2}\Delta u_k, \Delta u_{kt}) + (|\nabla u_k|^{q-2}\nabla u_k, \nabla u_{kt}) \\
= (|\nabla u_k|^{q-2}\nabla u_k \log|\nabla u_k|, \nabla u_{kt})
\end{aligned}$$

integrating with respect to t,

$$\int_0^t \|u_{kt}\|_{H_0^1}^2 dt + J(u_k(t)) = J(u_k(0)) \text{ for all } t \in [0,T]. \tag{14}$$

As contrast to that, a constant $C_6 > 0$ satisfying

$$J(u_k(0)) \le C_6 \text{ for all } k. \tag{15}$$

exists since $u_k(0) \to u_0$ and by the continuity of J. Then from (12),(13),(14) and (15) we can see that

$$\begin{aligned}
C_6 &\ge \int_0^t \|u_{kt}\|_{H_0^1}^2 dt + \frac{1}{p}\|\Delta u_k\|_p^p + \frac{q+1}{q^2}\|\nabla u_k\|_q^q - \frac{1}{q}\int_\Omega |\nabla u_k|^q \log|\nabla u_k| dx \\
&\ge \int_0^t \|u_{kt}\|_{H_0^1}^2 dt + \left(\frac{1}{p} - \frac{\epsilon}{q}\right)\|\Delta u_k\|_p^p + \frac{q+1}{q^2}\|\nabla u_k\|_q^q - \frac{C(\epsilon)}{q}\|u_k\|_{H_0^1}^{2\beta} \\
&\ge \int_0^t \|u_{kt}\|_{H_0^1}^2 dt + \left(\frac{1}{p} - \frac{\epsilon}{q}\right)\|\Delta u_k\|_p^p + \frac{q+1}{q^2}\|\nabla u_k\|_q^q - \frac{C(\epsilon)}{q}2^\beta C_T^\beta
\end{aligned}$$

Let $\tilde{C} = C_6 + \frac{C(\epsilon)2^\beta}{q}C_T^\beta$. Then we gain that

$$\begin{aligned}
\int_0^t \|u_{kt}\|_2^2 dt &\le \tilde{C} \\
\int_0^t \|\nabla u_{kt}\|_2^2 dt &\le \tilde{C} \\
\|\Delta u_k\|_p^p &< \tilde{C}\left(\frac{1}{p} - \frac{\epsilon}{q}\right)^{-1} \\
\|\nabla u_k\|_q^q &< \tilde{C}\frac{q^2}{q+1}
\end{aligned}$$

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Thus we have $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(0,T;W_0^{2,p}(\Omega))$ and $\{u_{kt}\}_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;H_0^1(\Omega))$. Hence there exists a subsequence, however indicated by $\{u_k\}_{k\in\mathbb{N}}$ which agrees,

$$\begin{aligned} u_k \to u \quad & \text{weakly* in } L^\infty(0,T;W_0^{2,p}(\Omega)) \\ u_{kt} \to u_t \quad & \text{weakly in } L^2(0,T;H_0^1(\Omega)) \\ u_k \to u \quad & \text{weakly* in } L^\infty(0,T;W_0^{1,q}(\Omega)) \end{aligned}$$

since

$$u_{kt} \to u_t \qquad \text{weakly in } L^2(0,T;L^2(\Omega))$$

by Aubin-Lions lemma we get,

$$u_k \quad \to \quad u \text{ strongly in } C(0,T;L^2(\Omega))$$

Therefore,

$$|\Delta u_k|^{p-2}\Delta u_k \quad \to \quad \xi_1 \text{ weakly* in } L^\infty(0,T;W_0^{-2,p'}(\Omega))$$

and,

$$|\nabla u_k|^{q-2}\nabla u_k \quad \to \quad \xi_2 \text{ weakly* in } L^\infty(0,T;W_0^{-1,q'}(\Omega))$$

where $W_0^{-2,p'}(\Omega)$ is the dual space of $W_0^{2,p}(\Omega)$ and $W_0^{-1,q'}(\Omega)$ is the dual space of $W_0^{1,q}(\Omega)$. Now from the theory of monotone operators, it concludes,

$$\xi_1 = |\Delta u|^{p-2}\Delta u \qquad \text{and} \qquad \xi_2 = |\nabla u|^{q-2}\nabla u.$$

Now let $\Phi(u) = |u|^{q-2}u\log|u|$. We have

$$\begin{aligned} \nabla u_k \to \nabla u \quad & \text{weakly* in } L^\infty(0,T;L^2(\Omega)) \\ \nabla u_{kt} \to \nabla u_t \quad & \text{weakly in } L^2(0,T;L^2(\Omega)) \end{aligned}$$

Therefore,

$$\nabla u_k \to \nabla u \text{ strongly in } C(0,T;L^2(\Omega))$$

and

$$\Phi(\nabla u_k) \to \Phi(\nabla u) \qquad a.e \text{ in } \Omega\times(0,T)$$

We again use Lemma(1) and Gagliardo-Nirenberg interpolation inequality to emerge the below.

$$\begin{aligned} \int_\Omega (\Phi(\nabla u_k))^{q'}dx &\le \int_{\{x\in\Omega:|\nabla u_k|\le 1\}} \left(|\nabla u_k|^{q-1}|\log|\nabla u_k||\right)^{q'} dx \\ &\quad + \int_{\{x\in\Omega:|\nabla u_k|\ge 1\}} \left(|\nabla u_k|^{q-1}|\log|\nabla u_k||\right)^{q'} dx \\ &\le (e(q-1))^{-q'}|\Omega| + (e\mu)^{-q'}\|\nabla u_k\|_r^r \\ &\le (e(q-1))^{-q'}|\Omega| + (e\mu)^{-q'}C_7^r\|\Delta u_k\|_p^{r\alpha}\|u_k\|_2^{r(1-\alpha)} \\ &< C_8 \end{aligned}$$

where $r=(q-1+\mu)q'$, $q'=\frac{q}{q-1}$ and $\alpha=\left(\frac{1}{n}+\frac{1}{2}-\frac{1}{r}\right)\left(\frac{2}{n}+\frac{1}{2}-\frac{1}{p}\right)^{-1}$. Hence,

$$\Phi(\nabla u_k) \to \Phi(\nabla u) \text{ weakly* in } L^\infty(0,T;L^{q'}(\Omega))$$

Now for a fixed i in (8) letting k tends to ∞ we get,

$$(u_t,w_i) + (\nabla u_t,\nabla w_i) + (|\Delta u|^{p-2}\Delta u,\Delta w_i) + (|\nabla u|^{q-2}\nabla u,\nabla w_i) = (|\nabla u|^{q-2}\nabla u\log|\nabla u|,\nabla w_i)$$

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for all $i = 1, 2, \ldots, k$. Then for all $\phi \in W_0^{2,p}(\Omega)$ and for a.e. $t \in [0, T]$,

$$(u_t, \phi) + (\nabla u_t, \nabla\phi) + (|\Delta u|^{p-2}\Delta u, \Delta\phi) + (|\nabla u|^{q-2}\nabla u, \nabla\phi) = (|\nabla u|^{q-2}\nabla u \log|\nabla u|, \nabla\phi)$$

and $u(x, 0) = u_0(x)$ in $W_0 2, p^2(\Omega)\backslash\{0\}$.

Uniqueness

Let u and $\tilde{u}$ be two weak solutions of problem (1). For any $\phi \in H_0^2(\Omega)$, it is noted that,

$$(u_t, \phi) + (\nabla u_t, \nabla\phi) + (|\Delta u|^{p-2}\Delta u, \Delta\phi) + (|\nabla u|^{q-2}\nabla u, \nabla\phi) = (|\nabla u|^{q-2}\nabla u \log|\nabla u|, \nabla\phi)$$

$$(\tilde{u}_t, \phi) + (\nabla\tilde{u}_t, \nabla\phi) + (|\Delta\tilde{u}|^{p-2}\Delta\tilde{u}, \Delta\phi) + (|\nabla\tilde{u}|^{q-2}\nabla\tilde{u}, \nabla\phi) = (|\nabla\tilde{u}|^{q-2}\nabla\tilde{u} \log|\nabla\tilde{u}|, \nabla\phi)$$

On subtraction of one equation from the other and taking $\phi = u - \tilde{u}$, the above yields that

$$\begin{aligned}(\phi_t, \phi) + (\nabla\phi_t, \nabla\phi) &+ \int_\Omega (|\Delta u|^{p-2}\Delta u - |\Delta\tilde{u}|^{p-2}\Delta\tilde{u})(\Delta u - \Delta\tilde{u})dx \\ &+ \int_\Omega (|\nabla u|^{q-2}\nabla u - |\nabla\tilde{u}|^{q-2}\nabla\tilde{u})(\nabla u - \nabla\tilde{u})dx \\ &= \int_\Omega (|\nabla u|^{q-2}\nabla u \log|\nabla u| - |\nabla\tilde{u}|^{q-2}\nabla\tilde{u}\log|\nabla\tilde{u}|)(\nabla u - \nabla\tilde{u})dx\end{aligned}$$

Then by the monotonicity of q-Laplacian $\operatorname{div}(|\nabla u|^{q-2}\nabla u)$ and the p-Biharmonic operator $\Delta(|\Delta u|^{p-2}\Delta u)$ and by the Lipschitz continuity of $|x|^{q-2}x\log|x|$ we get,

$$(\phi_t, \phi)_1 \leq L\int_\Omega (\nabla u - \nabla\tilde{u})^2 dx$$

where $L > 0$ is the Lipschitz constant. Thus we obtain,

$$(\phi_t, \phi)_1 \leq L\|\nabla\phi\|_2^2 \leq L\|\phi\|_{H_0^1}^2$$

By the integration from 0 to t with respect to t we obtain that,

$$\|\phi\|_{H_0^1}^2 - \|\phi(0)\|_{H_0^1}^2 \leq L\int_0^t \|\phi\|_{H_0^1}^2 dt.$$

Since $\phi(0) = u(0) - \tilde{u}(0) = 0$, apply Gronwall's inequality to gain,

$$\|\phi\|_{H_0^1}^2 = 0$$

Therefore, $\phi = 0$ a.e. in $\Omega \times (0, T)$. That is, $u = \tilde{u}$ a.e. in $\Omega \times (0, T)$.

Energy inequality

Let $\chi \in C[0, T]$ be a non-negative function. Then (14) implies

$$\int_0^T \chi(t)\int_0^t \|u_{kt}\|_{H_0^1}^2 dsdt + \int_0^T J(u_k(t))\chi(t)dt = \int_0^T J(u_k(0))\chi(t)dt$$

Since, we have the lower semi-continuity $\int_0^T J(u_k(t))\chi(t)dt$ with respect to the weak topology of $L^2(0, T; W_0^{2,p}(\Omega))$.

$$\int_0^T J(u(t))\chi(t)dt \leq \liminf_{k\to\infty}\int_0^T J(u_k(t))\chi(t)dt$$

also $\int_0^T J(u_k(0))\chi(t)dt \to \int_0^T J(u_0)\chi(t)dt$ as $k \to \infty$. Thus we get,

$$\int_0^T \chi(t)\int_0^t \|u_t\|_{H_0^1}^2 dsdt + \int_0^T J(u(t))\chi(t)dt \leq \int_0^T J(u_0)\chi(t)dt$$

Since $\chi(t)$ is arbitrary,

$$\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + J(u(t)) \leq J(u_0) \quad \text{for } 0 \leq t \leq T.$$

Hence the proof is complete.

Next theorem address the case of the initial energy of the system is sub-critical, i.e, $J(u_0) < d$. We will demonstrate the existence of weak global solutions.

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Theorem 2. *(Global Existence for $J(u_0) < d$)*
A unique global weak solution u satisfying the energy estimate,

$$\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + J(u(t)) \leq J(u_0) \quad \textit{for } 0 \leq t < \infty \tag{16}$$

exists for problem(1) if the conditions $J(u_0) < d$ and $I(u_0) > 0$ holds for the initial value $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}$.

Proof. Define $\{w_i\}_{i\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$ as in the proof of Theorem(1). Multiplying (8) by $a'_{k,i}(t)$ and summing over i and integrating with respect to t from 0 to t we identify,

$$\int_0^t \|u_{kt}\|_{H_0^1}^2 dt + J(u_k(t)) = J(u_k(0)) \text{ for all } t \in [0, T_{max}) \tag{17}$$

where T_{max} is the maximum time for solution $u_k(x,t)$ to exist.
We have $J(u_k(0)) \to J(u_0)$ as $k \to \infty$ and $J(u_0) < d$. Therefore,

$$\int_0^t \|u_{kt}\|_{H_0^1}^2 dt + J(u_k(t)) < d, \qquad t \in [0, T_{max}) \tag{18}$$

Since $I(u_0) > 0$ we have $I(u_k(0)) > 0$ for sufficiently large k. We claim that $I(u_k) > 0$ for sufficiently large k. Otherwise we can locate a t_0 such that $I(u_k(t_0)) = 0$, $u_k(t_0) \neq 0$. Then $u_k(t_0) \in \mathcal{N}$ and $J(u_k(t_0)) \geq d$, which is a contradiction to (18).
Therefore $I(u_k) > 0$ for appropriately large k.
Then we get,

$$J(u_k) = \frac{1}{q}I(u_k) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\Delta u_k\|_p^p + \frac{1}{q^2}\|\nabla u_k\|_q^q > 0$$

Therefore,

$$\int_0^t \|u_{kt}\|_{H_0^1}^2 dt < d$$

also

$$\left(\frac{1}{p} - \frac{1}{q}\right)\|\Delta u_k\|_p^p + \frac{1}{q^2}\|\nabla u_k\|_q^q < J(u_k) < d$$

Let $K_0 = \min\{\frac{1}{p} - \frac{1}{q}, \frac{1}{q^2}\}$ and $K = d + \frac{d}{K_0}$ then

$$\|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q < d/K_0$$

and

$$\int_0^t \|u_{kt}\|_{H_0^1}^2 dt + \|\Delta u_k\|_p^p + \|\nabla u_k\|_q^q < K \tag{19}$$

where $K > 0$. Hence we take $T_{max} = \infty$. Now it is noticeable that problem (1) has a weak global solution by applying identical ideas used to prove the Theorem(1), and the solution u also agrees with the energy inequality

$$\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + J(u(t)) \leq J(u_0), \qquad 0 \leq t < \infty.$$

We will explain the global existence of weak solutions in the following theorem for the critical initial energy. That is when $J(u_0) = d$.

Theorem 3. *(Global existence for $J(u_0) = d$)*
Observe the conditions $J(u_0) = d$ and $I(u_0) > 0$ holds for the initial value $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}$. Subsequently problem(1) possesses a unique global weak solution $u \in L^\infty(0,T; W_0^{2,p}(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ for $0 \leq t \leq T$ and it also accepts the energy estimate (16).

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Proof. Let $\eta_j = 1 - \frac{1}{j}$, $j = 1, 2, \ldots$ then $\eta_j \to 1$ when $j \to \infty$. Take into account the below problem:

$$\begin{cases} u_t - \Delta u_t + \Delta(|\Delta u|^{p-2}\Delta u) - \operatorname{div}(|\nabla u|^{q-2}\nabla u) = -\operatorname{div}(|\nabla u|^{q-2}\nabla u \log|\nabla u|) & \text{if } (x,t) \in \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{if } (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = \eta_j u_0(x) = u_0^j & \text{if } x \in \Omega. \end{cases} \tag{20}$$

Since $I(u_0) > 0$, lemma (2)(iv) gives a $\gamma^* > 1$ with $I(\gamma^* u_0) = 0$.
Again from lemma (2)(iii) and (iv) we gain $I(\eta_j u_0) > 0$ and $J(\eta_j u_0) < J(u_0)$ since $\eta_j < 1 < \gamma^*$.
Thus we have $J(u_0^j) < d$ and $I(u_0^j) > 0$.
Then by Theorem(2), for each j problem (20) has a global weak solution $u^j \in L^\infty(0,T; W_0^{p-2}(\Omega))$ with $u_t^j \in L^2(0,T; L^2(\Omega))$ which satisfies the energy inequality,

$$\int_0^t \|u_\tau^j\|_{H_0^1}^2 d\tau + J(u^j(t)) \leq J(u_0^j) \quad \text{for } 0 \leq t < \infty.$$

Thus we have

$$\int_0^t \|u_\tau^j\|_{H_0^1}^2 d\tau + J(u^j) < d \quad \text{for } 0 \leq t < \infty.$$

Now by applying ideas similar to the one used to prove Theorem(1), we obtain a subsequence of $\{u^j\}_{j\in\mathbb{N}}$ converging to a function u, which is a weak solution of problem (1). It also fulfils the energy inequality (16). The solution's uniqueness can also be proved as in Theorem(1).
Hence the proof is over.

The following theorem gives the blow-up of solutions for the subcritical initial energy and an upper bound for blow up time.

Theorem 4. *(Blow up for $J(u_0) < d$)*
Let $u_0 \in H_0^2(\Omega)\backslash\{0\}$, $J(u_0) < d$ and $I(u_0) < 0$. Then the weak solution u of problem (1) blows up in a finite time T_* in the notion, $\lim_{t\to T_*^-} \|u\|_{H_0^1}^2 = \infty$. Furthermore, the upper bound of blow-up time T_* is given by

$$T_* \leq \frac{4(q-1)\|u_0\|_{H_0^1}^2}{q(q-2)^2(d - J(u_0))}.$$

Proof. First we prove $J(u(t)) < d$ and $I(u(t)) < 0$ for $t \in [0,T]$, where T indicates the maximum time for which $u(x,t)$ exists.
We have $J(u(t)) < J(u_0) < d$ by (6).
If we can choose a $t_0 \in (0,T)$ with $I(u(t_0)) = 0$ or $J(u(t_0)) = d$, since $J(u(t_0)) < d$, we must have $I(u(t_0)) = 0$.
Which implies $u(t_0) \in \mathcal{N}$ and thus $d \leq J(u(t_0))$, a contradiction.
Hence, $J(u(t)) < d$ and $I(u(t)) < 0$ for $t \in [0,T]$. Now define

$$\mathcal{P}(t) = \int_0^t \|u\|_{H_0^1}^2 dt$$

Then,

$$\mathcal{P}'(t) = \|u\|_{H_0^1}^2$$

and

$$\mathcal{P}''(t) = 2(u, u_t)_1 = -2I(u) > 0$$

Hence for $t > 0$, $\mathcal{P}'(t) \geq \mathcal{P}'(0) = \|u_0\|_{H_0^1}^2 > 0$.
Now fix $t_1 > 0$. Then for $t_1 \leq t < \infty$,

$$\mathcal{P}(t) \geq \mathcal{P}(t_1) \geq t_1 \|u_0\|_{H_0^1}^2 > 0$$

By Holder's inequality, we have,

$$\frac{1}{4}(\mathcal{P}'(t) - \mathcal{P}'(0))^2 \leq \int_0^t \|u\|_{H_0^1}^2 dt \int_0^t \|u_t\|_{H_0^1}^2 dt \tag{21}$$

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Since $I(u(t)) < 0$, Lemma 2 *(iv)*, gives a γ^* with $0 < \gamma^* < 1$ and $I(\gamma^* u) = 0$. Therefore,

$$\begin{aligned} d &\leq \left(\frac{1}{p} - \frac{1}{q}\right)(\gamma^*)^p \|\Delta u\|_p^p + \frac{1}{q^2}(\gamma^*)^q \|\nabla u\|_q^q \\ &\leq \left(\frac{1}{p} - \frac{1}{q}\right)\|\Delta u\|_p^p + \frac{1}{q^2}\|\nabla u\|_q^q \end{aligned} \tag{22}$$

Now by using (4),(6) and (22) we see that,

$$\mathcal{P}''(t) \geq 2q(d - J(u_0)) + 2q\int_0^t \|u_t\|_{H_0^1}^2 dt \tag{23}$$

Then from (21) and (23) it follows that

$$\mathcal{P}''(t)\mathcal{P}(t) - \frac{q}{2}(\mathcal{P}'(t) - \mathcal{P}'(0))^2 \geq \mathcal{P}(t)2q(d - J(u_0)) > 0 \text{ for } t \in [t_1, \infty) \tag{24}$$

Now choose $\tilde{T} > 0$ large enough to introduce,

$$\mathcal{Q}(t) = \mathcal{P}(t) + (\tilde{T} - t)\|u_0\|_{H_0^1}^2 \text{ for } t \in [t_1, \tilde{T}]$$

Then $\mathcal{Q}(t) \geq \mathcal{P}(t) > 0$ for $t \in [t_1, \tilde{T}]$, $\mathcal{Q}'(t) = \mathcal{P}'(t) - \mathcal{P}'(0) > 0$ and $\mathcal{Q}''(t) = \mathcal{P}''(t) > 0$.
Hence from (24) we observe,

$$\mathcal{Q}(t)\mathcal{Q}''(t) - \frac{q}{2}(\mathcal{Q}'(t))^2 \geq \mathcal{P}(t)2q(d - J(u_0)) + \mathcal{P}''(t)(\tilde{T} - t)\|u_0\|_{H_0^1}^2 > 0 \tag{25}$$

Now define

$$\mathcal{R}(t) = \mathcal{Q}(t)^{-\frac{q-2}{2}}$$

Then,

$$\mathcal{R}'(t) = -\frac{q-2}{2}\mathcal{Q}(t)^{-\frac{q}{2}}\mathcal{Q}'(t)$$

and

$$\mathcal{R}''(t) = \frac{q-2}{2}\mathcal{Q}(t)^{-\frac{q+2}{2}}\left(\frac{q}{2}(\mathcal{Q}'(t))^2 - \mathcal{Q}(t)\mathcal{Q}''(t)\right) < 0$$

Hence $\mathcal{R}(t)$ is a concave function in $[t_1, \tilde{T}]$ for any sufficiently large $\tilde{T} > t_1$. Also since $\mathcal{R}(t_1) > 0$ and $\mathcal{R}''(t_1) < 0$, there appears a finite time $T_* > t_1 > 0$ having $\lim_{t \to T_*^-} \mathcal{R}(t) = 0$. That yields $\lim_{t \to T_*^-} \mathcal{Q}(t) = +\infty$, which in turn gives $\lim_{t \to T_*^-} \mathcal{P}(t) = +\infty$. Hence we get

$$\lim_{t \to T_*^-} \|u\|_{H_0^1}^2 = +\infty.$$

To obtain an upper limit for blow-up time we define,

$$\mathcal{S}(t) = \mathcal{P}(t) + (T_* - t)\|u_0\|_{H_0^1}^2 + \sigma(t + \varphi)^2 \text{ for } t \in [0, T_*]$$

where the constants $\sigma, \varphi > 0$ will be given later.
Then,

$$\mathcal{S}'(t) = \|u\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2 + 2\sigma(t + \varphi) > 2\sigma(t + \varphi) > 0 \tag{26}$$

also by (23) we get,

$$\mathcal{S}''(t) \geq 2q(d - J(u_0)) + 2q\int_0^t \|u_t\|_{H_0^1}^2 dt + 2\sigma \tag{27}$$

By Schwartz's inequality, we have,

$$\int_0^t \frac{d}{dt}\|u\|_{H_0^1}^2 dt \leq 2\int_0^t \|u\|_{H_0^1}^2 dt \int_0^t \|u_t\|_{H_0^1}^2 dt \tag{28}$$

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Therefore,

$$\begin{aligned}
(\mathcal{S}'(t))^2 &= 4\left(\frac{1}{2}\int_0^t \frac{d}{dt}\|u\|^2_{H_0^1}dt + \sigma(t+\varphi)\right)^2 \\
&\leq 4\left(\int_0^t \frac{d}{dt}\|u\|^2_{H_0^1}dt + \sigma(t+\varphi)^2\right)\left(\int_0^t \frac{d}{dt}\|u_t\|^2_{H_0^1}dt + \sigma\right) \\
&= 4\left(\mathcal{S}(t) - (T_* - t)\|u_0\|^2_{H_0^1}\right)\left(\int_0^t \frac{d}{dt}\|u_t\|^2_{H_0^1}dt + \sigma\right) \\
&\leq 4\mathcal{S}(t)\left(\int_0^t \frac{d}{dt}\|u_t\|^2_{H_0^1}dt + \sigma\right) \qquad (29)
\end{aligned}$$

Now by applying (27) and (29) we can see that,

$$\mathcal{S}(t)\mathcal{S}''(t) - \frac{q}{2}(\mathcal{S}'(t))^2 \geq \mathcal{S}(t)(2q(d-J(u_0)) - 2\sigma(q-1))$$

If $\sigma \in \left(0, \frac{q(d-J(u_0))}{q-1}\right)$, then

$$\mathcal{S}(t)\mathcal{S}''(t) - \frac{q}{2}(\mathcal{S}'(t))^2 > 0.$$

Also we have $\mathcal{S}(0) = T_*\|u_0\|^2_{H_0^1} + \sigma\varphi^2 > 0$ and $\mathcal{S}'(0) = 2\sigma\varphi > 0$. Then by Levine's Concavity approach, we obtain the upper bound for blow-up as,

$$T_* \leq \frac{\mathcal{S}(0)}{(\frac{q}{2}-1)\mathcal{S}'(0)} = \frac{T_*\|u_0\|^2_{H_0^1}}{(q-2)\sigma\varphi} + \frac{\varphi}{q-2}$$

Therefore,

$$T_* \leq \frac{\sigma\varphi^2}{(q-2)\sigma\varphi - \|u_0\|^2_{H_0^1}}$$

thus we must have

$$\varphi \in \left(\frac{(q-1)\|u_0\|^2_{H_0^1}}{q(q-2)(d-J(u_0))}, \infty\right)$$

Let $\upsilon = \sigma\varphi \in \left(0, \frac{q(d-J(u_0))\varphi}{q-1}\right)$, then $T_* \leq \frac{\varphi\upsilon}{(q-2)\upsilon - \|u_0\|_{H_0^1}}$.
Now let $h(\varphi,\upsilon) = \frac{\varphi\upsilon}{(q-2)\upsilon - \|u_0\|_{H_0^1}}$. Since h is monotonically decreasing concerning υ, we have

$$\begin{aligned}
\inf_{\{(\varphi,\upsilon)\}} h(\varphi,\upsilon) &= \inf_{\{\varphi\}} h\left(\varphi, \frac{q(d-J(u_0))\varphi}{q-1}\right) \\
&= \inf_{\{b\}} k(\varphi)
\end{aligned}$$

where,

$$k(\varphi) = h\left(\varphi, \frac{q(d-J(u_0))\varphi}{q-1}\right) = \frac{\varphi^2 q(d-J(u_0))}{q(q-2)(d-J(u_0))\varphi - (q-1)\|u_0\|_{H_0^1}}$$

now since $k(\varphi)$ takes the minimum at $\varphi^* = \frac{2(q-1)\|u_0\|_{H_0^1}}{q(q-2)(d-J(u_0))}$ we can conclude that,

$$T_* \leq k(\varphi^*) = \frac{4(q-1)\|u_0\|^2_{H_0^1}}{q(q-2)^2(d-J(u_0))}.\square$$

The following theorem show that the weak solution of the system blow-up when the initial energy of the system is critical.

Theorem 5. *(Blow up for $J(u_0) = d$)*
Let $u_0 \in W_0^{2,p}(\Omega)\backslash\{0\}, J(u_0) = d$ and $I(u_0) < 0$, then the weak solution $u(t)$ of problem (1) blows up in the sense, there appears a $T_* < \infty$ such that $\lim_{t\to T_*^-} \|u\|^2_{H_0^1} = \infty$.

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*Proof.*Since $J(u_0) = d > 0$ and $J(u)$ is continuous with respect to t, there appears a t_0 with $J(u(x,t)) > 0$ for $0 < t \leq t_0$. Also, it is easy to see $I(u(t)) < 0$ for every t. Therefore from the energy inequality, $\int_0^{t_0} \|u_\tau\|^2_{H_0^1} d\tau + J(u(t_0)) < J(u_0) = d$, it follows that $J(u(t_0)) < d$.
Now choose $t = t_0$ as initial time, we have $J(u(t_0)) < d$ and $I(u(t_0)) < 0$. Now define

$$\mathcal{P}(t) = \int_{t_0}^{t} \|u\|^2_{H_0^1} \qquad \text{for } t > t_0$$

and the rest of proof resembles the proof of Theorem (4). □

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/09/

EXTREME STATES, OPERATOR SPACES AND TERNARY RINGS OF OPERATORS

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ABSTRACT

In this survey article on extreme states of operator spaces in $C^{¡}$*-algebras and related ternary ring of operators an extension result for rectangular operator extreme states on operator spaces in ternary rings of operators is discussed. We also observe that in the spacial case of operator spaces in rectangular matrix spaces, rectangular extreme states are conjugates of inclusion or identity maps implemented by isometries or unitaries. A characterization result for operator spaces of matrices for which the inclusion map is an extreme state is deduced using the above mentioned results.*

KEYWORDS

keyword 1, Keyword 2,...

1 INTRODUCTION

Arveson's extension theorem [3, Theorem1.2.3] for completely positive(CP) maps in the context of operator systems in C^*-algebras is a remarkable result in the study of boundary representations of C^*-algebras for operator systems. The theorem asserts that any CP map on an operator system $\mathcal{S}$ in a C^*-algebra $\mathcal{A}$ to $B(\mathcal{H})$ can be extended to a CP map from $\mathcal{A}$ to $B(\mathcal{H})$. A natural non-self-adjoint counterpart of the set up Arveson worked with can be considered to be consisting of operator spaces in ternary rings of operators(TROs) and completely contractive maps on them. An important recent work in this scenario is [10] where rectangular matrix convex sets and boundary representations of operator spaces were introduced. An operator space version of Arveson's conjecture, namely, every operator space is completely normed by it's boundary representations is also established in [10].

Apart from Arveson's fundamental work [3–5], we refer to the work of Douglas Farenick on extremal theory of matrix states on operator systems [8, 9] and Kleski's work on pure completely positive maps and boundary representations for operator systems [12].

In this article we study an extension result for rectangular extreme states on operator spaces in TROs. A characterization result for rectangular extreme state on operator spaces of matrices with trivial commutants is deduced.

2 Preliminaries

In this section we recall the fundamental notions that we require for the discussions later on in this article.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $B(\mathcal{H}, \mathcal{K})$ be the space of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$. When $\mathcal{K} = \mathcal{H}$, we denote $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$. A (concrete) *operator space* is a closed subspace of the (concrete) C^*-algebra $B(\mathcal{H})$. The space $B(\mathcal{H}, \mathcal{K})$ can be viewed as a subspace of $B(\mathcal{K} \oplus \mathcal{H})$, and hence it is always an operator space which we study more in later parts of this article.

An abstract characterisation of operator spaces was established by Ruan [16]. A subclass of operator spaces called ternary rings of operators referred to as TROs is of special interest to us here. These were shown to be the injective objects in the category of operator spaces and completely contractive linear maps by Ruan [17].

Definition 1. *A ternary ring of operators (TRO) T is a subspace of the C^*-algebra $B(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ that is closed under the triple product $(x, y, z) \to xy^*z$ for all $x, y, z \in T$.*

A triple morphism between TROs is a linear map that preserves the triple product.

*A triple morphism between TROs can be seen as the top-right corner of a *-homomorphism between the corresponding linking algebras [11]; see also [6, Corollary 8.3.5].*

Clearly, an obvious, but important example of a TRO is the space $B(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces.

Definition 2. *Let X and Y be operator spaces. A linear map $\phi : X \to Y$ is called completely contractive (CC) if the linear map $\mathbb{I}_n \otimes \phi : \mathbb{M}_n \otimes X \to \mathbb{M}_n \otimes Y$ is contractive for all n.*

We denote the set of all CC maps from X to $B(\mathcal{H}, \mathcal{K})$ by $CC(X, B(\mathcal{H}, \mathcal{K}))$.

Definition 3. *A representation of a TRO T is a triple morphism $\phi : T \to B(\mathcal{H}, \mathcal{K})$ for some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.*

A representation $\phi : T \to B(\mathcal{H}, \mathcal{K})$ is *irreducible* if, whenever p, q are projections in $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively, such that $q\phi(x) = \phi(x)p$ for every $x \in T$, one has $p = 0$ and $q = 0$, or $p = 1$ and $q = 1$.

A linear map $\phi : \mathbf{T} \to B(\mathcal{H},\mathcal{K})$ is *nondegenerate* if, whenever p, q are projections in $B(\mathcal{H})$ and $B(\mathcal{K})$, respectively, such that $q\phi(x) = \phi(x)p = 0$ for every $x \in \mathbf{T}$, one has $p = 0$ and $q = 0$.

Definition 4. *A rectangular operator state on an operator space X is a non degenerate linear map $\phi : X \to B(\mathcal{H},\mathcal{K})$ such that $\|\phi\|_{cb} = 1$ where the norm here is the completely bounded norm.*

Several characterizations of nondegenerate and irreducible representations of TROs are obtained in [7, Lemma 3.1.4 and Lemma 3.1.5].

A concrete TRO $\mathbf{T} \subseteq B(\mathcal{H},\mathcal{K})$ is said to act irreducibly if the corresponding inclusion representation is irreducible.

3 systems and spaces

An operator system in a C^*-algebra is a unital selfadjoint closed linear subspace. Let $\mathcal{S}$ be an operator system in a C^*-algebra $\mathcal{A}$, and $\mathcal{B}$ be any other C^*-algebra.

A linear map $\phi : \mathcal{S} \to \mathcal{B}$ is called completely positive (CP) if the linear map $\mathbb{I}_n \otimes \phi : \mathbb{M}_n \otimes \mathcal{S} \to \mathbb{M}_n \otimes \mathcal{B}$ is positive for all natural numbers n.

We denote the set of all unital CP maps from $\mathcal{S}$ to $B(\mathcal{H})$ by $UCP(\mathcal{S}, B(\mathcal{H}))$.

One of the crucial theorems in this context is Arveson's extension theorem [3, Theorem1.2.3]. Arveson's extension theorem asserts that any CP map on an operator system $\mathcal{S}$ to $B(\mathcal{H})$ can be extended to a CP map from the C^*-algebra $\mathcal{A}$ to $B(\mathcal{H})$.

Given an operator space $X \subseteq B(\mathcal{H},\mathcal{K})$, we can assign an operator system $S(X) \subseteq B(\mathcal{K} \oplus \mathcal{H})$, called the *Paulsen system* [15, Chapter 8] which is defined to be the space of operators

$$\left\{ \begin{bmatrix} \lambda I_{\mathcal{K}} & x \\ y^* & \mu I_{\mathcal{H}} \end{bmatrix} : x, y \in X, \lambda, \mu \in \mathbb{C} \right\}$$

where $I_{\mathcal{H}}$, $I_{\mathcal{K}}$ denote the identity operators on $\mathcal{H}$, $\mathcal{K}$ respectively. It is well known that [15, Lemma 8.1] any completely contractive map $\phi : X \to B(\mathcal{H},\mathcal{K})$ on the operator space X extends canonically to a unital completely positive (UCP) map $S(\phi) : S(X) \to B(\mathcal{K} \oplus \mathcal{H})$ defined by

$$S(\phi)\left(\begin{bmatrix} \lambda I_{\mathcal{K}_0} & x \\ y^* & \mu I_{\mathcal{H}_0} \end{bmatrix} \right) = \begin{bmatrix} \lambda I_{\mathcal{K}} & \phi(x) \\ \phi(y)^* & \mu I_{\mathcal{H}} \end{bmatrix}.$$

4 Extreme states and boundary theorems

4.1 Commutant of a TRO and boundary theorem

We introduce the notion of commutant of a rectangular operator set $X \subseteq B(\mathcal{H},\mathcal{K})$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. In the context of Hilbert C^*-module by Arambašić [1] introduced a similar notion. We can now define commutants of operator spaces in general and TROs in particular. We will prove that in the case of TROs, the commutants satisfy the usual properties with respect to the relevant notions of invariant subspaces and irreducibility of representations and thereby justifying the term. We observe that commutant behaves well with respect to the Paulsen map which is crucial for us. Throughout the artilce unless mentioned otherwise T represents a TRO.

Definition 5. *For $X \subseteq B(H,K)$, the commutant of X is the set in $B(K \oplus H)$ denoted by X' and defined by*

$$X' = \{A_1 \oplus A_2 \in B(\mathcal{K} \oplus \mathcal{H}) : A_1 \in B(\mathcal{K}) \text{ and } A_2 \in B(\mathcal{H}), A_1 x = x A_2 \text{ and }$$
$$A_2 x^* = x^* A_1, \ \forall x \in X\}$$

where $(A_1 \oplus A_2)(\eta_1 \oplus \eta_2) = A_1\eta_1 \oplus A_2\eta_2$, $\eta_1 \in K$ and $\eta_2 \in H$.

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Remark 1. *For any non-empty set $X \subseteq B(\mathcal{H},\mathcal{K})$, the commutant X' is a von Neumann subalgebra of $B(\mathcal{K} \oplus \mathcal{H})$.*

Definition 6. *Let $\phi : T \to B(\mathcal{H},\mathcal{K})$ be a non-zero representation and $\mathcal{H}_1 \subseteq \mathcal{H}$ and $\mathcal{K}_1 \subseteq \mathcal{K}$ be closed subspaces. We say that the pair $(\mathcal{H}_1,\mathcal{K}_1)$ of subspaces is ϕ-invariant if $\phi(T)H_1 \subseteq K_1$ and $\phi(T)^*K_1 \subseteq H_1$.*

The following two results follow easily from the definitions above.

Lemma 1. *Let $\phi : T \to B(H,K)$ be a non-zero representation. Let p and q be the orthogonal projections on closed subspaces $H_1 \subseteq H$ and $K_1 \subseteq K$ respectively. Then (H_1,K_1) is a ϕ -invariant pair of subspaces if and only if $q \oplus p \in \phi(T)'$.*

Corollary 1. *Let $\phi : T \to B(H,K)$ be a representation. Then ϕ is irreducible if and only if ϕ has no ϕ-invariant pair of subspaces other than $(0,0)$ and (H,K).*

The following result illustrate that the term *commutant* used above is an appropriate one.

Proposition 1. *Let $\phi : T \to B(H,K)$ be a non zero representation of a TRO T. Then ϕ is irreducible if and only if $\phi(T)' = \mathbb{C}(I_K \oplus I_H)$.*

Proof. Assume ϕ is irreducible. Let $q \oplus p$ be a projection in $\phi(T)'$. From the definition of commutant we have $\phi(x)p = q\phi(x)$. By irreducibility of ϕ we have either $p = 0$ and $q = 0$, or $p = 1$ and $q = 1$. Hence $\phi(T)' = \mathbb{C}(I_K \oplus I_H)$.

Conversely let $\phi(T)' = \mathbb{C}(I_K \oplus I_H)$ and p, q be projections such that $\phi(x)p = q\phi(x)$. Applying adjoint we have $p\phi(x)^* = \phi(x)^*q$. Thus $q \oplus p \in \phi(T)'$. By assumption, either $p = 0$ and $q = 0$, or $p = 1$ and $q = 1$. Hence ϕ is irreducible.

The following result shows that commutant behaves well with respect to the Paulsen map.

Lemma 2. *For a rectangular matrix state $\phi : X \to \mathbb{M}_{n,m}(\mathbb{C})$ we have*

$$\phi(X)' = [S(\phi)(S(X))]'.$$

Proof. To show $\phi(X)' \subseteq [S(\phi)(S(X))]'$, consider $A_1 \oplus A_2 \in \phi(X)'$, then we have

$$\begin{aligned}
\begin{bmatrix} \lambda & \phi(x) \\ \phi(y)^* & \mu \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} &= \begin{bmatrix} \lambda A_1 & \phi(x)A_2 \\ \phi(y)^*A_1 & \mu A_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda A_1 & A_1\phi(x) \\ A_2\phi(y)^* & \mu A_2 \end{bmatrix} \\
&= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} \lambda & \phi(x) \\ \phi(y)^* & \mu \end{bmatrix}
\end{aligned}$$

Hence $A_1 \oplus A_2 \in [S(\phi)(S(X))]'$.

Conversely, since the Paulsen system contains a copy of scalar matrices, if $A \in [S(\phi)(S(X))]'$, then $A = A_1 \oplus A_2$, where $A_1 \in B(K)$ and $A_2 \in B(H)$. Using the commutativity relation of the Paulsen map on the matrices

$$\begin{bmatrix} 0 & \phi(x) \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ \phi(x)^* & 0 \end{bmatrix}$$

we can conclude that $A_1\phi(x) = \phi(x)A_2$ and $A_2\phi(x)^* = \phi(x)^*A_1$, $\forall x \in X$. Hence $A = A_1 \oplus A_2 \in \phi(X)'$. Thus $[S(\phi)(S(X))]' \subseteq \phi(X)'$.

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We prove a version of the rectangular boundary theorem proved in [10, Theorem 2.17] which is an analogue of boundary theorem of Arveson in the context of operator systems. Arveson's boundary theorem [4, Theorem 2.1.1] states that if $\mathcal{S} \subseteq B(\mathcal{H})$ is an operator system which acts irreducibly on $\mathcal{H}$ such that the C^*-algebra $C^*(\mathcal{S})$ contains the algebra of compact operators $\mathcal{K}(\mathcal{H})$, then the identity representation of $C^*(\mathcal{S})$ is a boundary representation for $\mathcal{S}$ if and only if the quotient map $B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is not completely isometric on $\mathcal{S}$. Our context here is that of an operator space and the generated TRO.

Throughout we assume that X is an operator space, and T is a TRO containing X as a generating set. We say that a rectangular operator state $\phi : X \to B(\mathcal{H}, \mathcal{K})$ has *unique extension property* if any rectangular operator state ϕ on T whose restriction to X coincides with ϕ is automatically a triple morphism. Boundary representations for operator spaces were introduced in [10]. For our purposes we consider the following definition which is slightly different from the definition given in [10, Definition 2.7] but we remark that both are the same.

Definition 7. *An irreducible representation $\theta : T \to B(\mathcal{H}, \mathcal{K})$ is a boundary representation for X if θ_X is a rectangular operator state on X and θ_X has unique extension property.*

An exact sequence of TROs induces an exact sequence of the corresponding linking algebras, which is actually an exact sequence of C^*-algebras. The decomposition result for representations of C^*-algebras is well known [5, section 7] and the result for TROs follows from it.

If $0 \to T' \to T \to T'' \to 0$ is an exact sequence of TRO's, then every non degenerate representation

$$\pi : T \to B(\mathcal{H}, \mathcal{K})$$

of T decomposes uniquely into a direct sum of representations $\pi = \pi_T' \oplus \pi_T''$ where π_T' is the unique extension to T of a nondegenerate representation of the TRO-ideal [7, Definition 2.2.7] T', and where π_T'' is a nondegenerate representation of T that annihilates T'. When $\pi = \pi_T'$ we say that π *lives on* T'.

With X and T as above let $\mathcal{K}(\mathcal{H}, \mathcal{K})$ denotes the set of compact operators from $\mathcal{H}$ to $\mathcal{K}$. Then $\mathcal{K}_T = T \cap \mathcal{K}(\mathcal{H}, \mathcal{K})$ is a TRO-ideal and we have an obvious exact sequence of TRO's. Denote $\mathcal{K}_T$ to be the set of all irreducible triple morphisms $\theta : T \to B(\mathcal{H}, \mathcal{K})$ such that θ lives on $\mathcal{K}_T$ and θ_X is a rectangular operator state on X. The following is a boundary theorem in this context.

Theorem 1. *Let $X \subseteq B(\mathcal{H}, \mathcal{K})$ be an operator space such that TRO T generated by X acts irreducibly and $\mathcal{K}_T \neq \{0\}$. Then $\mathcal{K}_T$ contains a boundary representation for X if and only if the quotient map $q : T \to T/\mathcal{K}_T$ is not completely isometric on X.*

Proof. Assume that the quotient map q is not completely isometric on X. If $\mathcal{K}_T$ contains no boundary representations for X, then every boundary representation must annihilate $\mathcal{K}_T$ and consequently it factors through q. By [10, Theorem 2.9], there are sufficiently many boundary representations π_l, $l \in I$, for X so that

$$\|(q(x_{ij}))\| \le \|(x_{ij})\| = \sup_{l \in I} \|\pi_l(x_{ij})\| \le \|(q(x_{ij}))\|$$

for every $n \times n$ matrix (x_{ij}) over X and every $n \ge 1$. This is a contradiction.

Conversely, if the quotient map is completely isometric on X, then we show that no $\pi \in \mathcal{K}_T$ can be a boundary representation for X. Let $\pi : T \to B(\mathcal{H}, \mathcal{K})$ be an irreducible representation that lives on $\mathcal{K}_T$ and consider the map $Q : q(T) \to B(\mathcal{K} \oplus \mathcal{H})$ defined by $Q(q(a)) = \pi(a)$. Then Q is well defined and

$$\|Q(q(a))\| = \|\pi(a)\| \le \|a\| = \|q(a)\|.$$

Similarly $\|Q_n(q(a_{ij}))\| \le \|(q(a_{ij}))\|$ and hence Q is completely contractive. Then the Paulsen's map $S(Q) : S(q(T)) \to B(\mathcal{K} \oplus \mathcal{H})$ is unital and completely positive. By Arveson's extension theorem $S(Q)$

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extends to a completely positive map $S.Q/ : C^{¡}.S.q.T/// \rightarrow B.\mathcal{K}\ \mathcal{H}/$. Define $\psi : T \rightarrow B.\mathcal{H}, \mathcal{K}/$ via

$$S.Q/\left(\begin{bmatrix} 0 & q.a/ \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} ¡ & \psi.a/ \\ ¡ & ¡ \end{bmatrix}$$

Then clearly ψ is linear. Also for all a Ë T,

$$\psi.a/ = \begin{bmatrix} 0 & \psi.a/ \\ 0 & 0 \end{bmatrix} \leq S.Q/\left(\begin{bmatrix} 0 & q.a/ \\ 0 & 0 \end{bmatrix}\right) \leq S.Q/q.a/ \leq a.$$

Since π Ë $\boldsymbol{\mathcal{K}}_T$ and $\psi_X = \pi_X$ we have $\psi_{cb} = 1$. Also we have $\psi.\boldsymbol{\mathcal{K}}_T/ = 0$ and hence $\pi.\boldsymbol{\mathcal{K}}_T/ = 0$ which is a contradiction to the fact that π lives in $\boldsymbol{\mathcal{K}}_T$.

Remark 2. *The following result is a specific form of rectangular boundary theorem given above. Here we give an independent proof which directly uses the corresponding version of Arveson's boundary theorem and Lemma 2.*

Theorem 2. *Assume that X is an operator space in $\mathbb{M}_{n,m}.\mathbb{C}/$ such that* $\dim.X^{Í}/ = 1$. *If $\phi : X \rightarrow \mathbb{M}_{n,m}.\mathbb{C}/$ is a rectangular matrix state on $\mathbb{M}_{n,m}.\mathbb{C}/$ for which $\phi.x/ = x$, Åx Ë X, then $\phi.a/ = a$, Å a Ë $\mathbb{M}_{n,m}.\mathbb{C}/$.*

Proof. By Lemma 2, $\phi.X/^{Í} = [S.\phi/.S.X//]^{Í}$. Hence $\dim[S.\phi/.S.X//]^{Í} = 1$. Since $\phi.x/ = x$, Åx Ë X, we have $S.\phi/.y/ = y$, Å y Ë $S.X/$. Then by [9, Thoerem 4.2] , $S.\phi/.z/ = z$, Å z Ë $\mathbb{M}_{n+m}$. In particular,

$$\begin{bmatrix} 0 & \phi.a/ \\ 0 & 0 \end{bmatrix} = S.\phi/(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}/ = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix},$$

and hence $\phi.a/ = a$, Å a Ë $\mathbb{M}_{n,m}.\mathbb{C}/$.

4.2 Rectangular operator extreme states

We prove an important extension result in this section. In this section we prove that any rectangular operator extreme state on an operator space in a TRO can be extended to a rectangular operator extreme state on the TRO. Extension results in the same spirit concerning operator systems and UCP maps were proved by Kleski [12]. The following definition which appeared in [10, Definition 2.9] is important for our further discussion.
We begin by defining rectangular operator convex combination, and rectangular operator extreme states.

Definition 8. *Suppose that X is an operator space, and $\phi : X \rightarrow B.\mathcal{H}, \mathcal{K}/$ is a completely contractive linear map. A rectangular operator convex combination is an expression $\phi = \alpha_1^{¡}\phi_1\beta_1 + \bigtriangledown + \alpha_n^{¡}\phi_n\beta_n$, where $\beta_i : \mathcal{H} \rightarrow \mathcal{H}_{\rangle}$ and $\alpha_i : \mathcal{K} \rightarrow \mathcal{K}_{\rangle}$ are linear maps, and $\phi_i : X \rightarrow B.\mathcal{H}_{\rangle}, \mathcal{K}_{\rangle}/$ are completely contractive linear maps for $i = 1, 2, \bigtriangledown n$ such that $\alpha_1^{¡}\alpha_1 + \bigtriangledown + \alpha_n^{¡}\alpha_n = 1$ and $\beta_1^{¡}\beta_1 + \bigtriangledown + \beta_n^{¡}\beta_n = 1$. Such a rectangular convex combination is proper if α_i, β_i are surjective, and trivial if $\alpha_i^{¡}\alpha_i = \lambda_i 1$, $\beta_i^{¡}\beta_i = \lambda_i 1$, and $\alpha_i^{¡}\phi_i\beta_i = \lambda_i\phi_i$ for some λ_i Ë $[0, 1]$.*

A completely contractive map $\phi : X \rightarrow B.\mathcal{H}, \mathcal{K}/$ is a rectangular operator extreme state if any proper rectangular operator convex combination $\phi = \alpha_1^{¡}\phi_1\beta_1 + \bigtriangledown + \alpha_n^{¡}\phi_n\beta_n$ is trivial.

Rectangular operator states will be referred to as *rectangular matrix states* if the underlying Hilbert spaces are finite dimensional. The following theorem illustrates a relation between linear extreme states and and rectangular operator extreme states.

Theorem 3. *Let X_1 X_2 be operator spaces. If a completely contractive map $\phi : X_2 \rightarrow B.\mathcal{H}, \mathcal{K}/$ is linear extreme in the set $CC.X_2, B.\mathcal{H}, \mathcal{K}//$ of all completely contractive maps from X_2 to $B.\mathcal{H}, \mathcal{K}/$, and ϕ_{X_1} is rectangular operator extreme, then ϕ is a rectangular operator extreme state.*

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Proof. Let $\phi : X_2 \to B.\mathcal{H}, \mathcal{K}/$ be linear extreme and ϕ_{X_1} be rectangular operator extreme. Then $S.\phi/ : S.X_2/ \to B.\mathcal{K}\ \mathcal{H}/$ is linear extreme in $UCP.S.X_2/, B.\mathcal{K}\ \mathcal{H}//$. For, let ψ_1, ψ_2 Ë $UCP.S.X_2/, B.\mathcal{K}\ \mathcal{H}//$ and $0 < t < 1$ be such that

$$S.\phi/ = t\Psi_1 + .1 * t/\Psi_2.$$

Then $S.\phi/ * t\Psi_1$ is UCP, so by ([10], Lemma 1.11) there exists a completely contractive map $\phi_1 : X_2 \to B.\mathcal{H}, \mathcal{K}/$ such that

$$t\Psi_1 \left(\begin{bmatrix} \lambda & x \\ y^{¡} & \mu \end{bmatrix} \right) = w^{1˙2} \begin{bmatrix} \lambda I_{\mathcal{K}} & \phi_1.x/ \\ \phi_1.y/^{¡} & \mu I_{\mathcal{H}} \end{bmatrix} w^{1˙2},$$

where

$$w = t\Psi_1 \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = t \begin{bmatrix} I_{\mathcal{K}} & 0 \\ 0 & I_{\mathcal{H}} \end{bmatrix}.$$

So

$$t\Psi_1 \left(\begin{bmatrix} \lambda & x \\ y^{¡} & \mu \end{bmatrix} \right) = tS.\phi_1/ \left(\begin{bmatrix} \lambda & x \\ y^{¡} & \mu \end{bmatrix} \right).$$

Thus $\Psi_1 = S.\phi_1/$. Similarly $\Psi_2 = S.\phi_2/$ for some cc map $\phi_2 : X_2 \to B.\mathcal{H}, \mathcal{K}/$. Hence

$$\begin{aligned} S.\phi/ &= tS.\phi_1/ + .1 * t/S.\phi_2/ \text{ and therefore} \\ \phi &= t\phi_1 + .1 * t/\phi_2. \end{aligned}$$

Since ϕ is a linear extreme point, we have $\phi = \phi_1 = \phi_2$. Thus

$$\begin{aligned} S.\phi/ &= S.\phi_1/ = S.\phi_2/ \\ S.\phi/ &= \Psi_1 = \Psi_2. \end{aligned}$$

Thus $S.\phi/ : S.X_2/ \to B.\mathcal{K}\ \mathcal{H}/$ is a linear extreme point. Since $S.\phi/_{S.X_1/}$ is pure [10, Proposition 1.12], by [12, Proposition 2.2] $S.\phi/$ is a pure UCP map on $S.X_2/$. Hence ϕ is a rectangular operator extreme state on X_2.

Here we consider the set of bounded operators from X to $B.\mathcal{H}, \mathcal{K}/$ with the weak¡ topology called the bounded weak topology or BW-topology, identifying this set with a dual Banach space. In its relative BW-topology, $CC.X, B.\mathcal{H}, \mathcal{K}//$ is compact (see [3, Section 1.1] or [15, Chapter 7] for details).

Theorem 4. *Let X_1 X_2 be operator spaces. Then every rectangular operator extreme state on X_1 has an extension to a rectangular operator extreme state on X_2.*

Proof. Let ϕ Ë $CC.X_1, B.\mathcal{H}, \mathcal{K}//$ be a rectangular operator extreme state and let

$$\mathcal{F} = ˆ\psi \text{ Ë } CC.X_2, B.\mathcal{H}, \mathcal{K}// : \psi_{X_1} = \phi‘.$$

Then clearly $\mathcal{F}$ is linear convex, and BW-compact. We claim that it is a face. For let ψ_1, ψ_2 Ë $CC.X_1, B.\mathcal{H}, \mathcal{K}//$ and $0 < t < 1$ be such that $t\psi_1 + .1 * t/\psi_2$ Ë $\mathcal{F}$.
Then

$$t\psi_{1_{X_1}} + .1 * t/\psi_{2_{X_1}} = \phi$$

$$tS.\psi_1/_{.S.X_1/} + .1 * t/S.\psi_2/_{S.X_1/} = S.\phi/.$$

Since $S.\phi/$ is pure,

$$S.\psi_1/_{S.X_1/} = S.\psi_2/_{S.X_2/} = S.\phi/.$$

$$\psi_{1_{X_1}} = \psi_{2_{X_2}} = \phi.$$

$\Rightarrow \psi_1, \psi_2$ Ë $\mathcal{F}$ and hence $\mathcal{F}$ is a face. Thus $\mathcal{F}$ has a linear extreme point say $\phi^{Í}$ which is a linear extreme point of $CC.X_2, B.\mathcal{H}, \mathcal{K}//$. By Theorem 3, it follows that $\phi^{Í} : X_2 \to B.\mathcal{H}, \mathcal{K}/$ is a rectangular operator extreme state.

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Now, in view of the above results, extension of rectangular operator extreme state from an operator space to the generated TRO is immediate.

Corollary 2. *If X is an operator space and T is the TRO generated by X and containing X, then any rectangular operator extreme state on the operator space X can be extended to a rectangular operator extreme state on the TRO T.*

Proof. Follows directly from the Theorem 4 by taking $X_1 = X$ and $X_2 = T$.

4.3 Rectangular matrix extreme states

Here we take up the special case of finite dimensional rectangular operator states and show that they are isometric or unitary 'conjugates' of the identity map. Here we investigate the relation between matrix extreme states on operator spaces and commutants of images of operator spaces under rectangular matrix extreme states. For operator spaces in rectangular matrix algebras with trivial commutants, we deduce that rectangular matrix extreme states are certain 'conjugates' of the identity state.

Proposition 2. *If $\phi : X \to \mathbb{M}_{n,m}.\mathbb{C}/$ is a rectangular matrix extreme state on the operator space X, then* $\dim.\phi.X/^{\text{Í}}/ = 1$.

Proof. The commutant $\phi.X/^{\text{Í}}$ is a unital ¡-subalgebra of $\mathbb{M}_{n+m}.\mathbb{C}/$ and is therefore the linear span of its projections. Choose any nonzero projection $p\ \ q$ Ë $\phi.X/^{\text{Í}}$. Then $\phi = q\phi p + .I * q/\phi.I * p/$. Since ϕ is a rectangular matrix extreme point, we have $p^{\text{¡}}p = \lambda_1 I$, $q^{\text{¡}}q = \lambda_1 I$ and $.I * p/^{\text{¡}}.I * p/ = \lambda_2 I$, $.I * q/^{\text{¡}}.I * q/ = \lambda_2 I$ and $q\phi p = \lambda_1\phi$, $.I * q/\phi.I * p/ = \lambda_2\phi$, for some λ_1, λ_2 Ë $[0, 1]$. Thus $\lambda_1^2 = \lambda_1$ and $\lambda_2^2 = \lambda_2$. This gives $p = I$ and $q = I$. Therefore $\phi.X/^{\text{Í}} = \mathbb{C}I$. Hence $\dim.\phi.X/^{\text{Í}}/ = 1$.

Theorem 5. *Assume that X is an operator space in $\mathbb{M}_{n,m}.\mathbb{C}/$ and that* $\dim.X^{\text{Í}}/ = 1$.

1. *If $\phi : X \to \mathbb{M}_{r,s}.\mathbb{C}/$ is a rectangular matrix extreme state on X, then $r \leq n$, $s \leq m$ and there are isometries $w : \mathbb{C}^r \to \mathbb{C}^n$ and $v : \mathbb{C}^s \to \mathbb{C}^m$ such that $\phi.x/ = w^{\text{¡}}xv$, Å x Ë X.*
2. *A rectangular matrix state ϕ on X with values in $\mathbb{M}_{n,m}.\mathbb{C}/$ is rectangular matrix extreme if and only if there exist unitaries v Ë $\mathbb{M}_n.\mathbb{C}/$ and u Ë $\mathbb{M}_m.\mathbb{C}/$ such that $\phi.x/ = v^{\text{¡}}xu$, Å x Ë X.*

Proof. (1): Let $X\ \ \mathbb{M}_{n,m}.\mathbb{C}/$, and $\dim.X^{\text{Í}}/ = 1$. Let $\phi : X \to \mathbb{M}_{r,s}.\mathbb{C}/$ be a rectangular matrix extreme state. By Corollary 2, ϕ can be extended to a rectangular extreme state $\Phi : \mathbb{M}_{n,m}.\mathbb{C}/ \to \mathbb{M}_{r,s}.\mathbb{C}/$. Then $S.\Phi/ : S.\mathbb{M}_{n,m}.\mathbb{C}// \to B.\mathbb{C}^r\ \ \mathbb{C}^s/$ is pure. So by [12, Theorem 3.3] there exists a boundary representation $w : \mathbb{M}_{n+m}.\mathbb{C}// \to B.\mathcal{L}/$ for $S.\mathbb{M}_{n,m}.\mathbb{C}//$ and an isometry $u : \mathbb{C}^r\ \ \mathbb{C}^s \to \mathcal{L}$ such that

$$S.\phi/.y/ = u^{\text{¡}}w.y/u, \text{ for all } y \text{ Ë } S.\mathbb{M}_{n,m}.\mathbb{C}//.$$

By [10, Proposition 2.8] we can decompose $\mathcal{L}$ as an orthogonal direct sum $\mathcal{L} = K_w\ \ H_w$ in such a way that $w = S.\pi/$ for some irreducible representation $\pi : \mathbb{M}_{n,m}.\mathbb{C}/ \to B.\mathcal{H}_w, \mathcal{K}_w/$.

From the construction of $\mathcal{H}_w$ and $\mathcal{K}_w$, it follows that u maps $\mathbb{C}^r\ 0$ to $\mathcal{K}_w$ and $0\ \mathbb{C}^s$ to $\mathcal{H}_w$. By defining the maps u_1 and u_2 as $u_1.x/ = u.x\ \ 0/$, x Ë $\mathbb{C}^r$ and $u_1.y/ = u.0\ \ y/$, y Ë $\mathbb{C}^s$ we see that $u_1 : \mathbb{C}^r \to \mathcal{K}_w$ and $u_2 : \mathbb{C}^s \to \mathcal{H}_w$ are isometries such that $u = u_1\ u_2$. Then $\Phi.x/ = u_1^{\text{¡}}\pi.x/u_2$, Å$x$ Ë $\mathbb{M}_{n,m}.\mathbb{C}/$ and thus Φ is a compression of an irreducible representation of $\mathbb{M}_{n,m}.\mathbb{C}/$. Since every irreducible representation of $\mathbb{M}_{n,m}.\mathbb{C}/$ is unitarily equivalent to the identity representation [7, Lemma 3.2.3] we have that ϕ is a compression of the identity representation. That is, there are isometries $w : \mathbb{C}^r \to \mathbb{C}^n$ and $v : \mathbb{C}^s \to \mathbb{C}^m$ such that

$$\phi.x/ = w^{\text{¡}}xv, \text{ Å } x \text{ Ë } X.$$

Since v and w are isometries, we conclude that $r \leq n$ and $s \leq m$.

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(2) Let $\phi : X \to \mathbb{M}_{n,m}.\mathbb{C}/$ be a rectangular matrix extreme state. Then by part $.a/$, there are isometries (unitaries in this case) u Ë $\mathbb{M}_n.\mathbb{C}/$, v Ë $\mathbb{M}_m.\mathbb{C}/$ such that

$$\phi.x/ = v^{\text{¡}}xu, \text{ Å}x \text{ Ë } X.$$

Conversely let $\phi.x/ = v^{\text{¡}}xu$, Åx Ë X for unitaries u and v then

$$S.\phi/ \left(\begin{bmatrix} \lambda & x \\ y^{\text{¡}} & \mu \end{bmatrix} \right) = \begin{bmatrix} u^{\text{¡}} & 0 \\ 0 & v^{\text{¡}} \end{bmatrix} \begin{bmatrix} \lambda & x \\ y^{\text{¡}} & \mu \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}, \text{ Å } x, y \text{ Ë } X.$$

Then by [9, Theorem 4.3] $S.\phi/$ is pure. Hence ϕ is a rectangular matrix extreme state by ([10, Proposition2.12]).

The following result is now an immediate consequence of Proposition 2 and Theorem 5.

Theorem 6. *Let X $\mathbb{M}_{n,m}.\mathbb{C}/$ be an operator space. Then the inclusion map $i.x/ = x$, Åx Ë X, is a rectangular matrix extreme state if and only if $dim.X^{\text{Í}}/ = 1$.*

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/10/

SHAPLEY VALUES TO EXPLAIN MACHINE LEARNING MODELS OF SCHOOL STUDENT'S ACADEMIC PERFORMANCE DURING COVID-19

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ABSTRACT

In this work we perform an analysis of distance learning format influence, caused by COVID-19 pandemic on school students' academic performance. This study is based on a large dataset consisting of school students grades for 2020 academic year taken from "Electronic education in Tatarstan Republic" system. The analysis is based on the use of machine learning methods and feature importance technique realized by using Python programming language. One of the priorities of this work is to identify the academic factors causing the most sensitive impact on school students' performance. In this work we used the Shapley values method for solving this task. This method is widely used for the feature importance estimation task and can evaluate impact of every studied feature on the output of machine learning models. The study-related conditional factors include characteristics of teachers, types and kinds of educational organization, area of their location and subjects for which marks were obtained.

KEYWORDS

Data Science, Python, education, Machine Learning, Feature Importance.

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1. INTRODUCTION

Failure to achieve educational goals negatively affects society as a whole and is a serious problem. This problem can manifest itself most significantly during periods of drastic changes, one of which was the introduction of distance learning during the COVID-19 pandemic. To quantify the influence of this event on educational system, a variety of quantitative models based on modern statistical methods in combination with Big Data approaches can be used, as has shown in Li et al. [2021].

Machine learning (ML) is one of the new and actively developing methods of analysis, combining approaches that can "learn" based on the received data, which allows to perform a wide range of different tasks. ML can be used to solve problems of detection, recognition, prediction, prediction, diagnostics, and optimization.

A large number of huge datasets has been accumulated recently in educational system, which can be used to analyze and then improve educational process, as was demonstrated by Park [2020]. For example, Livieris et al. [2019] analyze a dataset consisting of performance of 3716 students in course of Mathematics of the first 5 years of secondary school. They develop two semisupervised machine learning algorithms to predict students' performance in the final examinations and then evaluate methods' accuracy. Authors compare these two methods with supervised machine learning method and as a result, these approaches outperform it, and the final accuracy exceeds 80%.

Jeslet et al. [2021] used well-known algorithms of machine learning Logistic Regression and Support Vector Machine to predict whether student is eligible to acquire a degree or not. Authors analyzed dataset of 1460 students' final year's results and obtained a model trained to 99.27% and 99.72% accuracy. Also, Nuanmeeseri et al. [2022] analyzed dataset of 1650 university students' academic performance. As a result, after adjusting model's parameters, authors achieved accuracy of 96.98%, so their model outperformed other considered machine learning methods and can be effectively used to evaluate significant academic performance factors in drastically changing period.

In our work, we study changes of academic performance of whole school grades in the framework of a variety of machine learning methods with the following feature importance analysis to identify significant parameters that affect academic performance the most after the introduction of distance learning format due to the COVID-19 pandemic.

2. MACHINE LEARNING METHODS AND FEATURE IMPORTANCE

2.1 MACHINE LEARNING TECHNIQUES

Hastie et al. [2009] introduce Machine learning as a set of mathematical techniques that give computer algorithms an ability to learn. This methodology is based on the input and required output of the algorithms and can automate the way how humans are able to carry out the task, as stated by Mnih et al. [2015].

Ensemble methods are groups of algorithms that use several machine learning methods at once and makes correction of each other's errors. Bostanabad et al. [2016] define supervised learning as a type of algorithms where the method is supplied with example inputs along with the required output, which then allows it to learn a rule that maps inputs to outputs. Bengio et al. [2013] state that in unsupervised learning, on the contrary, only the inputs are supplied, and the learning algorithm is required to determine the structure of the input and perform according to unknown characteristics [10].

In this work we use supervised machine learning methods: Decision Tree, Gradient Boosting, K-nearest neighbors (KNN) Regressor, Lasso Regression, Linear Regression and MultiLayer Perceptron neural networks, Support Vector Regressor; and ensemble method: Random Forest.

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In our study, we solved the regression task to predict Cohen's effect size, defined by Cohen [1988], based on subsets of school grades' marks in February and March, and April and May. Cohen's effect size measures the difference between mean values of two variables Cohen [1988].

2.2 SHAP FEATURE IMPORTANCE IMPLEMENTATION

Usually, machine learning models are difficult to interpret and it's hard to identify which features affect the output of the models the most. SHAP method (Shapley additive explanations) is one of the techniques used to solve this problem. This method is based on cooperative game theory, explained by Shapley [1953], and is used to increase transparency and interpretability of machine learning models. Absolute SHAP value shows us how much a single feature affected the prediction. SHAP values can represent the local importance of features and how it changes with lower and higher values, as shown by Sahakyan et al. [2021].

3. EXPERIMENTAL DATA DESCRIPTION

In this work, we study the influence of COVID-19 pandemic on school students' academic performance by analyzing a large dataset consisting of data from all schools in Tatarstan Republic, introduced by Ustin et al. [2022]. The dataset includes marks of entire grades of school students for main subjects for grades from 2 to 11.

During the preprocessing of original data, for the following analysis by machine learning methods, the initial dataset was modified into a new dataset consisting of features describing different parameters. These parameters included teachers' characteristics (age, sex, and educational category), mean mark of grade for February and March of 2020, school characteristics (location in or out of town, region of location, organization kind and type, subject). Data was filtered to consider school grades with at least 60 school grades in certain time periods (February and March, April and May 2020). For every row in dataset, Cohen's effect size was calculated. Figure 1 shows histograms for certain grades that represent whole dataset. It should be noted that most parameter values are positive, i.e., after the introduction of distance learning format, grades have generally increased.

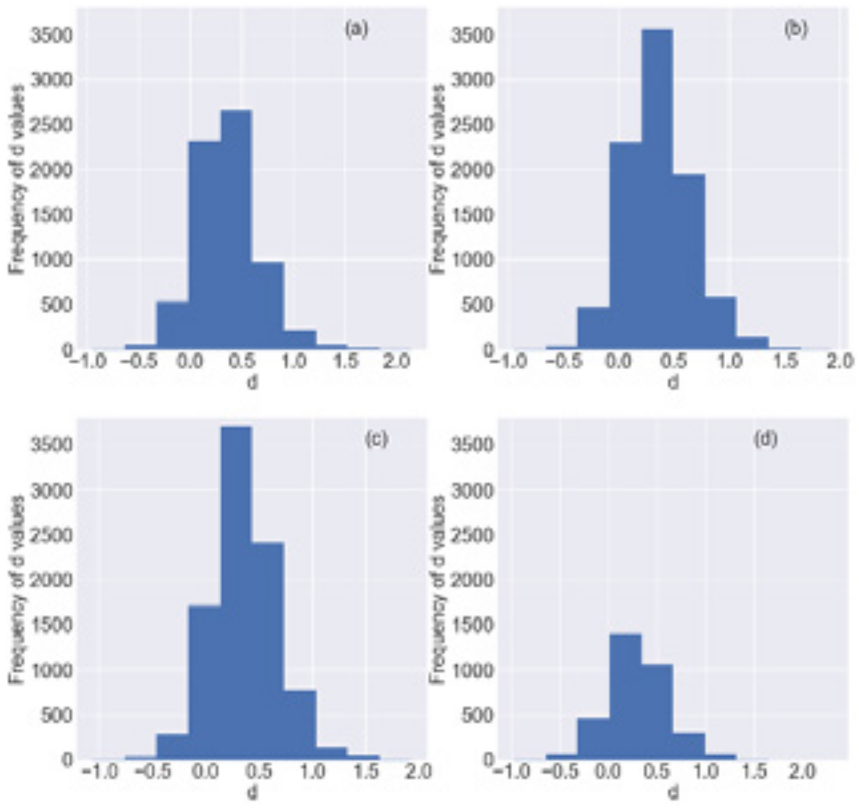

Fig. 1. Histograms of parameter d for: (a) 5th grade; (b) 7th grade; (c) 8th grade; (d) 11th grade.

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4. APPLICATION OF MACHINE LEARNING METHODS IN THE ANALYSIS OF THE SCHOOL STUDENTS' ACADEMIC PERFORMANCE

The analysis was performed in two stages. At the first stage, we implemented a variety of machine learning methods for a comparative analysis of machine learning methods in the regression problem of predicting the values of the Cohen's effect size based on a large set of features. At the second stage, we performed evaluation of the importance of explanatory variables in the predictive model.

4.1 MACHINE LEARNING METHODS IMPLEMENTATION

In our work we applied machine learning techniques realized in PyTorch and scikit-learn frameworks in Python. Among the applied methods: one-layer Linear regression and MultiLayer Perceptron realized in Pytorch; Decision Tree, Gradient Boosting, K-nearest neighbors algorithm, Lasso regression, Random Forest and Support Vector Regression realized in scikit-learn framework.

MultiLayer Perceptron consisted of the input layer, two hidden layers with 64 neurons and output layer with 1 neuron. We used ReLU as activation function, Adam as optimizer with learning rate equal to 0.00005 and Mean Squared Error (MSE) as loss function. Figure 2 shows the learning curve of one-layer Linear regression and MultiLayer Perceptron.

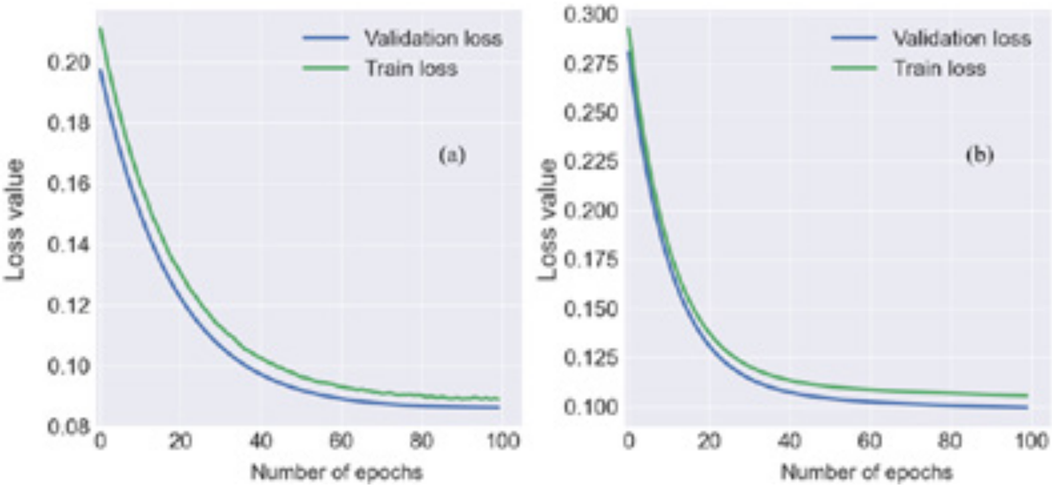

Fig. 2. Learning curve of: (a) MultiLayer Perceptron for 6th grade; (b) one-layer Linear regression for 6th grade.

Figure 3 shows the resulting plot of minimal MSE values for each method of machine learning for every studied school grade. It should be noted that the most precise algorithms are Random Forest, Lasso Regression, K-nearest neighbors and Support Vector Regression. Decision Tree and Gradient Boosting, on the other hand, have high values of error function. Also we obtained that for 8th and 10th grade values of MSE are increased significantly of the methods, and hence the Cohen's effect size values are more difficult to predict, while for 4th and 9th these values are decreased. Therefore, marks of students in 8th and 10th grade after the introduction of distant learning format due to the COVID-19 pandemic changed more randomly than the marks of students in other grades, especially in 4th and 9th grades.

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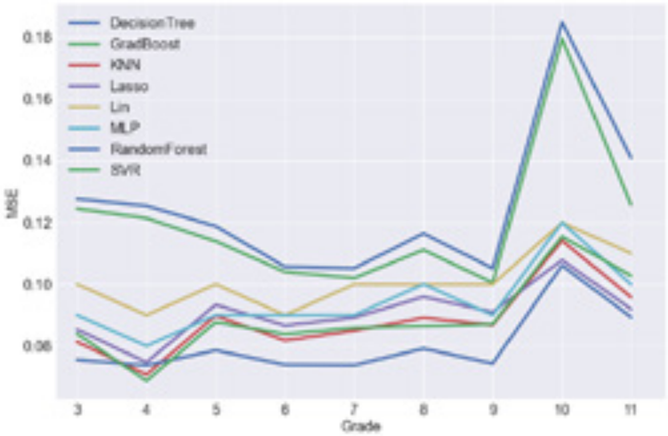
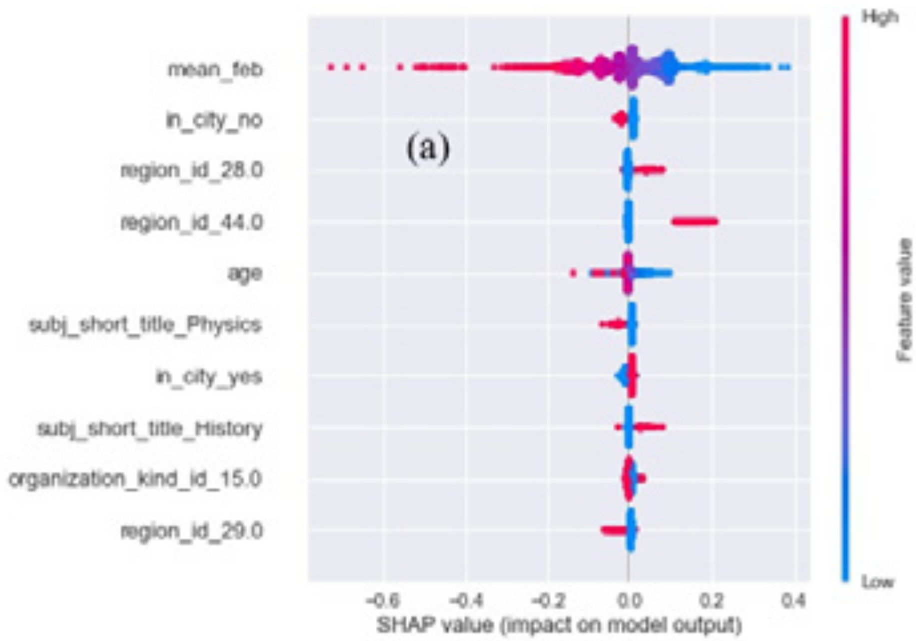


Fig. 3. The values of MSE for machine learning algorithms for each studied school grade.

4.2 EVALUATION OF THE IMPORTANCE OF EXPLANATORY VARIABLES

At the second stage of our analysis, we evaluated importance of our explanatory features for predicting values of Cohen's effect size. Figure 4 shows the distribution of Shapley values, i.e., influence on the value of parameter exerted by the studied explanatory features. Analysis was performed for Gradient Boosting, Random Forest and MultiLayer Perceptron models with primary Explainer and Tree Explainer.



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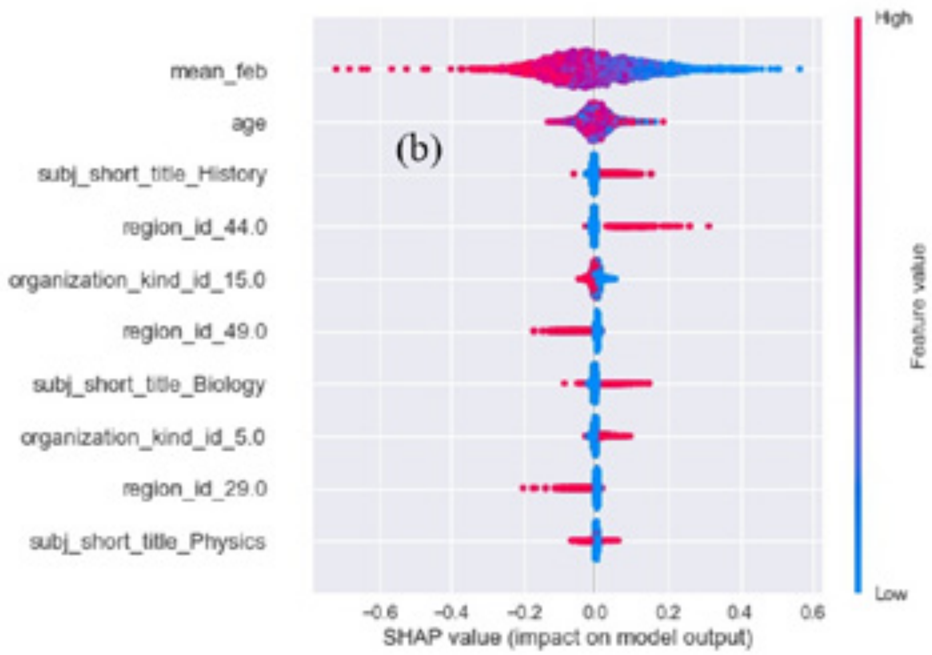

Fig. 4. The distribution of SHAP values (impact on parameter d) of explanatory features for predictive models: (a) for Gradient Boosting model with primary Explainer; (b) for Random Forest model with Tree Explainer. Cases with high values are shown in red, and those with low values are shown in blue. The variables are ranked in descending order. The horizontal location indicates whether the effect of that value is associated with a higher or lower prediction.

The main influence on the prediction of the Cohen's effect size value is exerted by the mean value of school grade in February and March, which obviously follows from the formula for the parameter . Also, significant contribution to the prediction of the parameter Cohen's effect size value is also made by the age of teachers: usually it is either not defined, or also negative (with an increase of age value, the value of the parameter decreases), which means that young teachers were more likely to give higher grades after introduction of distance learning format.

There exists also a significant improvement in school marks for the lessons of history, biology, while for such important subjects as physics, mathematics and Russian language, grades decreased after the introduction of distance learning. Besides that, location in certain regions: Naberezhnye Chelny, Kazan's Vakhitovsky, Novo-Savinovsky and Privolzhsky districts, also made significant positive contribution to the value of effect size . And in opposite, for schools located in Nizhnekamsk and Sovetsky district of Kazan, mean marks decreased significantly. Location of schools in the town also made positive contribution to the value of parameter d, while location outside of the town had a negative impact.

Besides that, different kinds of schools also played a special role as the used models features. The most significant influence was due to whether the educational organizations were secondary schools, lyceums, gymnasiums, or boarding schools. In case of lyceums, gymnasiums and boarding schools, the influence was strictly positive and increased the value of the Cohen's effect size , which means that after the introduction of distance learning into them, the marks of school grades increased. A different situation has developed in secondary schools: on average, the impact of the introduction of the distance learning format was mixed and did not affect academic performance in a certain way.

The influence of all the above factors may be explained by the fact that, depending on the characteristics of teachers, subjects taught and geographical location, the approach and time of transition to a new, previously practically unused format of education varied in different schools.

5. CONCLUSIONS

In this paper, we performed analysis of variation of academic performance in a large set of schools in Tatarstan Republic in the period before and during distance learning caused by COVID-19 pandemic.

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We used eight different machine learning methods to solve the regression task of forecasting value of Cohen's effect size . We determined the values of the error function corresponding to all applied algorithms and established school classes for which prediction is easier and the ones for which prediction is more difficult.
We discovered impact of age of teachers to the forecasting of parameter; lessons for which marks were more significant in the studied task and areas of Tatarstan Republic, location of school in which increased or decreased Cohen's effect size. Moreover, we discovered that the kind of educational organization also plays a special role in the forecasting task and identified the ones which had a significant impact on the value of Cohen's effect size. The impact of these study-related factors may indicate that different schools, school types and teacher had different periods of adaptation to a rapidly changing learning format, and these changes can be evaluated using feature importance method in combination with machine learning algorithms.
The results obtained during the research, after appropriate verification, may be used to evaluate the influence on academic performance of school students after introduction of distance learning.

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BENCHMARKING FOR RECOMMENDER SYSTEM (MFRISE)

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ABSTRACT

The advent of the internet age offers overwhelming choices of movies and shows to viewers which create need of comprehensive Recommendation Systems (RS). Recommendation System will suggest best content to viewers based on their choice using the methods of Information Retrieval, Data Mining and Machine Learning algorithms. The novel Multifaceted Recommendation System Engine (MF-RISE) algorithm proposed in this paper will help the users to get personalized movie recommendations based on multi-clustering approach using user cluster and Movie cluster along with their interaction effect. This will add value to our existing parameters like user ratings and reviews.

In real-world scenarios, recommenders have many non-functional requirements of technical nature. Evaluation of Multifaceted Recommendation System Engine must take these issues into account in order to produce good recommendations. The paper will show various technical evaluation parameters like RMSE, MAE and timings, which can be used to measure accuracy and speed of Recommender system. The benchmarking results also helpful for new recommendation algorithms.

The paper has used MovieLens dataset for purpose of experimentation. The studied evaluation methods consider both quantitative and qualitative aspects of algorithm with many evaluation parameters like mean squared error (MSE), root mean squared error (RMSE), Test Time and Fit Time are calculated for each popular recommender algorithm (NMF, SVD, SVD++, SlopeOne, Co-Clustering) implementation. The study identifies the gaps and challenges faced by each above recommender algorithm. This study will also help researchers to propose new recommendation algorithms by overcoming identified research gaps and challenges of existing algorithms.

KEYWORDS

Comparing recommender system, bench-marking recommendation system algorithms, comparing recommendation algorithms, challenges of various recommendation algorithms, Performance evaluation of Recommendation algorithms.

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1. INTRODUCTION

Availability of internet and global resources has increased number of availability of movies and shows which can be viewed by users. Recommendation Systems are tools used to give movie recommendations to the end-users based on their likes or likes of the similar users [1]. Recommender systems are good for both service providers as well as users. They reduce the time to find and selecting correct item on internet. A recommender system is an information filtering system which recommends the best movies to the user by considering some similarity between users or movies or user ratings for movies. The existing types of recommendation systems algorithms are Collaborative Filtering (CF) and Content-Based Filtering (CB).

Multiple well-known recommendation algorithms based on above categories are already proposed, KDD algorithm, SVD algorithms, SlopeOne and Co-Clustering algorithm. In this paper we have implemented and analyzed their comparative performance, as it can be used for benchmarking performance of our proposed multifaceted recommender system (MFRISE). This paper also explained the challenges and limitations of each algorithm. Such, challenges can be used to improve performance quality recommendations by modifying algorithm.

2. ARCHITECTURE OF MULTIFACETED RECOMMENDER SYSTEM (MFRISE)

The general recommendation system algorithm will use the mathematical function to suggest recommendations based on past similarity between users and movies [3]. The algorithm must be able to measure the usefulness of movie to user. In order to get good recommendations, we need lot of implicit and explicit data. Data coming from user ratings is acts as explicit data. Implicit data fetched from social data and watch history. The MFRISE is hybrid recommender system introduced by our paper which is used for Improvement Recommendations with help of identifying similar movies using content based (CB) filtering and perform multi-clustering and find the community impact on recommendations using text analytics,

Step 1 : Data Prepossessing

Step 2 : Similarity based recommendations

Step 3 : Clustering using Items similarity

Step 4 : Clustering using User similarity

Step 5 : Find Social impact on items

Step 6 : Multi-cluster & interactions

Step 7 : Ranking recommendations to user

Step 8 : Validation and testing

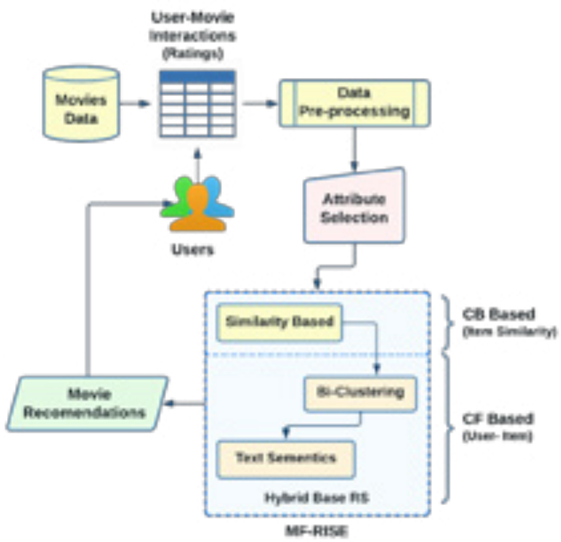

Fig. 2. MS-RISE Proposed Architecture.

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The detailed Implementation of Multifaceted Recommendation System Engine (MF-RISE) is included in upcoming paper on our proposed work. The proposal of method and

experimentation on benchmarking algorithms is proposed in this paper.

2.1 RECOMMENDER SYSTEM (RS) EVALUATION METHODS

The evaluation of recommendation system algorithm is not as easy as evaluation of any other machine learning algorithms, as the recommendation output for each user is different than other user [3]. This is main reason for which we cannot simply divide dataset into training and testing data. The methods used for evaluation RSs are,

a) Train - Test Split [6]: In RS algorithms it is not possible to take separate Training data set and testing data set, since the training data used to fit algorithm and test data set is used for evaluating RS algorithm. But, the user in training data may not be available in test data so, it is difficult to use separate test data. We have used masking method, rating values for some users are masked and then rating are predicted using algorithm then we can compare these ratings for checking accuracy.

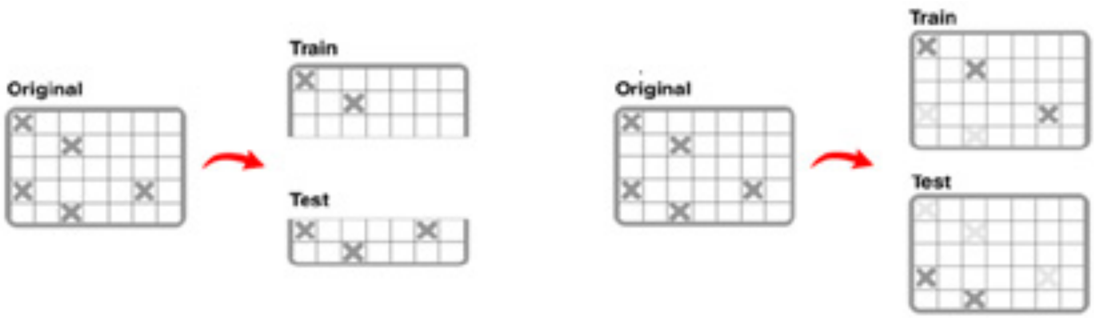

Fig. 3. Train-Test Split (masking).

b) K-Fold Cross Validation Method [7]: Cross-validation is a statistical method used to estimate the performance of RS algorithms or any machine learning algorithm.

- Dataset divided in k Groups
- Accept that group as Test Data
- Take remaining groups as Training Data
- Fit Model on Training Data
- Evaluate on Test Data
- Calculate Evaluation Score

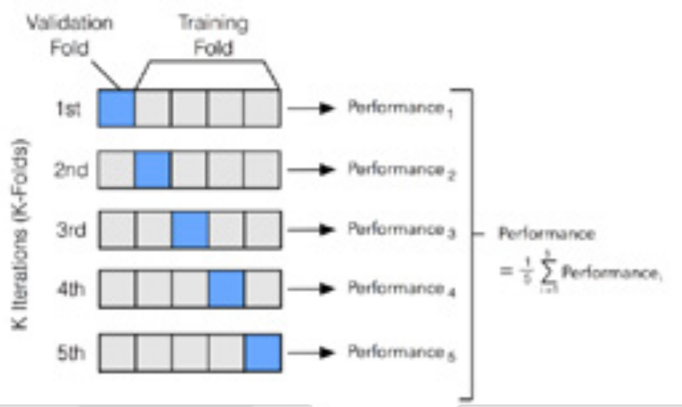

Fig. 4. K-Fold Cross Validation Concept.

3. BENCHMARKING RECOMMENDER SYSTEM (RS) ALGORITHMS

We usually categorize recommendation engine algorithms as collaborative filtering models and content-based models. In this paper, we are going to study and discuss few advantages and drawbacks of some popular recommender algorithms to compare their performances based on various evaluation

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metrics. This paper will set a benchmark for our proposed implementation of MF-RISE with below popular RS algorithms,

a) Similarity Based Algorithm

- Baseline algorithm

b) Neighborhood Algorithm

- K-Nearest Neighbors Algorithm (KNN)

c) Hybrid Methods

- Co-Clustering
- Slope-One

d) Matrix Factorization Method

- Single value Decomposition (SVD)
- Advanced SVD (SVD++)
- Negative Matrix Factorization (NMF)

4. EVALUATION METRICS FOR RECOMMENDER SYSTEMS

The most important thing for RS is to evaluate the performance of algorithm. The traditional algorithm evaluation metrics used to measure errors may not be effective for recommendation algorithms, as there are different recommendations for each user and no recommendations can be same for even same user. We need to take help of various traditional and modern methods for validating recommendation results.

A. Accuracy Metrics

Recommendation accuracy will measure difference between recommender's estimated ratings and actual user ratings.

a) Mean Absolute Error(MAE): [5] Absolute Error is the amount of error in prediction and actual rating.

$$\text{AbsoluteError} = |r_{ui} - \hat{r}_{ui}|$$

r_{ui} = Rating from the proposed algorithm
$\hat{r}_{ui}$ = Actual user Rating

The mean value of absolute errors can be given as Mean Absolute Error(MAE),

$$\text{MAE} = \frac{1}{|\hat{R}|} \sum_{r_{ui} \in \hat{R}} |r_{ui} - \hat{r}_{ui}|$$

b) Mean Square Error(MSE): [5] The measure of the average of the squares of the errors is called as Mean Square Error(MSE). MSE is not as small as MAE. MSE can be calculated as,

$$\text{MSE} = \frac{1}{|\hat{R}|} \sum_{r_{ui} \in \hat{R}} (r_{ui} - \hat{r}_{ui})^2$$

c) Root Mean Square Error(RMSE): [5] The Root Mean Squared Error(RMSE) is better in terms of performance when dealing with larger error values. RMSE is more useful when lower residual values

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are preferred. MSE is highly biased for higher values. Therefore, RMSE is more preferred accuracy measure.

B. Classification Metrics: Recommendation accuracy can also be measured by traditional precision and recall metrics. Recommended items has a high interaction value i.e. number of ratings, can be considered as most accurate predictions.

i) Precision: Precision is the ratio of true positives (number of relevant results) and total positives recommended items.

ii) Recall: A Recall is essentially the ratio of number of relevant items that are recommended to all relevant items.

C. Ranking Metrics [4]: Recommendation accuracy can also measure by Top-N results given by RS algorithm.

i) Hit Rate: The Hit occurs, if a user rated one of the top-10 recommended movie. So, first we find the Top 10 movie recommendations. then, we find movies rated by user. If user rates a movie which is already recommended, we consider that as one hit. Finally, ratio of Number of hits and total recommended movies is Hit Ratio.

ii) Miss Ratio: The Miss occurs, if a user rated movie not present in the top-10 recommended movie. If user rates a movie which is not recommended, we consider that as Miss. Finally, ratio of Number of Misses and total recommended movies is Miss Ratio.

D. Execution Time Metrics

Recommendation algorithm speed can be one of the important metrics, as we are dealing with very large set of data. The time required for algorithm to calculate the recommendation from input dataset is used as execution time. The time required for fitting algorithm is Fit Time and the time taken to run it on test data is Test time.

5. EXPERIMENTAL SETUP

We build experiments based on MovieLens datasets provided by Group Lens [11]. MovieLens datasets contain user ratings for multiple movies. The dataset contains 2113 users, 10197 movies and 855598 user ratings including tag assignments. The datasets contain only users that have rated at least 20 movies. They have conventional ratings which is preferred when predicting ratings. Since the Root Mean Squared Error(RMSE) and Mean Squared Error(MAE) values are depended on the rating scale, the results will be more comparable. We have used 5-Fold cross validation method for selecting training and testing dataset more effectively. The performance after each fold is analyzed and decided to work on 5 Folds for memory and time optimization. After 5-Folds, the performance is not improved considerably hence, decided to work with 5-Fold method [7]. The comparison of MovieLens datasets,

Table I. Comparisons of Datasets.

Dataset	HetRec [6]	ml latest [6]	100k [6]
Movies	10197	9742	1682
Users	2113	610	943
Ratings	855598	100836	1000000

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6. PERFORMANCE EVALUATION

A. Baseline Algorithms [12]

The similarity based algorithms are content based algorithms used for predicting a random rating based on the distribution, this algorithm assumes user ratings are normally distributed. The prediction is generated from a normal distribution N (μ, σ²) where μ and σ are estimated from the training data using Maximum Likelihood Estimation [3], If user u is unknown, then the bias b_u is assumed to be zero. The same applies for item i with b_i.

$$\hat{\mu} = \frac{1}{|R_{train}|} \sum_{r_{ui} \in R_{train}} r_{ui}$$

$$\hat{\sigma} = \sqrt{\sum_{r_{ui} \in R_{train}} \frac{(r_{ui} - \hat{\mu})^2}{|R_{train}|}}$$

Predicted Rating is,

$$\hat{r}_{ui} = b_{ui} = \mu + b_u + b_i$$

The best part of algorithm is simple implementation and useful for comparing algorithm accuracy. The points to improve is need more personalized predictions and less execution time for complex predictions. This algorithm also faces the problem of cold start for novice system users.

B. Matrix factorization Algorithm [16]

The Single Value Decomposition (SVD) is a Matrix factorization algorithm popularized by Simon Funk during the Netflix Prize. This is equivalent to Probabilistic Matrix Factorization algorithm. It Constructs a matrix with the row of users and columns of items and the elements are given by the users' ratings The singular value decomposition [15] is a method of decomposing a matrix into three other matrices.

$A = U\ S\ V^T$

Where A= m x n utility matrix

U= m x r rating singular matrix

The prediction r_{ui} is set as,

$$\hat{r}_{ui} = \mu + b_u + b_i + q_i^T p_u$$

If user u is unknown, then the bias b_u and the factors p_u are assumed to be zero.

The SVD is good for with few datasets and it can improve performance on many algorithms. It majorly uses the Principal component analysis(PCA) which is useful for dimensional reduction.

C. SVD++ Algorithm [20]

The Single Value Decomposition (SVD++) is extension of SVD algorithm, with considering implicit ratings This is equivalent to Probabilistic Matrix Factorization algorithm. It Constructs a matrix with the row of users and columns of items and the elements are given by the users' ratings.

The prediction r_{ui} is set as,

$$\hat{r}_{ui} = \mu + b_u + b_i + q_i^T \left(p_u + |I_u|^{-\frac{1}{2}} \sum_{j \in I_u} y_j \right)$$

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Where, the y_j terms are a new set of item factors that capture implicit ratings. Here, an implicit rating describes the fact that a user u rated an item j, regardless of the rating value. If user u is unknown, then the bias b_u and the factors p_u are assumed to be zero.

D. Non-Negative Matrix Factorization(NMF) [22]

The Non-Negative Matrix Factorization(NMF) is equivalent to Non negative Matrix Factorization algorithm. It Constructs a matrix with the row of users and columns of items and the elements are given by the users' ratings.

The NMF algorithm can improves performance on many algorithms. NMF based methods used in for solving problems in computer vision. The computational complexity of CF based algorithm is very high and it results in many missing ratings. Some major improvements are required to achieve high computational efficiency and prediction accuracy.

E. Co-Clustering Algorithm [25]

A Co-Clustering is based on collaborative filtering algorithm. This approach is based on simultaneous clustering of users and movies (items) for efficient CF based algorithm.

In Co-clustering method, every users and movies are assigned some clusters C_u, C_i, and some co-clusters C_{ui} The prediction r_{ui} is set as,

$$r_{ui} = C_{ui} + (\mu_u - C_u) + (\mu_i - C_i)$$

Where, If the user is unknown, the prediction is $r_{ui} = \mu_i$, If the item is unknown, the prediction is $r_{ui} = \mu_u$, If both are unknown, the prediction is $r_{ui} = \mu$

The co-clustering algorithm has good control over learning and can consider multiple dimensions of data. but, needs more execution time in few cases for some critical recommendation. The cold start issue become major issue in this algorithm.

F. Slope One Algorithms [27]

Slope One algorithm is based on the movie-user rating matrix based on the linear model y=xb+c. Where, parameter y is the rating of the predicted target user on the target movie, parameter x is the rating of the target user on the reference movie, and parameter b is the deviation value of the user's score of different movies. Slope One algorithm calculates enter of the evaluation of excessive user ratings mean the score difference between the movies, and then at the time of target users recommend, uses the linear relationship, estimate the prediction score of the movies y according to the target user's score of project x and the deviation value b , that is, generate the prediction by using the deviation value of all users among different movies. Slope One algorithm is simple in calculation and having good performance. It can handle cold start issue well by predicting ratings. But, the fit time will be higher as compare to other algorithms.

7. RESULTS

All experiments are run on a Desktop with Intel Core i5 8th gen (CPU@2.30GHz) and 8GB RAM, all data stored on solid state Memory (SSD) for faster access and optimum performance. In this paper, we present the various evaluation parameter like average RMSE, MAE and total execution time of various algorithms (used in study) with a 5-fold cross-validation procedure.

Table II. Evaluation Metrics for Benchmark Algorithms.

Parameter	RMSE	MAE	FIT TIME	TEST TIME
BaseLine	1.52	1.2224	0.9	0.13
KNN	0.9793	0.7736	0.41	2.32
SVD	0.9379	0.7394	3.97	0.13
CO-CLUSTERING	0.9654	0.7555	1.41	0.09
SLOPEONE	0.9434	0.74527	0.55	1.71

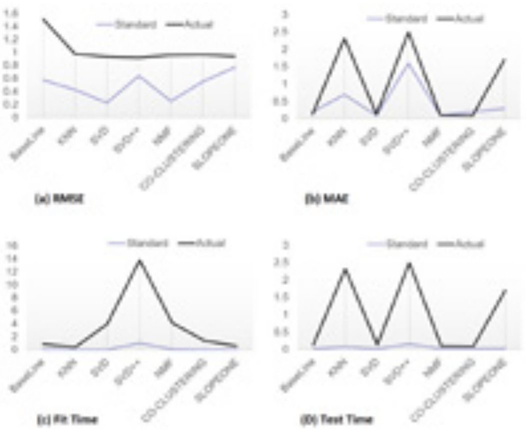

Fig. 15. Benchmark Algorithms Performance Analysis.

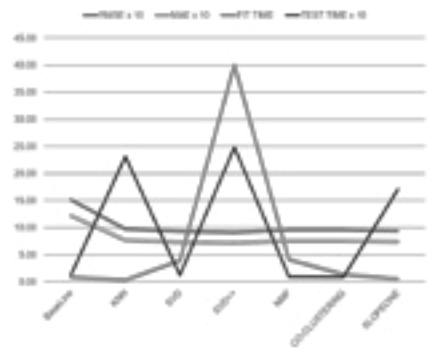

Fig. 17. Comparative Performance Analysis.

8. CONCLUSION AND FUTURE WORK

In this paper, we present a real-world benchmark for our new recommendation system algorithm. The prediction accuracy of a recommender system is dependent on various parameters. In study, we have seen all algorithms are optimized for the MovieLens dataset. SVD, NMF and co-clustering algorithm performs better on the larger dataset than the other collaborative filtering algorithm. To obtain more detailed results, testing algorithms on datasets with more similar properties can be performed.

We have deployed many important RS algorithms to study their performance comparisons, which was ubiquitous and crucial in recommendation scenarios. After comparing all algorithms, we found that SVD++ algorithm need highest Fit Time due to complexity of calculations. There is lower RMSE calculations in all algorithms except Baseline algorithm. Overall execution time of SVD algorithm and co-clustering algorithm is very lower. So, we are planning to can plan to use SVD, NMF and Co-Clustering algorithms for efficient implementation of movies recommendation process.

We can conclude that the SVD, NMF and Co-Clustering algorithm is seemingly more accurate than other the Item-based collaborative filtering algorithm for larger datasets.

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RICE QUALITY ANALYSIS USING IMAGE PROCESSING AND MACHINE LEARNING

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ABSTRACT

Object Detection and its analysis are used in various fields. Rice quality evaluation subtask in Agricultural industries is not exception for object Detection. Manual identification using image processing techniques, Machine Learning Techniques and Deep Learning is also used for the rice quality analysis. Due to Feature identification challenge machine Learning and Deep Learning are in the demand. As rice is mostly used agricultural product so it is important to have the proper analysis of the crops. In this study we proposed the used of image processing method with the help of Machine Learning model. Rice grain morphological characteristics are what define a grain's quality analysis. The suggested method can operate efficiently with little expense.

KEYWORDS

Object Detection, Machine Learning, Morphological characteristics, Deep Learning.

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1. INTRODUCTION

The oldest and largest sector of the global economy is agriculture. Traditionally, a human sensory panel uses physical and chemical features of food products to determine their quality (Mahale and Korde, 2014)(Shah, Jain, and Maheshwari, 2013). In Asian nations, rice is a popular and widely eaten cereal grain. It is widely accessible anywhere in the world. When rice is used for human consumption, several items with value added are created. Quality is a major factor in the milled rice business. With the growth of the import and export industries, quality assessment becomes increasingly crucial. The dispensable items found in rice samples include paddy, chaff, broken grains, weed seeds, stones, etc. These impurity levels affect the quality of the rice. As a complex problem it is solved by using image processing techniques. There have been major advancements in the essential and cutting-edge technology field of image processing such as canny edge detection algorithm (Mahale and Korde, 2014), Artificial Neural Network(Hamzah and Mohamed, 2020).

The approach of image processing is intended to preserve the integrity of the specifications. Image manipulation involves applying certain procedures to a target image in order to produce a better and more appealing image. And extract some useful data from the supplied image. Genetic algorithm based LS-SVM(Chen, Ke, Wang, Xu, and Chen, 2012) was used which provides good result but required lots of processing and complex operation. Later machine Learning based algorithms are used for this classification and Analysis. ANN(Chen et al., 2012), SVM(Philip and Anita, 2017) is mostly used algorithm which provides the good quality of results.

The main purpose of the proposed method is, to offer an alternative way for quality control and analysis which reduce the required effort, cost and time by using other Machine Learning algorithms and with object Detection. As rice quality analysis is control the diet and business of agriculture industry, proper analysis is required. Image Processing Techniques and Machine Learning are tested for the analysis. In this work we apply the object detection machine learning algorithm Region-based Convolutional Neural Networks (R-CNN) with dimension reduction techniques. Results depict that the results are less difference in the above methods.

This paper is organized as follows, and Section II describes the work of various researchers on Rice quality analysis or detection. Section III gives a detailed overview of the technology used to select good Rice and analyse it. Section IV uses this method to elaborate on the results. Section V provides an overview of the results description, better performance than other results, and shortcomings.

2. LITERATURE REVIEW

Philip and Anita (2017), proposed new characteristics for rice grain categorization. For the categorization of nine types of commercially accessible grains in the South Indian area, both spatial and frequency-based criteria were applied. The classification is carried out in two phases, with the first stage utilising the NB Tree classifier and the second stage utilising the SMO classifier.

Authors archives remarkable accuracy to the spatial features and suggested two work on real time. Images. Parveen, Alam, and Shakir (2017) proposed image processing algorithm based some characteristics with colour images. Characteristics wised results are obtained to user. Author suggested applying the same with large data with more feature or characteristics. Kuchekar and Yerigeri (2018) attempted to grade rice grains using image processing and morphological methods. Segmenting the individual grains comes first, followed by pre-processing of the picture.

The grain's geometrical characteristics, such as its area and the lengths of its main and minor axes, are extracted and classified. The results have been determined to be positive. Rice is graded according to the length of the grain. As a future scope it can be expanded in the future by focusing on moving images and identifying additional qualities of rice grains. Kong, Fang, Wu, Gong, Zhu, Liu, and Peng (2019) suggested to use an automated approach for extracting rice thickness based on edge properties.

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The solution addressed the issue brought on by the structural similarity of edges by matching the appropriate edge points based on the form of the edge rather than its texture. As a future scope authors suggested to go with edge features extraction, with epipolar geometry to match the corresponding points on the rice edge.

Avudaiappan and Sangamithra () analyse the visual features with image processing and MLP.

Authors used the SVM and Na¨ıve bays algorithm with 90% accuracy. As a future scope authors focused on Non-Uniform Illumination with Transformation using top-hat so that rice can be classify in long, normal or small category. Using a k-NN classifier, Wah, San, and Hlaing (2018) suggested an image processing method and assessed three classes (30 images for each class).

Other studies Xiaopeng and Yong (2011), Yao, Chen, Guan, Sun, and Zhu (2009), Tahir, Hussin, Htike, and Naing (2015) put more focus on identifying the grain's apparent chalkiness. A grain with a partly opaque or milky white kernel is said to be chalky. One of the key markers in the evaluation process is the level of chalkiness. High levels of chalkiness in rice grains make them more likely to shatter during milling, which will alter how they taste. An automated system for grading milled rice is suggested by Wyawahare, Kulkarni, Dixit, and Marathe (2020).

Broken rice is an essential factor in rice grading. This technique may be used to determine the percentage of broken rice from a sample's picture. The relevant characteristics are retrieved from the coloured pictures of the samples using particular preprogrammed processing procedures, and the regression model is created. Estimating the percentage of broken rice requires less runtime than other approaches since basic regression models are used. Lin, Li, Chen, and He (2018) offered a comparison of two approaches—CNN and conventional methods—to identify rice grains with three distinct forms (medium, round, and long grain). 5,554 photos were examined for calibration, and 1,845 images were examined for validation. In the CNN approach, the experiments changed training parameters like batch size and epochs. In a separate trial, they used conventional statistical techniques, and the categorization accuracy they obtained ranged from 89 to 92%. As opposed to the conventional approaches, the experiment employing the CNN method obtained a classification accuracy of 95.5%. Benefits from the interplay between CNN and hyperspectral imaging were employed by Chatnuntawech, Tantisantisom, Khanchaitit, Boonkoom, Bilgi¸c, and Chuangsuwanich (2018). Their research used two sets of data, 414 samples from paddy rice and 232 samples from six different types of milled rice. The accuracy of the suggested approach was 86.3%. In contrast, the SVM method on the paddy seed dataset produced a result of 79.8%, whilst the accuracy of the other set was somewhat off. By combining three machine learning techniques—kNN, SVM, and CNN—with hyper spectral imagery, Qiu, Chen, Zhao, Zhu, He, and Zhang (2018) were able to identify four different types of rice seeds. Two distinct spectral ranges were used in the experiment, and there were various numbers of training samples.

In various studies, a hyperspectral camera was used to address the issue of classifying rice types. The gadget, however, was expensive and complicated. Additionally, a quick computer, sensitive detectors, and ample data storage were needed.

3. PROPOSED FRAMEWORK

By observing the various literature we Identified architecture. Process is as follows.

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Fig 1: Architecture of the proposed system.

1. Training the data: To train the data Rice seeds image scanning needed then it Seed area segmentation is done then Seed orientation was performed then proper data frame of images is form for training of data.
2. Training of data Training is done for both RCNN and statistical classical model.
3. Feature Extraction: For better accuracy we have to find the Features of the data frames. We used Greedy Filter method to get the proper features.
4. Again training was performing and tests the accuracy on both RCNN and statistical classical model.
5. Dimension Reduction: to speed the process at each epoch we reduce the dimension by using PCA.

4. RESULTS

The below table gives average aspect ratio and classification which is based on kind of rice grain used. It will show the exact value of parameters for the rice grain used in a bar graph where x-axis belongs to particles and y-axis is average aspect ratio of parameters. Following Figure shows the ranges which generally known as classification of rice grains that have been identified and training was done.

Table 1: classification of rice grains.

Long Slender(LS)	Length >= 6mm, L/B Ratio>=3mm
Short slender(SS)	Length< 6mm, L/B Ratio>=3mm
Medium Slender (MS)	Length >= 6mm, 2.5< L/B Ratio <3mm
Long Bold (LB)	Length >= 6mm, L/B Ratio<3mm
Short Bold (SB)	Length < 6mm, L/B Ratio<3mm

As per Architecture we have applied RCCN object detection method for identification of edges of the rice so that classification can be done by training machine learning model. Here we train it using Recurrent neural network (RNN) with activation function as Relu in hidden layers which can be helpful for RCNN. The Rice quality analysis of application b uisng this application is shown below.

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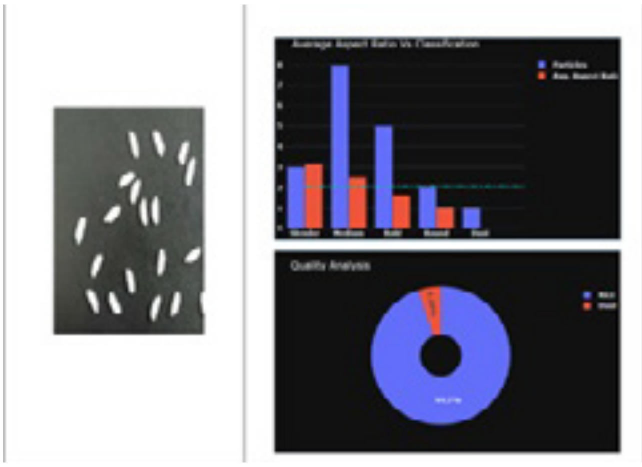

Fig 2: Quality Analysis Avg. Aspect Ratio VS Classification with single proper image.

5. CONCLUSION AND FUTURE SCOPE

In this work we applied machine learning and image processing techniques for identification and rice quality analysis of work. In the process we used RNN model as machine: earning model for classification. To detect the edge and identify its type RCNN object detect ion method where used. Accuracy of the model is 92.36% and analysis was done on the rice grace classification. We have successfully executed all the steps proposed. Last two steps include calculating the size of the grains and then classifying them according to the Table provided. As a future challenge we can try with another edge Detection algorithm which detect in less time so that above accuracy will be increased and reduced the time of analysis.

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RFM ANALYSIS FOR CUSTOMER SEGMENTATION USING MACHINE LEARNING: A SURVEY OF A DECADE OF RESEARCH

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ABSTRACT

Customer segmentation is a method of categorizing corporate clients into groups based on shared characteristics. In this study, we looked at the different customer segmentation methods and execute RFM analysis by using various clustering algorithms. Based on RFM values (Recent, Frequency, and Cost) of customers, the successful classification of company customers is divided into groups with comparable behaviors. Customer retention is thought to be more significant than acquiring new clients are analyzed on two different databases. Results show the significance of each method. Comparison is helps for selection of better customer segmentation.

KEYWORDS

Clustering, Classification, RFM, Customer segmentation.

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1. INTRODUCTION

Customer Segmentation is way of organization of customers with respect to the various features. In recent years there has been a huge boom in opposition between companies to stay in the field. The income of the organization may be stepped forward through a patron segmentation model. According to the Pareto principle (Srivastava, 2016), 20% of the clients make a contribution greater to the sales of the organization than the relaxation Customer segmentation is the exercise of dividing an organization's clients into agencies that mirror similarity amongst clients in every group. The intention of segmenting clients is to determine how to narrate to clients in every section that allows you to maximize the fee of every patron to the business. Customer segmentation has the ability to permit entrepreneurs to cope with every patron withinside the simplest way. Using the massive quantity of statistics to be had on clients (and ability clients), a patron segmentation evaluation lets in entrepreneurs to identify discrete agencies of clients with an excessive diploma of accuracy primarily based totally on demographic, behavioral and different indicators.

Evaluation of RFM (Recency, Frequency, and Monetary) is a famend approach is worn for comparing the clients primarily based totally on their shopping for behavior. Scoring method was developed to test Recent, Frequency, and Finance ratings. Finally, ratings of all three variables are strengthened as RFM ratings from different ranges (Haiying and Yu, 2010) which are compiled to anticipate recants trends for studying existing and higher sponsor transactions history. Next step is defined as the remaining time the consumer buys. The latest currency is the type of days the sponsor takes between purchases. The latest small payment means that the sponsor visits the organization frequently in a timely manner. Similarly, extra money means that the sponsor is less likely to go to the organization soon. Frequency is described because the variety of transaction a patron makes in a selected period. The better the fee of frequency the greater unswerving are the clients of the organization.

Cash is defined as the amount spent by the investor over a period of time in a favorable period. The improvement in the amount of money spent by the large sales they provide to the organization. Each sponsor is given 3 different ratings of the latest, frequency, and economic volatility. Score points are used within a range from five to 1. The core quintile is given a five-point scale, while the others are given 4, 3, 2 and 1.

In recent years, there has been a significant increase in the number of opposition groups among companies in care within the arena. Customer retention is more important than purchasing the latest customers. Customer segregation allows people's messages to speak more to target audiences.

2. LITERATURE REVIEW

Segmentation is middle of the advertising and marketing approach due to the fact exclusive consumer organizations mean the want for exclusive advertising and marketing mixes primarily based totally on consumer conduct and its needs. Many authors give the segamentaion methods to increase the profit and sustain the company position. (Jiang and Tuzhilin, 2009) proposed K-Classifiers Segmentation algorithm which recognized that each client segmentation and consumer focused on are important to enhance the marketing performances. K-Classifiers Works as optimizer who have two tasks. Above method more resources to the ones clients who supply greater returns to the company. (He and Li, 2016) proposed a 3-dimensional approach to improving consumer health (CLV), customer pride and customer behavior. The authors conclude about the customers and the requirements for a better service. A segment used to meet customer expectations and suggest better service. (Sheshasaayee and Logeshwari, 2017) used RFM Analysis which provides the usage of CRM (Customer Relationship Management). Authors analyzed the customers by segmenting them which helps to increase company profits. Further they enhanced the segmentation by using Fuzzy Clustering Method which classified them into the appropriate scoring strategies based on their needs.

(Shah and Singh, 2012) provides the K-means algorithm and K-medoids algorithms for clustering.

The presented techniques do not always yield the best answer, but they do minimise the cluster error criterion. They came to the conclusion that when the number of clusters grows, the new method takes less time to run than existing methods. (Sheshasaayee and Logeshwari, 2017) developed hybrid method which combine RFM and LTV methods. Authors used K-means and Neural Network algorithms for segmentation with two phase models. They suggested having better optimizer for customer categorization. Using logistic regression, (Liu, Chu, Chan, and Yu, 2014) proposed predicting customer attrition. Individual marketing methods can be used to identify customers with similar churn value and to keep them. Benefit customer segmentation using various methodologies allows customers to be classified based on their relationships, allowing marketers to focus their marketing efforts on their strengths and target benefit categories accordingly.

3. TYPES OF CUSTOMER SEGMENTATION

Customer segmentation models come in a range of shapes and sizes, ranging from simple to complex, and they can be used for a variety of purposes. Demographic, Recency, Frequency Monetary (RFM), High-Value Customers, Customer Status, Behavioral, and Psychographic models are some of the most common models.

3.1 DEMOGRAPHIC

It is a method of segmenting customers based on characteristics such as age, gender, ethnicity, income, education, religion, and career (Lu, Lin, Lu, and Zhang, 2014).

3.2 RFM

It is a direct segmentation strategy whose main goal is to categories clients based on the time since their previous purchase, the total number of purchases they've made (frequency), and the amount they've spent (monetary) (Sheshasaayee and Logeshwari, 2017).

3.3 HVCS (HIGH-VALUE CUSTOMER)

It's an extended RFM segmentation for any firm, focusing on what traits they have in common so you can get more of them.

3.4 CUSTOMER STATUS

It is a mechanism which check the status of customer which categories as active and lapsed. The focus of this method is to how the customer engaged by the company on the time period as a status.

3.5 BEHAVIORAL SEGMENTATION

It is a mechanism which check the status of customer which categories as active and lapsed. The focus of this method is to how the customer engaged by the company on the time period as a status.

3.6 PSYCHOGRAPHIC SEGMENTATION

It is allows grouping the customers based on attitudes, beliefs, or even personality traits. For this we require good data analysis method. Analysis done on all above attributes.

3.7 GEOGRAPHIC SEGMENTATION

It is allows grouping the customers based on geographical location i.e region, city, country etc. It is used when target area is location wise improvement of services and increased the profit.

4. MAJOR CLUSTERING TECHNOLOGIES FOR CUSTOMER SEGMENTATION

4.1 K –MEANS

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It is a popular unsupervised method that accepts parameters and k value as number of clusters as inserting and separating data into clusters with high intra-cluster similarities. K-Means is a method that repeats itself, adding the number of centroids before each multiplication. Depending on the inches calculated for each multiplication, data points are allocated among distinct sets. Using min max normalization, the RFM values are normalized (Lee and Memon, 2016).

4.2 FUZZY C-MEANS

It is a Method (?, ?) that allows a specific piece of data to appear to numerous clusters. It no longer determines a cluster's club records for a given information factor. Rather, the probabilities of a specific information factor with similarities are determined. The advantage of this method over previously discussed K-Means is that the final result obtained of a large and comparable database is most suitable than a set of K-method rules, because in the KMeans method, as cluster formation of based on data element. The normalization in this approach is done using min max normalization. Cluster RFM value is based on cluster (Zahrotun, 2017) value.

4.3 REPETITIVE MEDIAN K-MEANS

It's a novel approach to determining the initial centroids for the K-Means method. The traditional K–Means algorithm's range of iterations and computational time is reduced by choosing preliminary centroids with it's proposed distribution. RFM values will be combined and sorted into three vectors, R', F', and M ', respectively. The initial centroids are calculated using the median value of each vector. The median values are derived k times iteratively from the R', F', and M' values, depending on the value of k. (number of segments).

5. RFM ANALYSIS USING ON ONLINE SHOPPING DATA

RFM: according to following definitions. The RFM method divides clients into segments. It categorises clients based on their previous purchase transactions, taking into account criteria like as Recency (R): Last purchase date in specified session. Frequency (F):Purchase count in the specified session. Monetary (M):Value of Purchase in the specified session Based on forms needs, we define a season with different intervals for this model and we calculate RFM values for each customer. Working with a set of customer activity data in an online retail store year-round from the University of California Irwin (UCI) repository was used to evaluate system performance.
The following is a sample customer separation process. STEP 1: Sort the customer by recency.
STEP 2 AND 3: Sort the customer from most to least frequent customer and summarized the F, M score. STEP 4: Rank customers by combining R,F, and M ranking

CUSTOMER ID	RECENCY (DAY)	FREQUENCY (NUMBER)	MONETARY (TOTAL)
1	4	6	540
2	6	11	940
3	46	1	35
4	23	3	65
5	15	4	179
6	32	2	56
7	7	3	140
8	50	1	950
9	34	15	2630
10	10	5	191
11	3	8	845
12	1	10	1510
13	27	3	54
14	18	2	40
15	5	1	25

Fig1: Sample Data.

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CUSTOMER ID	RECENCY	RANK	R SCORE
12	1	1	5
11	3	2	5
1	4	3	5
15	5	5	4
2	6	5	4
7	7	6	4
10	10	7	3
5	15	8	3
14	18	9	3
4	23	10	2
13	27	11	2
6	32	12	2
9	34	13	1
3	46	14	1
8	50	15	1

Fig2: Calculation of Recency.

CUSTOMER ID	FREQUENCY	F SCORE
9	15	5
2	11	5
12	10	5
11	8	4
1	6	4
10	5	4
5	4	3
13	3	3
7	3	3
4	3	2
14	3	3
6	2	2
15	1	1
8	1	1
3	1	1

CUSTOMER ID	MONETARY	M SCORE
9	2630	5
12	1510	5
8	950	5
2	940	4
11	845	4
1	540	4
10	191	3
5	179	3
7	140	3
4	65	2
6	56	2
13	54	2
14	40	1
3	35	1
15	26	1

Fig3: Calculation of F and M Score.

CUSTOMER ID	RFM CELL	RFM SCORE
1	5,4,4	4.3
2	4,5,4	4.3
3	1,1,1	1.0
4	2,2,2	2.0
5	3,3,3	3.0
6	2,2,2	2.0
7	4,3,3	3.3
8	1,1,5	2.3
9	1,5,5	3.7
10	3,4,3	3.3
11	5,4,4	4.3
12	5,5,5	5.0
13	2,3,2	2.3
14	3,2,1	2.0
15	4,1,1	2.0

Fig4: Customer Segmentation.

6. ANALYSIS OF VARIOUS ALGORITHMS

For evaluation of commonly used clustering algorithm was done on two different open source databases like the transactional data set of the customers of an online retail store available at UCI repository and on e-commerce datasets which is available at UCI Machine Learning Repository.

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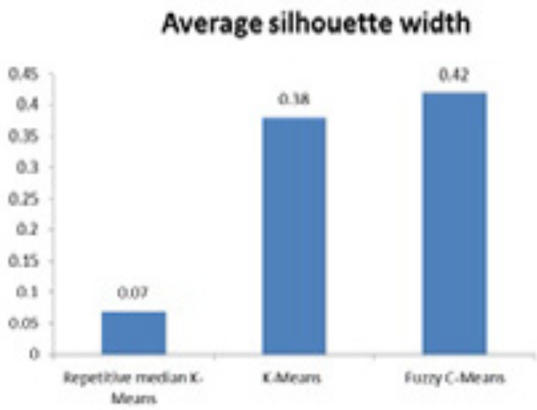

Fig5: Result Analysis of various Algorithms on online retail.

From above analysis it is observed that Fuzzy C means perform well on the basis of average Silhouette width but the time taken is more and number of iterations are also more as compared to others algorithms.

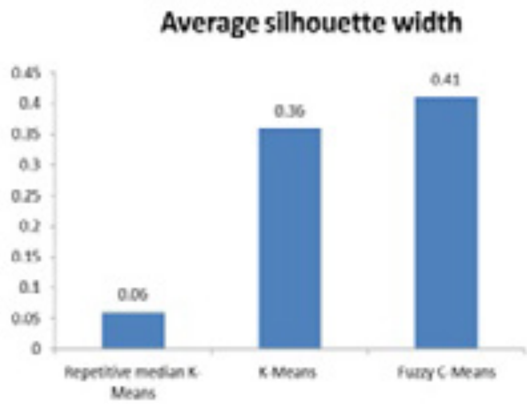

Fig6: Result Analysis of various Algorithms on E- commerce data.

7. CONCLUSION AND FUTURE SCOPE

This paper provides the overview of customer Segmentation and its different type's with types. RFM segmentation allows to group based on the requirements and target the different marketing strategies. In future more complex methods would be designed to target specific customers and the methods would also be more flexible if the company wants to target a different audience for a particular time or wants to permanently change their customers based on the needs of the company or priorities of the customer. The RFM should be made more flexible according to the needs of the different companies.

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/14/

VIRTUAL EMOTION DETECTION BY SENTIMENT ANALYSIS

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ABSTRACT

As websites, social networks, blogs, and online portals proliferate on the internet, authors are producing reviews, opinions, ideas, ratings, and feedback. The emotional content of this writer may be about things like books, people, hotels, items, studies, events, and so on. These emotions have great value for businesses, for governments, and for people. The majority of the writer-generated material requires the usage of text mining algorithms and sentiment analysis, even if this information is meant to be instructive. Sentiment Analysis is a technique in Natural Language Processing (NLP) that tries to identify and extract assessments communicated within a given text. This paper intends to execute different content handling strategies in NLP and use of Valence Aware Dictionary for Sentiment Reasoning (VADER) Model that is sensitive to both polarity (positive/negative) and intensity (strength) of emotion.

KEYWORDS

Natural Language Processing (NLP), Polarity Intensity, Sentiment Analysis, Virtual Emotion Detection, VADER.

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1. INTRODUCTION

The emotions are of primarily of six types - love, happiness, anger, sadness, surprise and fear. Sentiment analysis is used to analyze the emotions present in a text. It is a method for determining if a given piece of writing is positive, negative, or neutral. The goal of sentiment analysis is to determine the writer's attitude, sentiments, and emotions in a written text using a computational treatment of subjectivity in the text. By application of sentiment analysis one can decipher the given sentence, paragraph or a document contains a positive or negative emotions or expressions in it. In sentiment analysis we classify the polarity of given text, it results by telling about opinion whether it's positive, negative or neutral.

2. LITERATURE SURVEY

Sentiment analysis has multiple ways and Vader is one of the best way (Mozetic et al., 2016) which is being used. Vader stands for Valence Aware Dictionary and Sentiment Reasoning. It works on a ruled based sentiment analysis and it contains list of lexical features which are labeled as per semantic orientation. By analyzing the intensity of wordings in the text, the sentiment score can be obtained of that text. Vader is smart enough to extract meaning of these words or texts as positive sentiments and words like Sad, bad, awful as negative sentiments. Vader only cares about the expressions in the text. Opinions like positive negative or neutral are the expression which is concerned for Vader.

Steven Bird & Loper (2009) express when a text is changed into its canonical or standard form, then it is called as Text normalization. A few processes have to be done to standardize the content and convert it into fitting structure which would then be given to the machine learning (ML) model. This helps in reducing unnecessary information that the computer does not require, thus subsequently improving efficiency. Library utilized for this is given in. Steps associated with this cycle are shown in Figure 1 and explained briefly in the following sections.

Fig 1. Text Normalization.

2.1 REMOVING STOPWORDS

A stopword is an ordinarily utilized word that can be disregarded, both when ordering sections for looking and while recovering them (Saif et al., 2014). Using pre-compiled stopword lists or more complex algorithms for dynamic stopword recognition, removing stopwords from textual data is a popular procedure for reducing noise. The Natural Language Toolkit (NLTK) in Python includes a list of stopwords for 16 different dialects.

2.2 TOKENIZATION

The method of breaking text into smaller units called tokens is commonly known as Tokenization (Stanford NLP Group, 2015). Tokens can be words, characters, or subwords in this case. As a result, tokenization can be divided into three categories: word, character, and sub-word tokenization.

2.3 DERIVING ROOT WORD

Nicolai & Kondrak (2016) explains in the zones of Natural Language Processing we come across circumstances where a word has many offshoots Stemming and lemmatization are the two main NLP

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processes for generating root words. The root type of an inflected word is produced by both stemming and lemmatization (Samir & Lahbib, 2018). Stemming computation works by removing the word's postfix. Lemmatization thinks about morphological examination of the words. It restores the lemma which is the base type of all its inflectional structures.

2.4 FEATURE EXTRACTION

Machine learning calculations are unable to work on Crude content legitimately. The procedure of component extraction requires converting content into matrix or in vector form. The module can be used to extract features from database consisting of formats such as text and images and extract features in a format supported by machine learning algorithms the most popular strategies that includes feature extraction are Bag-of-Words and TF-IDF Vectorizer. Normalize with diminishing important tokens that appears in majority samples/documents (Mahajan et al., 2020).

2.5 POLARITY AND INTENSITY SCORE IN EMOTIONAL ANALYSIS

A key element of emotional analysis is to examine the body of a text to understand the concept it expresses. Emotional analysis is appropriate for positive or negative values, known as polarity. The perfect situation usually ends up being good, neutral or bad with the help of a polarity point calculation. In general, emotional analysis works better in a submissive text than in a single context text of purpose. Emotional analysis is widely used, as part of the analysis of social media in any domain, to understand the functioning of any system, to be controlled by people and that their response is based on their opinions.

Text textual analysis data can be calculated at most levels, either at the level of each sentence, at the paragraph level, or throughout the document. There are two major theories in emotional analysis. First is Learning Prescribed machine reading or in-depth learning: In this approach, traditional machine learning techniques with a TF-IDF model using the n-gram method. These divisions are the mindless Bayes of many lands, the orderliness of things, the closest neighbor k and the uninhabited forest. In all four classes, orderliness is achieved with minimal accuracy. Second is unsupported dictionary control: This method is to use a large learning process. The accuracy we get from reading a lot is much less than how to learn by machine. After obtaining excellent performance and fragmentation, the next step is to create a final model for back-to-work using certain advanced machine learning methods.

2.6 VALENCE AWARE DICTIONARY FOR SENTIMENT REASONING (VADER)

VADER is a model used for analysis of text sentiment from which it can detect both polarity (positive/negative) and emotion intensity or strength. VADER majorly relies on a dictionary that matches the lexical features to emotion intensities also known as sentiment scores for sentimental analysis (Beri, 2020). By summing up the intensity of each word in the text we can get the sentiment score of a text. Sentiment analysis statistically detects whether the polarity of a piece of text is negative or positive. Sentimental analysis is based on two approaches: polarity-based analysis, in which texts are classed as either negative or positive, and valence-based analysis, in which the intensity of the emotion or sentiment is considered.

2.7 WORKING OF VADER

VADER is a sentiment or emotion analysis method that uses lexicons of sentiment-related words. Each word in the lexicon is classed as positive or negative, and the strength of positivity or negativity is also examined using this method. Table 1 depicts the sentiment rating of an excerpt from VADER's lexicon, with higher positive ratings for more positive words and lower negative ratings for more negative terms.

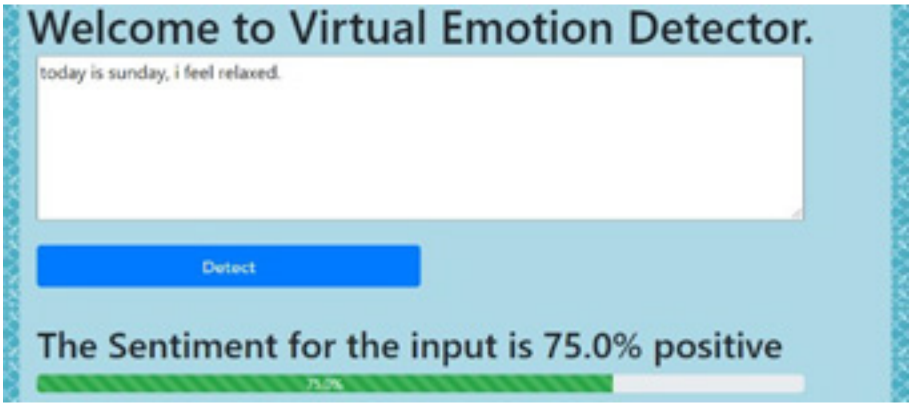
Table I. Sentiment rating of the various words in a text.

Word	Sentiment rating
tragedy	-3.3
rejoiced	2.0
insane	-1.6
disaster	-3.2
great	3.3

According to Hutto & Gilbert in 2014, to determine if these terms are favorable or negative, a group of individuals must personally rate them, which is both time-consuming and costly (Hutto & Gilbert, 2014). The lexicon must contain a good coverage of the words of interest in your text; otherwise, accuracy will be poor. When the lexicon and the text are well-matched, this method is quite precise, and it even produces speedy results on vast volumes of text. VADER not only matches the words in the text with its lexicon, but it also takes into account certain aspects of the way the words are written as well as their context meaning.

3. EXPERIMENTATION AND PERFORMANCE ANALYSIS

In this paper, sentiment analysis by the utilization of VADER is performed. Firstly, utilize standardization methods in Natural Language Processing (NLP) for converting the test text into its vector form. The implementation of emotion analysis evaluated in Python using Natural Language Toolkit (NLTK) library. Figure 2 shows the virtual emotion analysis of the text where the emotional content is adjudged as 75% positive. Figure 3 shows the virtual emotion analysis of the text where the emotion is adjudged as 26% positive which is shown in red color depicting 26% negative. Figure 4 shows the sentiment analysis of long text where the sentiment is adjudged as 92% positive. By adding



this feature of detecting gibberish, we are able to detect gibberish text where the algorithm returns 50% as can be seen in Figure 5.

Fig 2. Virtual emotion detection of text showing 75% positive polarity.

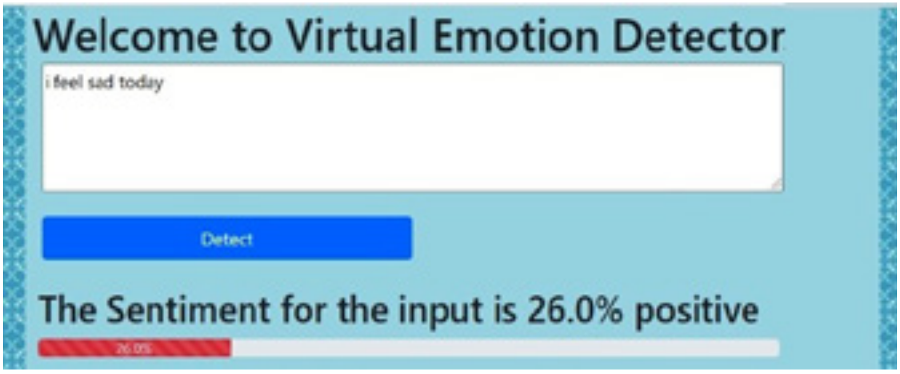


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Fig 3. Virtual emotion detection of text showing 26% negative polarity.

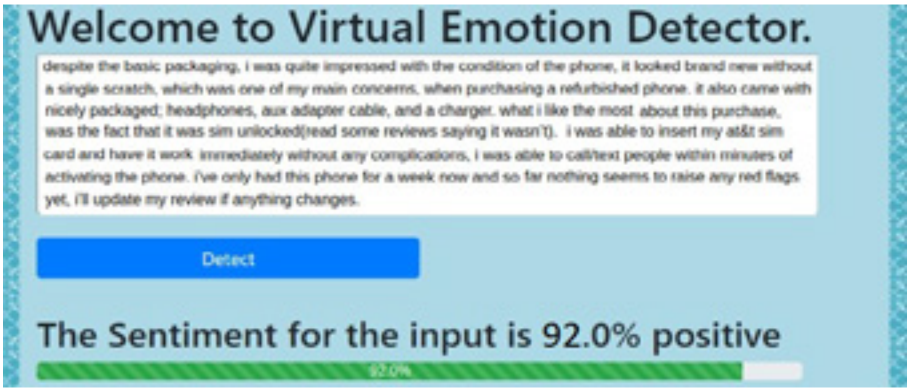

Fig 4. Virtual emotion detection of long text showing 92% positive polarity.

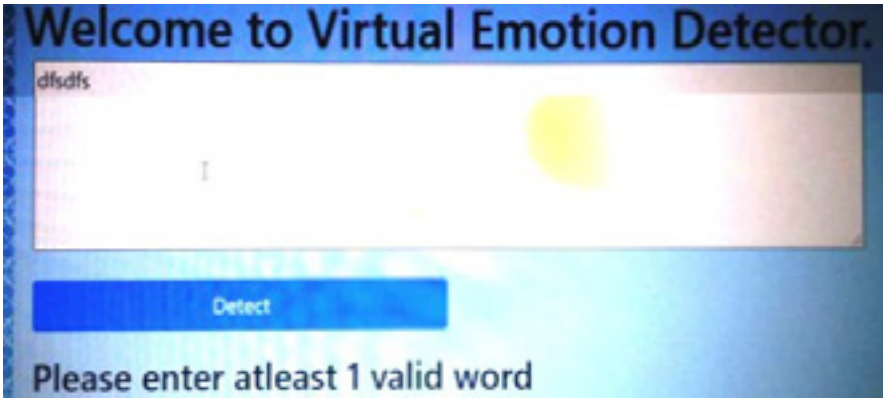

Fig 5. Emotion detection of gibberish text.

4. CONCLUSION

The main focus of this paper is to calculate two scores of text: polarity and its intensity by using machine learning. The polarity range is between -1 to 1(negative to positive) and which help us to find whether the text is positive or negative. In this paper we have tested different emotions of a text by using natural language processing toolkit in python and quantified this emotion as positive or negative. Also we were able to identify a gibberish text i.e. words which does not make any sense or are rubbish.

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PERFORMANCE ANALYSIS OF NOMA IN RAYLEIGH AND NAKAGAMI FADING CHANNEL

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ABSTRACT

Cellular connectivity is expanding rapidly in the modern world. The multiple access strategy is one of the highly used methods for allocating the range of users in cellular network. Spectrum allocation is a crucial element to take into account since cellular communication is becoming more and more popular. NOMA is a channel access mechanism used in 5G mobile communication. It is also known as non-orthogonal multiple access. NOMA is a potential strategy for enhancing spectral efficiency and sum rate. Using the NOMA method, we evaluated the BER versus transmitted power of two users in rayleigh and nakagami fading channels. In this NOMA setup, a single antenna is shared by two users. Two users can accept the same frequency using 5G NOMA technology, but at different power levels. The results of the MATLAB simulation show that the two user NOMA in the Nakagami channel performs better than the Rayleigh channel in terms of Bit Error Rate vs. Transmitted Power.

KEYWORDS

NOMA, Rayleigh, Nakagami Fading, BER, Transmitted Power and Probability Density Function.

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1. INTRODUCTION

One of the crucial requirements for 5th generation mobile systems is expanded data networks. The development in this field aims to boost system throughput and capacity. This is a must-have requirement given the rapid increase in mobile traffic that has just occurred. The network's increasing traffic should be able to be handled by the multiple access strategy that is suggested Aditi Agrawal et al. [2022] M. W. Baidas et al. [2018]. Early versions of multiple access strategies distributed users and resources in an orthogonal way. But, in 5G, NOMA has been the centre of study. A mechanism called as NOMA is used to make sure that there is equity in the deployment of forthcoming radio access resources. The fifth generation must offer high connectivity, dependability, and low latency, therefore this is necessary. In this type of design, NOMA surpasses OMA by roughly 30% B. Kim et al. [2019. Superposition coding is used by the base station (BS) to broadcast in NOMA, while SIC is used to decode the signals. Combining this SIC with an interference cancellation combining receiver will boost capacity A. Benjebbour et al.[2013]. Non-orthogonal multiple access may be divided into 2 categories which are the domains of code and power C. Hsiung et al. [2019]. In NOMA domain, users having poor channel conditions receive high power, while users having acceptable channel conditions receive low power. While the less powerful user performs SIC, the more powerful user directly decodes their own signal in the receiver. Because users may communicate in both good and bad channel conditions, NOMA is more democratic than OMA P. N. Thakre et al. [2022]. In comparison to other orthogonal users, the throughput of cell edge users improves owing to the intra beam SIC's removal of interference D. K. Hendraningrat et al. [2020].

When fifth-generation systems connect large devices, improving spectrum allocation is essential. NOMA helps to improve spectrum utilization. In this study, the simulation results of BER vs transmitted power in two different fading channels are shown. This article has the advantage of illustrating the BER of a system employing NOMA in two fading channels. To enhance the spectral efficiency in 5-G communications, NOMA has been proposed. Spectrum distribution becomes significant in a number of methods as user numbers increase A. Benjebbour et al. [2013]. In order to improve a network's overall rate, outage probability, and ergodic capacity, multiple access approaches are deployed. The importance of power allocation in network performance also increases along with the number of users K. Wang et al. [2019] K. Higuchi et al. [2015] Y. Kishiyama et al. [2012] Harada et al. [2014] Prasheel Thakre et al. [2022] Y. Saito et al. [2013]. Performance is assessed using the users' BER calculations. Since the performance is usually acceptable, power domain NOMA is properly assessed. Performance will be better with the distribution to NOMA users than with regular OMA. Applications, like visible light communications, are also where NOMA is most commonly employed. The most frequent issue with VLC is blockages; the dynamic user pairing strategy helps with distribution of resource, which right away enhances the performance of the system Z. Xiao et al. [2019] Y. Yin at al. [2019] . Combining user pairing with power allocation, the fractional transmit power that results in low performance is used for resource allocation Y. Yin at al. [2019]. BER vs SNR of two users was compared for various fading channels using NOMA approach and it was found that Nakagami channel performs considerably better when compared to the Rayleigh and Rician channel in BER vs SNR but transmit power isn't considered for the different channels K. Higuchi et al. [2015]. Moreover, comparison of NOMA against OMA networks conveys that NOMA outperforms OMA and provide better spectral efficiency and user fairness (C. Hsiung et al. [2019]. Closed-form expressions of BER at near and far users of the considered downlink NOMA are calculated in the presence of SIC over Nakagami fading channel A. Benjebbour et al. [2013]. The equations for average SNR, achievable rate, and outage probability show that network users' ordered channel gains are equal to their diversity orders M. W. Baidas et al. [2018]. NOMA has developed independently in every aspect of wireless communication. Whenever there is a rise in users, the allocation of resources is also considered to account for the effectiveness of the system.

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2. NOMA SYSTEM MODEL

We consider Non-Orthogonal Multiple Access Scheme. Here, the BS superimposes the information waveforms for its serviced users. Each user equipment employs Successive Interference Cancellation to detect their own signals.

In a NOMA system with two users , suppose User 1 is a faraway user with a weak signal and User 2 is a close user with a good signal.

The BS serves both users on the same frequency spectrum. h_1 and h_2 be the channels of far user and near user respectively. The base station's signal can be described as follows:

$$x = \sqrt{P}(\sqrt{\alpha_1}x_1 + \sqrt{\alpha_2}x_2)$$

where, P is the transmitted power, α is fractional coefficient of total power such that $\alpha_1 > \alpha_2, \alpha_1 + \alpha_2 = 1$

At the User 1, the received vector is expressed as:

$$y_1 = h_1\sqrt{P}(\sqrt{\alpha_1}x_1 + \sqrt{\alpha_2}x_2) + w_1 \tag{2}$$

or,

$$y_1 = \underbrace{h_1\sqrt{P}\sqrt{\alpha_1}x_1}_{\text{Desired dominating}} + \underbrace{h_2\sqrt{P}\sqrt{\alpha_2}x_2}_{\text{Interference low power}} + \underbrace{w_1}_{\text{Noise}} \tag{3}$$

Desired dominating | Interference low power | Noise

Now, direct decoding is performed to estimate x_1

The SINR for decoding the 1st (far) user signal is given by:

$$\gamma_1 = \frac{\alpha_1 P|h_1|^2}{\alpha_2 P|h_1|^2 + \sigma^2} \tag{4}$$

The achievable rate(bps/Hz) of the User 1 is given as:

$$R_1 = \log_2\left(1 + \frac{\alpha_1 P|h_1|^2}{\alpha_2 P|h_1|^2 + \sigma^2}\right) \tag{5}$$

For User 2, the received vector is given as:

$$y_2 = \underbrace{h_2\sqrt{P}\sqrt{\alpha_1}x_1}_{\text{Interference dominating}} + h_2\sqrt{P}\sqrt{\alpha_2}x_2 + \underbrace{w_2}_{\text{Noise}} \tag{6}$$

Interference dominating | Desired low power | Noise

Firstly, direct decoding for the x1 signal, then the concept of the SIC is applied as:

$$y_2' = h_2\sqrt{P}\sqrt{\alpha_1}x_1 + h_2\sqrt{P}\sqrt{\alpha_2}x_2 + w_2 - h_2\sqrt{P}\sqrt{\alpha_1}\hat{x}_1 \tag{7}$$

Now, direct decoding for the near user signal x2.

The SINR for decoding far user's signal at near user is given by:

$$\gamma_{1,2} = \frac{\alpha_1 P\left|h_2\right|^2}{\alpha_2 P\left|h_2\right|^2 + \sigma^2} \tag{8}$$

Hence the achievable rate(bps/Hz) will be:

$$R_{1,2} = \log_2\left(1 + \frac{\alpha_1 P\left|h_2\right|^2}{\alpha_2 P\left|h_2\right|^2 + \sigma^2}\right) \tag{9}$$

After the far user's signal has been cancelled, the near user's SINR for decoding its own signal is:

$$\gamma_2 = \frac{\alpha_2 P\left|h_2\right|^2}{\sigma^2} \tag{10}$$

The corresponding achievable rate(bps/Hz) is given as:

$$R_2 = \log_2\left(1 + \frac{\alpha_2 P\left|h_2\right|^2}{\sigma^2}\right) \tag{11}$$

The three types of NOMA schemes that are now being employed in the broad spectrum are PD-NOMA, waveform domain NOMA, and CD-NOMA, according to the survey and resources that are currently accessible. The focus of most NOMA research is PD-NOMA, which necessitates a substantial power differential between the signals allocated to various users. At the transmitter's side of a PD-NOMA system, superposition coding (SC) is used to create the signals of numerous users on each subcarrier, which are dispersed over several users (In SC, although sharing the same time-frequency-code resources, each user has their own power level. Each user's power level is determined by the channel gain value; those with lower channel gain values receive greater power levels, and vice versa. At the receiver side, the SIC approach is used to filter out extra user signals that interfere with that band.

In terms of spectrum efficiency, NOMA's SIC method outperforms OMA. Between users 1 and 2, the power is split. The power distribution to users has a considerable impact on the throughput of users in the NOMA domain. The fairness of the users' Power allocation largely determines throughput.

3. FADING CHANNELS

During wireless propagation, fading refers to the degradation of the transmitted signal power caused by a variety of factors. These variables include geographic location, time, radio frequency, and atmospheric conditions like rainfall and lightning. There are different types of fading channels and they are Rayleigh, Rician, Nakagami, Weibull fading channel, etc.

In this paper two fading channels Rayleigh and Nakagami are considered. Only NLOS components between the transmitter and receiver are modelled in the Rayleigh. It is assumed that there is an absence of Line-Of-Sight route between the transmitter and receiver. When multipath scattering occurs with relatively high delay time spans and various groups of reflected waves, Nakagami fading takes place. Table 1 shows a comparative study of Rayleigh and Nakagami fading channels taking few key parameters into consideration.

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Table I. Comparison between rayleigh and nakagami fading channel.

Parameters	Rayleigh fading channel	Nakagami fading channel
SNR	For 1000 samples, SNR=0.9	For 1000 samples, SNR=0.35
Power Consumption	More power consumption	Less power consumption
BER vs SNR	BER vs SNR values for Rayleigh are lower as compared to Nakagami	BER vs SNR values for Nakagami are higher as compared to Rayleigh

4. PERFORMANCE OF NOMA IN RAYLEIGH AND NAKAGAMI FADING CHANNEL

From Fig. 1, we can infer that for a two user NOMA to maintain user fairness, the system has given the distant user more power and the close user less. Secondly, we conclude that as power increases, the BER for both users decreases. Also, at a particular value of transmit power, BER value for user who is located distantly from the base station is more compared to user nearer.

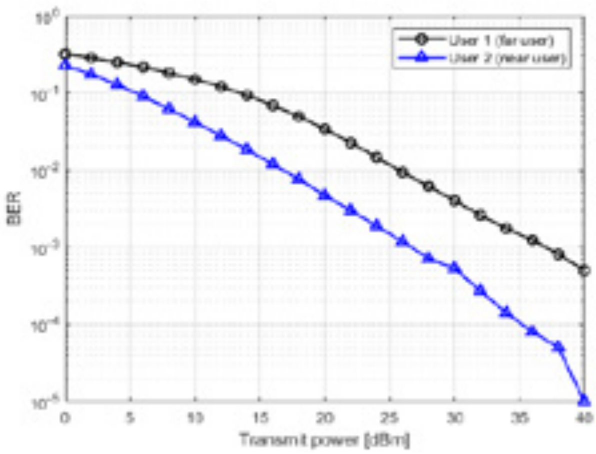

Fig. 1. BER vs Transmitted power graph for rayleigh channel.

Fig. 2 shows the Probability Density Function for Nakagami channel for different m values. From the literature survey, we infer that mu should be greater than 1 so that it corresponds to lesser fading than Rayleigh fading and omega is mostly taken 1.

mu=2 , w=1 and mu=5 and w=1 might be considered as the best values.

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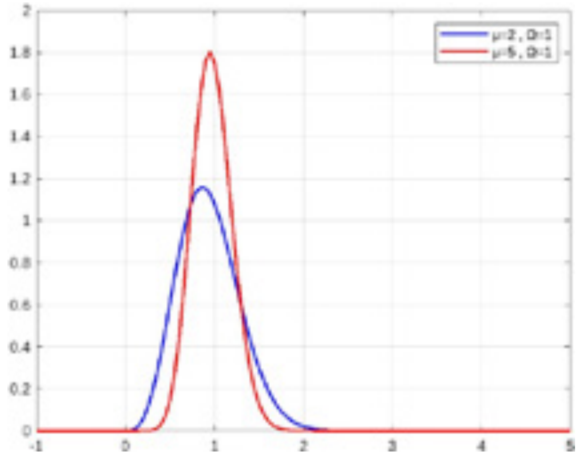

Fig. 2. PDF curve for nakagami-m channel for different values of mu and omega.

In Fig. 3, BER vs Transmit power has been plotted for Nakagami fading channel using different values of mu and omega as shown.

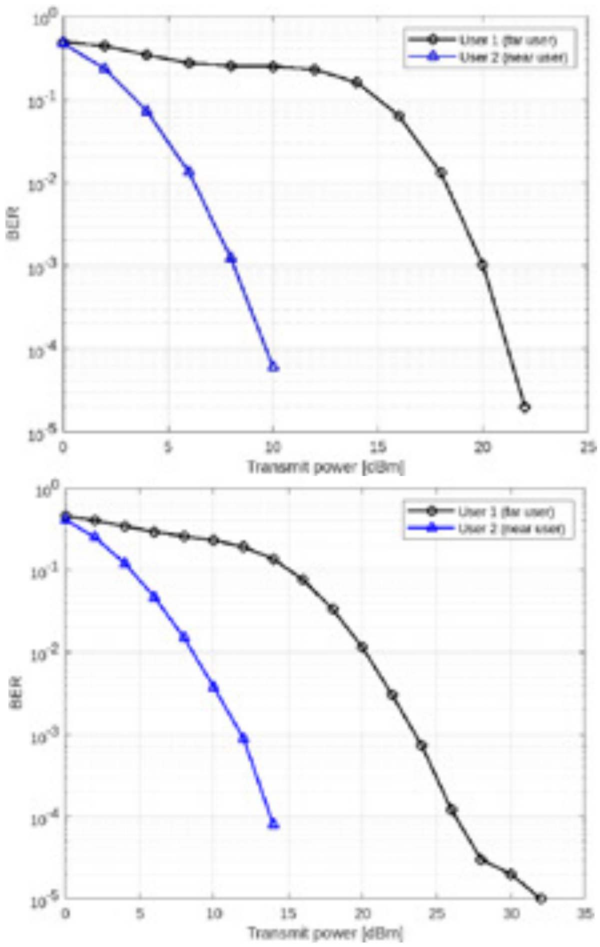

Fig. 3. BER vs Transmitted power graphs for nakagami channel (mu=5, omega=1) and (mu=2, omega=1).

5. CONCLUSION

NOMA is favored for 5G communications because it offers a strong connection, stability, and minimal latency. Large-scale networking is facilitated by this NOMA's improved spectrum efficiency. In this study, the performance analysis of two different fading channels—Rayleigh and Nakagami—for wireless NOMA communication is assessed. According to our research, when power increases in a Rayleigh fading channel, the bit error rate drops, greater power is distributed to far users while low power is distributed to nearby users, ensuring user fairness. Similar findings are obtained with Nakagami fading, although the BER performance is superior to that of Rayleigh fading. The Nakagami channel performs better than the Rayleigh channel in terms of Bit Error Rate. A rise in BER might be reduced by the employment of several coding techniques or differentiation strategies.

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BACKGROUND REMOVAL OF VIDEO IN REALTIME

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ABSTRACT

Background removal for video is a computer-vision based system to remove the background from a video with ease. Creating a professional background when at home, i.e., not in a very professional environment, can be a tedious task. Not everyone has time to learn editing and the technicalities involved in having an entire setup for creating a sophisticated background and it is not practical that normal people buy green screens or blue screens just for their everyday formal meets. Our goal is to create a quick and easy solution to that by removing background in real time while also maintaining the quality of the call, having the additional benefit of adding custom backgrounds and enabling users to add effects like adjust lighting, contrast etc., we are combining all 4-5 steps in 1 single step. SelfieSegmentation module of Mediapipe helps us achieve this. The Selfie Segmentation API creates an output mask from an input image. The mask will be the same size as the input image by default. A float integer with a range of [0.0, 1.0] is assigned to each pixel of the mask. The higher the confidence that the pixel depicts a person, and vice versa, the closer the number is to 1.0.

KEYWORDS

Categories and Subject Descriptors: G.4 [Mathematics of Computing]: Mathematical Software - User Interfaces; H5.2 [Information Interfaces and Presentation]: User Interfaces - User-centered design; Interaction styles; Theory and methods.

Background Removal, Online Meetings, Professional Background, Streaming.

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1. INTRODUCTION

This pandemic has made it clear that virtual meets are here to stay. They are the new normal. But often virtual meetups from home fail to create the professional atmosphere due to backgrounds in the video. The professionalism required can be achieved by proper background in videos and minimal disturbances. Unnecessary and visually unpleasing objects need to be removed. This can be achieved by background removal, changing background color and adding virtual backgrounds. Background removal can help avoid the cumbersome task of arranging a good backdrop in some cases even eliminates the use of green screens which are used by many youtubers. It helps add to the overall aesthetics and pleasantness of video conferences or even live streams.

The technique of mimicking human intelligence in robots programmed to think and behave like humans is known as artificial intelligence (AI). Computer vision is a branch of artificial intelligence (AI) that allows computers and systems to extract useful information from digital photos, videos, and other visual inputs and act or make recommendations based on that data. A computer vision technique called object detection is used to locate and identify objects in pictures and movies. Using this type of identification and localization, object detection can be used to count the items in a scene, locate and track them precisely, and accurately label them.

Image segmentation is the technique of dividing a digital image into numerous pieces for use in digital image processing and computer vision (sets of pixels, sometimes known as image objects). Various portions of a movie can be discovered using object detection and image segmentation, and different adjustments can be performed to each of the components. We'll use these ideas to remove the backdrop from a video using AI.

Open Broadcaster Software (OBS) helps you set your scene as a virtual camera to which we will be feeding our video directly in real time using pyvirtualcam and mediapipe. We'll use Google Meet to recognize OBS as a video source and output it as a virtual camera, resulting in enhanced quality of video calls.

2. REVIEW OF LITERATURE

2.1 PRESENT SYSTEM

The apps that currently provide such a feature for removing the background of a video in real time tends to worsen the quality of the video and do not provide us with smooth edges of the main object which is not ideal for a professional use. Also, the process to remove or change the background in these apps consists of various steps to be followed, sometimes we are even required to cut the video call and rejoin with a changed background. Through our project we have tried to overcome these problems by providing a system which removes the background while maintaining a proper quality of the video with fps greater than 30. And we also provide an easy way to change the backgrounds by just clicking a button on the keyboard.

Grzegorz Szwoch, in his paper presented, implementation of a background subtraction algorithm using the OpenCL platform. The algorithm works using a live stream of video frames from an on-line surveillance camera. A host machine and a parallel computing device are used to execute the processing. The research focuses on optimizing OpenCL algorithm implementation for GPU devices by taking into consideration specific GPU architecture aspects including memory access, data transfers, and work group structure. The technique is designed to work on any OpenCL-enabled device, including DSP and FPGA platforms. Several algorithm optimizations are presented and tested on a variety of devices with variable processing power. The work's major goal is to figure out which optimizations are required for on-line video processing in the surveillance system.

A relatively inexpensive background subtraction method is proposed by Hasup Lee et al., in their study employing background sets with im-age- and color-space reduction. Background sets are used to recognize objects from dynamic backdrops like waves, trees, and fountains. The image space is decreased to handle jittered and unstable frames, such as those from handheld mobile devices. The color space is shrunk to account for color noise, such as the scattered RGB values from a digital camera. To reduce expenses, a combination of color-space reduction and hash-table look-up operations is used. The results, when compared to other methods, suggest that the proposed technology is feasible: it may also be used in mobile or embedded environments.

S. Joudaki, et al., in their paper, they presented a comparison of numerous existing background subtraction methods, ranging from basic background subtraction to more complicated providential techniques. The purpose of this research is to provide an overview of the advantages and disadvantages of commonly utilized approaches. The approaches are compared based on how much memory they demand, how long they take to compute, and how well they handle different types of films. Finally, other criteria such as processing time and memory needs were used to compare the existing approaches. Baoxin Li, et al., in their paper they proposed a video background replacement algorithm, this is based on adaptive background modelling and background subtraction. It can be accomplished with a pre-recorded background scene image rather of a blue screen. Identifying statistical outliers in respect to a specific background is the challenge. A two- pass approach is utilized to modify initial segmentation based on statistics about a pixel's vicinity, which lowers false positives in the background area while raising detection rates for foreground objects. Experiments with real image sequences, as well as comparisons with other existing approaches, are shown to demonstrate the benefits of the proposed methodology.

S. Brutzer, et al., in their paper, presented one of the most important approaches for automatic video analysis, particularly in the field of video surveillance, is background subtraction. Despite their usefulness, reviews of recent background removal algorithms in relation to video surveillance challenges include several flaws. To address this problem, we must first identify the major obstacles to background subtraction in video surveillance. We then evaluate the performance of nine back-ground subtraction algorithms with post-processing depending on how well they over-come those challenges. As a result, a fresh evaluation data set is presented that includes shadow masks and precise ground truth annotations. This enables us to offer a thorough evaluation of the advantages and drawbacks of various background sub-traction techniques.

In their study, R. J. Qian et al., presented an algorithm for altering video backgrounds without a blue screen physically. Pre-recording a backdrop image of the scene free of any foreground objects is required for the operation. Based on the color difference between the pixels in an input frame and their corresponding pixels in the background image, the method computes a probability map that contains the likelihood for each pixel to be classified into the foreground or background. The probability map is further improved using anisotropic diffusion, which reduces classification mistakes without adding a lot of artefacts. The foreground pixels from the input frames are then feathered onto a brand- new background video or image based on the enhanced probability map to create the output video. The method requires only a little amount of CPU resources and is designed to work in real time. Experiment findings are also reported.

A. Ilyas, et al., in their paper, presented a Modified Codebook Model-Based Real Time Foreground-Background Segmentation. The initial step in object tracking is the essential process of segmenting the scene in real time into the foreground and background. beginning with the codebook approach. Authors suggested certain changes that show notable improvements in the majority of the typical and challenging conditions. For accessing, removing, matching, and adding codewords to the codebook as well as moving cached codewords into the codebook, they included frequency options. They also suggest an evaluation procedure based on receiver operating characteristic (ROC) analysis, precision and recall methodology, to impartially compare various segmentation techniques. Authors suggested expressing the quality factor of a method as a single value based on a harmonic mean between two related features or a weighted Euclidean distance.

Rudolph C. Baron, et al., in their paper, presented a solution for managing a video conference. When establishing a video conference with a second person, a first participant can choose from among

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several stored virtual backgrounds and use that background. One or more characteristics of the first and/or second participant, one or more characteristics of the video conference, and/or similar considerations may be used to choose the virtual background. The virtual backgrounds can be used, for instance, to provide people outside of a company organization a desired perception, message, and/or the like while they communicate with its employees via video conferencing. The virtual background can incorporate static image data, live or recorded video feeds, static business entity web pages, and dynamic business entity web pages.

Jian sun, et al., Effective techniques and approaches in a video sequence isolate the focus from the background, according to their paper. In one instance, a system creates an accurate real-time backdrop cut of live video by reducing the background contrast while maintaining the contrast of the segmentation boundary itself. This method enhances the border between the foreground and background images. The live video may then combine the fragmented foreground with another background. An adaptive background color mixture model can be used by the system to distinguish foreground from background more effectively when there are changes in the backdrop, such as camera movement, lighting changes, and the movement of small objects in the background.

Juana E. Santoyo-Morales, et al., in their paper presented a Background sub-traction models based on a Gaussian mixture have been widely employed in a range of computer vision applications for detecting moving objects. Background sub-traction modelling, on the other hand, remains a challenge, especially in video sequences with dramatic lighting changes and dynamic backdrops (complex backgrounds). The goal of this research is to make background subtraction models more resilient to complicated situations. The following enhancements were proposed as a result: Redefining the model distribution parameters (distribution weight, mean, and variance) involved in the detection of moving objects; enhancing pixel classification (background/foreground) and variable update mechanisms using a new time-space dependent learning rate parameter; and c) substituting a new space-time region-based model for the pixel-based model that is currently used in the literature.

According to Yiran Shen et al., background subtraction is a typical first step in many computer vision applications, including object localization and tracking. Its objective is to pick out the moving parts of a scene that match to the important things. Researchers in the field of computer vision have been working to increase the reliability and accuracy of such segmentations, but most of their techniques require a lot of computation, making them unsuitable for our target embedded camera platform, which has a much lower energy and processing capacity. In order to create a new background subtraction method that overcomes this issue while retaining an acceptable level of performance, authors added Compressive Sensing (CS) to the often-used Mixture of Gaussian. The results imply that their technique can significantly reduce the eventual time taking.

Semi-supervised video object segmentation should be considered, which is the process of creating precise and consistent pixel masks for objects in a video sequence based on ground truth annotations from the first frame, according to a suggestion made by Jonathan Luiten et al. To do this, they provided the PReMVOS algorithm (Proposal- generation, Refinement and Merging for Video Object Segmentation). The method separates the problem into two steps to specifically address the difficult issues related to segmenting multiple objects across a video sequence: first, generating a set of precise object segmentation mask proposals for each video frame; and second, choosing and merging these proposals into precise and object tracks that are pixel-wise and consistently timed inside a video sequence.

Thuc Trinh Le, et. al., demonstrated a method for removing items from videos. The technique simply requires a few input strokes on the first frame that roughly delineate the deleted objects. Authors claims that this is the first method which enables semi-automatic object removal from videos with intricate backgrounds. The following are the main phases in their system: Segmentation masks are improved after setup and then automatically distributed throughout the film. Video inpainting techniques are then used to fill in the gaps. Authors claim that their system can handle several, potentially intersecting objects, complex motions, and dynamic textures. As a result, a computational tool that can automate time-consuming manual tasks for editing high-quality videos has been developed.

Thanarat H. Chalidabhongse, et. al., in their paper, showed how to use color pictures to detect moving items in a static background scene with shading and shadows. They created a reliable and efficient background subtraction method that can deal with both local and global lighting variations, such as shadows and highlights. The approach is based on a proposed computational color model that distinguishes between the brightness and chromaticity components. This technology has been used to create real-world image sequences of both indoor and outdoor locations. The results, which demonstrate the system's performance, are also provided, as well as several speed-up techniques used in their implementation.

Yannick Benezeth, et. al., in their paper, presented comparison of different state-of-the- art background subtraction approaches is presented. On several movies containing ground truth, there have been developed and tested methods ranging from straightforward background subtraction with global thresholding to more complex statistical algorithms. The purpose is to lay a solid analytic foundation on which to highlight the benefits and drawbacks of the most extensively used motion detection methods. The approaches are compared in terms of their ability to handle various types of videos, memory requirements, and computing effort. A Markovian prior, along with several postprocessing operators, are also considered. Most of the films are from modern benchmark collections and highlight a range of issues, including low SNR, background motion in many dimensions, and camera jitter.

Yi Murphey, et. al., in their paper, describes their work on image content-based indexing and retrieval, which is an important technique in digital image libraries. Image features used for indexing and retrieval in most extant image content-based approaches are global, meaning they are computed over the full image. Background features can readily be mistaken for object features, which is the fundamental drawback of retrieval techniques based on global picture features. Users typically refer to the color of a particular object or objects of interest in an image while searching for photos using color attributes. The technique described in this article uses color clusters to analyze image backgrounds. After being identified, the background regions are deleted from the image indexing process, so they won't interfere with it anymore. Three main calculation processes make up the algorithm: fuzzy clustering, color picture segmentation, and background.

Zhao Fei Li, et. al., in their paper, presented background noise removal is a key stage in the picture processing and analysis process. Researchers use a variety of techniques to remove background noise from images. For instance, grey threshold techniques are frequently used to eliminate noises that have a strong contrast to the object of interest. However, there are a lot of noises in the grey scale that don't change as the interesting objects do. These noises cannot be reduced using the grey level-based noise removal technique, but the contour feature is excellent at doing so. The contour feature-based image background removal approach depends on the contour model. The contour characteristic of the interest items is modelled using a revolutionary method proposed in this study. A unique background noise with the same grey level as the background noise is completely eradicated using this method.

Thuc Trinh Le, et. al., demonstrated a method for eliminating objects from videos in their study. A few strokes in at least one frame are all that are needed for the technique to roughly delimit the items to be eliminated. These undeveloped masks are then polished and automatically broadcast throughout the video. The corresponding areas are synthesized again using video inpainting methods. Authors claim that their system is capable of navigating several, perhaps crossing objects, intricate motions, and dynamic textures. As a result, a computational tool has been created for editing high-quality videos that can take the place of laborious human work.

3. PROPOSED TECHNIQUE

The main objective of the proposed technique is to create a simpler process of removing background from a live or saved video, to simplify the process of creating an aesthetic background. Our project allows the user to completely remove the background or put a different background and be able to switch between multiple backgrounds or colors.

The main objective of our project is to eliminate the need to physically change the arrangement of room for a better and professional background. Thus, providing users an easy way of applying or changing backgrounds while they are in some online meet or live streams while maintaining the quality of the video.

3.1 Flow of Proposed Technique

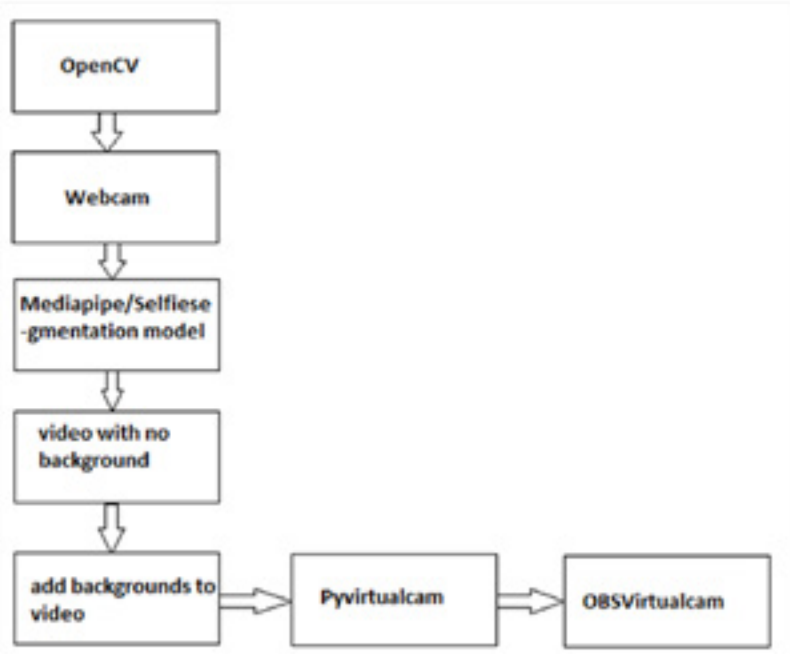

Fig. 1. Flowchart of the proposed system.

Fig. 1 shows the flow of the proposed system, starting with the accessing of the live video from webcam to the removal/changing of background and feeding the output to applications like google meet using OBS Virtual Camera.

3.2. IMPLEMENTATION

Our goal was to remove the background in real-time and with FPS more than 30.

3.2.1. STARTING THE WEBCAM

We should be able to access the webcam by simply running the code so that the videoi.e., real time video can be directly taken as input. A computer vision library is called OpenCV (Open-Source Computer Vision). with a variety of image and video manipulation tools. The OpenCV library can be used to manipulate films in a variety of ways. To capture a video, you'll need a *VideoCapture* object. The index of the device or the video file's name is stored in *VideoCapture*. The device index is just a number that identifies which of the camera device is being used.

Syntax: *capt = cv2.VideoCapture(0)*

Now a pop-up window will open if we have a webcam. We have set the frame size to 640X 480. Therefore, background-replacing images should be 640 x 480, which is the same size as the frame.

Creating a dataset for background images Make a folder called 'BackgroundImages' inside the project directory. You can download and store any image, or any number of images, in this directory.

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3.2.2. BACKGROUND REMOVAL

We have used the *SelfieSegmentationModule* from *cvzone* package which uses OpenCV and *Mediapipe* libraries at its core and makes AI operations on videos and images very easy. *SelfieSegmentation* is a technique for removing the frame's background and replacing it with photos from our directory. It is based on MobileNetV3 but has been tweaked to be more efficient. It uses a 256x256x3 (HWC) tensor as input and outputs a 256x256x1 tensor as the segmentation mask. Before feeding it into the ML models, *MediaPipe SelfieSegmentation* automatically resizes the input image to the necessary tensor dimension. We use the webcam for input and frame width should be set to 640 x

480. Then we utilize the *cvzone* to execute *SelfiSegmentation*(), which carries out object identification, image segmentation, and ultimately background removal. The output frames can show the frames per second (fps) using the FPS() method.

Syntax: seg = SelfiSegmentation() Setfps = cvzone.FPS()

SelfieSegmentation() converts the image into RGB and sends it to the SelfieSegmentation model to process and then it checks if the image is colored; if yes it changes the color of the background and if not, it then changes the color of the image. As a result, we can see the background successfully removed.

3.2.3. STORE BACKGROUND IMAGES IN A LIST

Then, after creating a list of every image in the *BackgroundImages* folder, we iterate through it, reading each one and adding it to an empty list. At the beginning, the index is set at zero.

3.2.4. REPLACE BACKGROUND WITH DESIRED BACKGROUND

The frames are read from the camera using a while loop, and the background is then removed from the frames using the *seg.removeBG()* method and replaced with images from the directory. The camera's image *frame (img),* the directory's collection of photos, and with an index of image (*imgList[indexImg]*), are all passed to the seg.*removeBG()* function along with the threshold. To improve the edges, we additionally modify the threshold setting.

3.2.5. FUNCTIONALITY TO CHANGE BACKGROUND USING KEYBOARD SHORTCUTS

Using *cvzone.stackImages*, we stack the images and retrieve the output of the frames or background-replaced image. Then, by means of a straightforward if statement, we assign keys to change the background. The principle is to sequentially remove the indexes according to the key that was pressed to display the image for the resulting index. This lets you change the backgrounds quickly.

3.2.6. SEND THE FRAMES TO OBS VIRTUAL CAMERA

We then send the resulting frames to OBS Virtual Camera using *Pyvirtualcam().* It sends frames to a virtual camera from Python. The OBS virtual camera is detected by various platforms, and we have our very own background removal with whichever backgrounds, or no backgrounds, as required.

4. RESULTS AND DISCUSSIONS

With our proposed implementation, we successfully removed the background and add any other desired background in real time with FPS more than 30.

Fig. 2. Removal of background in real time.

Fig. 2. shows that the left half of the image has normal real-time video with backgroundand the right side has video with no background with FPS 34.

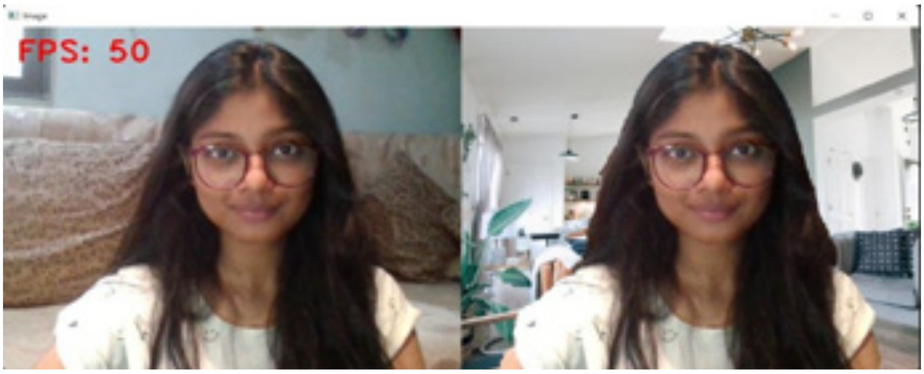

Fig. 3. Background changed in real time successfully with FPS = 50.

Fig. 3. shows that left side of the image has normal real-time video, and the right side hasvideo with desired background with FPS 50.

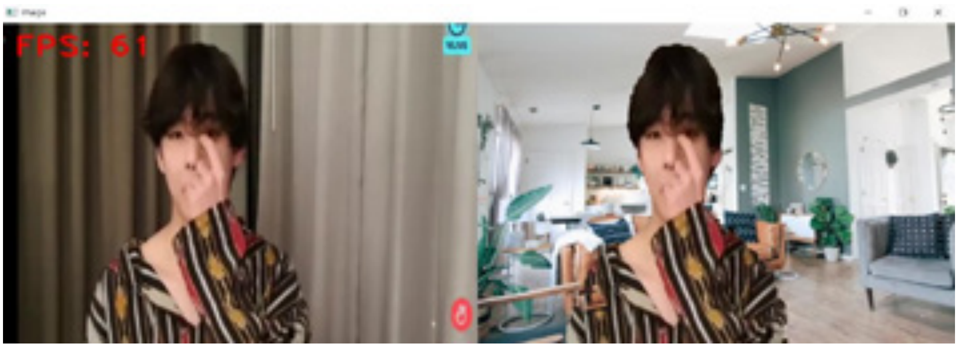

Fig. 4. Background changed of pre-recorded video successfully with FPS= 61.

Fig. 4. shows that left side of the image has normal pre-recorded video, and the right sidehas the video with desired background with FPS 61.

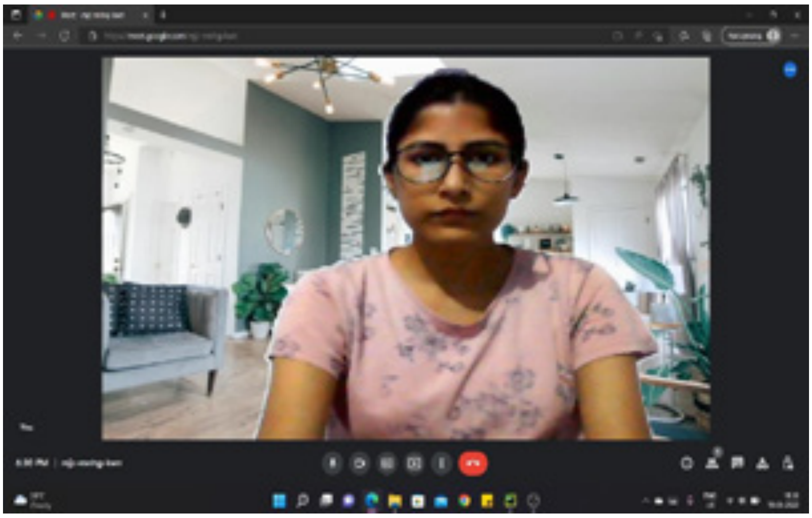

Fig. 5. Background changed in real time successfully on Google Meet.

Fig. 5. shows that we were able to change the background of our live video in a Google meet.

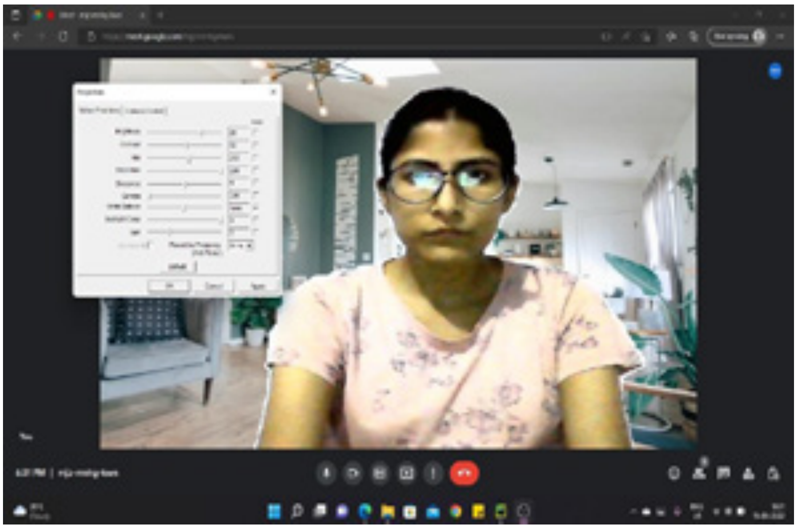

Fig. 6. Background changed in real time successfully and able to configure in Google Meet.

Fig. 6. shows that we were able to change the background of our live video as well as configure the features such as brightness, contrast, saturation, etc. of our real time videoin a Google meet.

4. CONCLUSION

We have implemented a computer-vision based system to remove the background from a video in real-time which enabled creating a professional background when not in a very professional environment. Our goal was to create solution for removing background in real time while also maintaining the quality of the call, having the additional benefit of adding custom backgrounds and enabling users to add effects like adjust lighting, contrast etc. We used *SelfieSegmentation* module of *Mediapipe* in this implementation. Our results shows that our technique successfully removed backgrounds from live videos as well as prerecorded videos at frame rate between 30 and 60. We also changed background of videos in reals time and prerecorded videos seamlessly. Our implementation also worked very well on streaming platforms like Google Meet.

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REVIEW ON DEEP LEARNING BASED TECHNIQUES FOR PERSON RE-IDENTIFICATION

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ABSTRACT

In-depth study has recently been concentrated on human re-identification, which is a crucial component of automated video surveillance. Re-identification is the act of identifying someone in photos or videos acquired from other cameras after they have already been recognized in an image or video from one camera. Re-identification, which involves generating consistent labelling between several cameras, or even just one camera, is required to reconnect missing or interrupted tracks. In addition to surveillance, it may be used in forensics, multimedia, and robotics.Re-identification of the person is a difficult problem since their look fluctuates across many cameras with visual ambiguity and spatiotemporal uncertainty. These issues can be largely caused by inadequate video feeds or low-resolution photos that are full of unnecessary facts and prevent re-identification. The geographical or temporal restrictions of the challenge are difficult to capture. The computer vision research community has given the problem a lot of attention because of how widely used and valuable it is. In this article, we look at the issue of human re-identification and discuss some viable approaches.

KEYWORDS

Person re-identification, Supervised Learning, Unsupervised Learning.

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1. INTRODUCTION

T The process of Person Re-identification(Re-ID) has been thoroughly studied as a distinct person retrieval problem among non-overlapping cameras [1].Re-goal ID's is to determine whether a person of interest has ever been at a location at a different time that was captured by the same camera at a different time instant or even the same camera placed somewhere else [2].A photograph [3], a video clip [4], or even a written explanation [5] might be used to illustrate the subject.Person Re-ID is essential in smart surveillance technology with substantial academic effect and practical advantage due to the pressing need for community security and the growing number of security cameras.

The procedure of re-ID is difficult because of a variety of camera motions [6], poor picture resolutions [7, 8, 9, 10, 11, heterogeneous modalities [11, 12], complicated camera surroundings, background clutter [12], inaccurate bounding box creation, etc. These provide a lot of variations and uncertainty. Other elements that significantly increase the difficulties for realistic model deployment include the dynamically network of upgraded cameras [13], a massive gallery that offers effective restoration [14], group ambiguity [15], important domain change [16], unknown examining situations [17], and updating a model progressively [18], and changing clothes [19]. Re-ID still presents a problem as a result of these issues. This encourages us to carry out an extensive survey, establish a solid baseline for various Re-ID efforts,

and discuss a wide range of potential future paths. Person Although Re-ID is a difficult process, enhancing the semantic integrity of the analysis depends on it. Re-ID is crucial for programs that make use of single-camera surveillance systems. For instance, to find out if a person regularly visits the same place or if a different person or the same one picks up an abandoned box or bag. In addition to tracking, it has uses in robotics, multimedia, and more well-known technologies like automatic photo labeling and photo surfing [20].It is not difficult to comprehend the Person's Re-ID process. Being human, we always do it with ease. Our sights and minds have been conditioned to locate, identify, and then re-identify things and people in the actual world. Re-ID, which can be shown in Fig. 1, is the idea that a person who has been earlier seen would be identified as soon as they make an appearance using a specific description of the individual.

Even if hand-crafted features had some early success [21] and metric learning [22], the most advanced Re-ID algorithms currently available are constructed using convolutional neural networks (CNNs), which, when trained under supervision, need a significant amount of annotated (labelled) data to learn a stable embedding subspace. Recent deep learning approaches and detailed investigations on person Re-ID utilizing custom systemsprovided in [23], respectively. Large-scale dataset annotation for Re-ID is exceedingly labor intensive, time-consuming, and expensive, especially for techniques needing numerous bounding boxes for each individual to increase accuracy by making generalisations between two separate activities. One-shot learning and unsupervised learning are combined, for instance, in [24] and, which employ the Resnet50 [25] architecture with pre-trained networks on ImageNet [26]. Although it has been empirically demonstrated that pre-training and transfer learning significantly boost neural network performance, they are not appropriate for adjusting parameters across a wide range of domains or topologies. This article highlights the obstacles and unresolved problems in human re-identification and datasets, deep learning algorithms, and current research in these areas.

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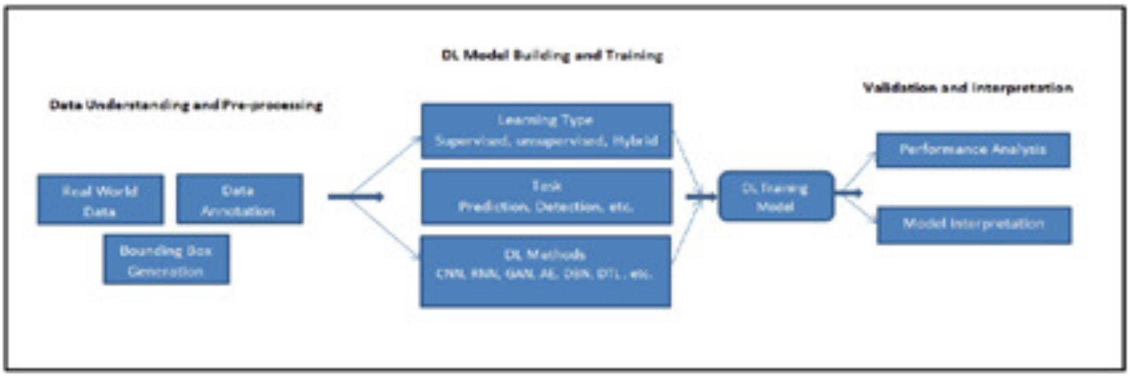

Figure 1. Shows an example of a common DL workflow that involves following five stages (i) Data Collection (ii) Bounding Box Generation (iii)Data Annotation (iv) Model Training (v) Validation [67].

2. DEEP LEARNING MODERN RESEARCH

In today's Era, intelligent systems and tech sophisticated automation are the main focuses across a diverse range of domains, including smart cities, e-Health, enterprise intelligence, innovative treatment, cyber security intellectual ability, and many more [27].Particularly when it comes to security technologies as a wonderful approach to disclose complicated data structures in high dimensions, deep learning techniques have substantially improved in terms of effectiveness across a wide range of applications. In order to create intelligent data-driven systems that satisfy current expectations, DL techniques might be extremely important because to their exceptional learning capabilities from past data. DL has the ability to change both the world and how people live since it can automate procedures and learn from mistakes.

3. DEEP LEARNING TECHNIQUES

This section discusses the various deep neural network techniques. These strategies frequently use hierarchical structures with numerous levels of information processing to learn. Among the numerous hidden layers that are frequently observed in deep neural networks are the input and output layers. Reviewing the different training exercises available, that is (i) Supervised, an approach that is challenging and utilizes the use of tagged training data, and (ii) Unsupervised, an approach that examines unlabeled sets of data, is important before diving into the details of DL approaches.

3.1 SUPERVISED OR DISCRIMINATIVE LEARNING NETWORK

The term "supervised learning" refers to a process in which a supervisor doubles as an educator. The technique of instructing or training a computer system using labelled data is known as supervised learning. This suggests that the appropriate response has already been given to the given data. The machine is then given a new collection of examples so that the supervised learning algorithm may examine the training data (set of training examples) and provide an accurate output from labelled data.Discriminative deep architectures are frequently developed to give discriminative capability for pattern classification by modelling the posterior distributions of classes conditioned on observable data [29]. The three main categories of discriminative architectures are Multi-Layer Perceptron (MLP), Convolutional Neural Networks (CNN or ConvNet), Recurrent Neural Networks (RNN), and their variations. Here, we'll briefly discuss these techniques.

3.1.1 MULTI-LAYER PERCEPTRON (MLP)

The feed-forward artificial neural network known as the Multi-layer perceptron (MLP) [30] is a technique for supervised instruction (ANN). The deep neural network (DNN) or deep learning base architecture are other names for it. Several activation functions, often referred to as transfer functions, including ReLU (Rectified Linear Unit), Tanh, Sigmoid, and Softmax determine an MLP network's output [32].

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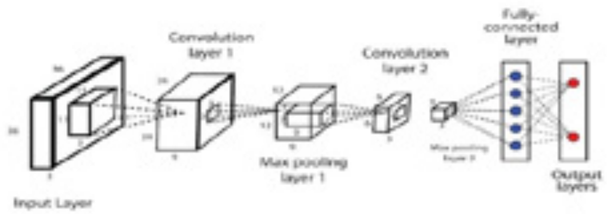
The most popular technique for training MLP is back-propagation [31], a supervised learning method that is frequently referred to as the fundamental component of a neural network. Throughout the training phase, a variety of optimization techniques are employed liked Stochastic Gradient Descent (SGD), Limited Memory BFGS (L-BFGS), and Adaptive Moment Estimation (Adam).

3.1.2 CONVOLUTIONAL NEURAL NETWORK (CNN)

The commonly used deep learning architecture known as a convolutional neural network [33] was modelled after the visual brain of animals [34]. As seen in fig 2, Originally it had been used extensively for tasks involving object recognition, but it is currently also being investigated in areas such as object tracking [35], pose estimation [36], text detection and recognition [37], visual saliency detection [38], action recognition [39], scene labelling , and many more [40].

Since CNNs are particularly made to control the various 2D forms, they are frequently employed in visual identification, analyze clinical data, segmenting an image, processing language naturally, and many other applications [41].Several CNN versions, including visual geometry group (VGG) [42], AlexNet [43], Xception [44], Inception [45], ResNet [46], etc., may be used in various application sectors depending on their learning capacity.



1. An illustration of Convolutional Neural Network.

3.1.3 RECURRENT NEURAL NETWORK (RNN)

Using sequential or time-series data, a different well-known neural network provides the result of one stage as input to the following step. The name for this neural network is recurrent neural network (RNN) [47]. Recurrent neural networks, like CNN and feed forward, learn from training input, but they stand out due to their "memory," which enables them to affect current input and output by consulting information from earlier inputs.While an RNN's output is reliant on what came before it in the sequence, a typical DNN assumes that inputs and outputs are independent of one another.However, because to the issue of declining gradients, standard networks with recurrence have difficulty in learning long data sequences. The popular recurrent network versions thatthe problems and perform effectively across a variety of real-life application areas are explored next.

3.1.4 LONG SHORT-TERM MEMORY (LSTM)

LSTMs are frequently used in video-based individual task re-ID and are capable of extracting temporal characteristics. Network for recurrent feature aggregation based on LSTM efficiently reduced interference brought on by background noise, shadowing, and recognition failure [48]. Among the first and shallowest LSTM nodes, it gathered cumulative discriminative characteristics. The temporal and geographical characteristics of the sections that include the probe pictures put were learned by the breakdown of a video sequence into multiple pieces [49]. The number of identical pedestrians in the sample is decreased by using this strategy, which also makes it simpler to identify similarity traits. Both of the aforementioned methods process each video frame independently.The duration of the video sequence typically has an impact on the characteristics that LSTM extracts. The RNN cannotcatch the temporalsignals of small details in the picture because it only creates temporal connections on high-level characteristics [50]. Therefore, research into a more effective technique for extracting spatial-temporal characteristics is still necessary.

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3.1.5 GATED RECURRENT UNITS (GRUS)

A popular gating-based variation of the recurrent network techniques to monitor and regulate the flow of information between neural network units is the Gated Recurrent Unit (GRU) [51]. A reset gate and an update gate are all that the GRU has, as seen in Fig. 3, making it less complex than an LSTM.The primary difference between the two devices is the number of gates: an LSTM has three gates compared to a GRU's two (the reset and update gates). The GRU's characteristics allow dependencies from long data sequences to be collected adaptively without removing information from previous portions of the sequence.GRU is a little more compact approach as a consequence, often providing comparable results, and is significantly faster to compute [52].

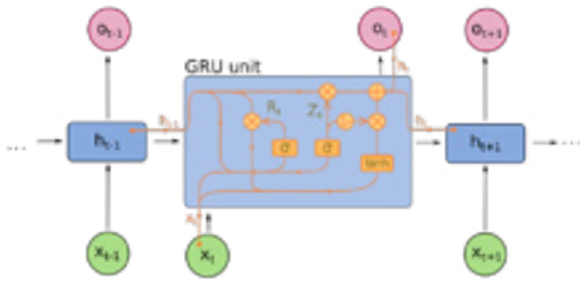

Figure 3. A Gated Recurrent Unit's Basic Structure with a Reset and Update Gate.

3.2 UNSUPERVISED LEARNING NETWORK

DL approaches are widely used to explain the combined statistical parameters of the available data and the classes that they belong to, as well as the higher predictive properties or features for pattern recognition or synthesis [53].Since the methods under this category include frequently used to learn features or data generation and representation, they are fundamentally utilized for unsupervised learning [54]. Since generative modeling maintains the correctness of the discriminative model, it may be used as a preliminary step for supervised learning tasks as well. For generative learning or unsupervised learning, deep neural network algorithms including the Generative Adversarial Network (GAN), Autoencoder (AE), Restricted Boltzmann Machine (RBM), Self-Organizing Map (SOM), and Deep Belief Network (DBN), as well as its variations, are often utilized.

3.2.1 GENERATIVE ADVERSARIAL NETWORK (GAN)

.GANs make use of neural networks' capacity to train a function that can simulate a distribution as closely as feasible to the real thing. They are particularly capable of producing synthetic pictures with great visual fidelity and do not rely on prior assumptions about the distribution of the data. This important characteristic enables the application of GANs to any imbalance issue in computer vision tasks. GANs give a technique to alter the original picture in addition to being able to create a false image. There are several GANs with different strengths that have been published in the literature to address the imbalance issue in computer vision tasks. For example, a specific type of GANs called AttGAN [55], IcGAN [56], ResAttrGAN [57], etc., is frequently employed for tasks involving modifying face attributes.

GANs are comprised of two neural networks, as shown in Fig. 4. The discriminator D predicts the chance that a following sample will be taken from real data as opposed to data given by the generator G, which generates new data with features similar to the original data.The generator and discriminator in GAN modeling are then instructed to interact with one another.Healthcare, computer vision, data augmentation, video production, voice synthesis, epidemic prevention, traffic control, network security, and many more fields might all benefit from the utilization of GAN networks. In general, GANs have demonstrated to be a solid field of independent data expansion and a solution to problems requiring generative techniques.

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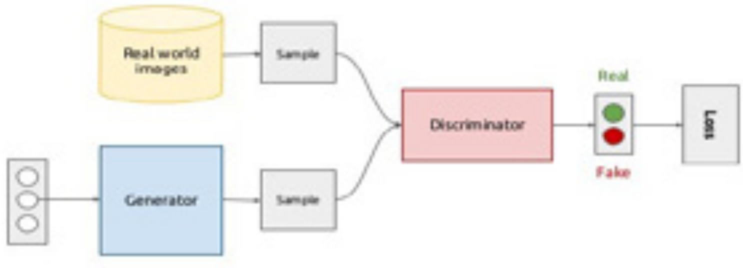

2. Generative Adversarial Networks Framework [58].

3.2.2 AUTO-ENCODER (AE)

A well-known auto-encoder (AE) unsupervised learning approach that makes use of neural networks to learn representations [59]. Data reduction describes the depiction of a set of data.High-dimensional data are often processed using auto-encoders. Three parts make up an autoencoder: an encoder, a code, and a decoder. The encoder creates the code that the decoder uses to reproduce the input by compressing it.Furthermore, generative data models have been learned using the AEs [60].Numerous unsupervised learning techniques, including dimension reduction, extraction of features, useful coding, dynamic modeling, noise removal, outlier or predictive modeling, etc., primarily rely on the auto-encoder. [59, 61]

3.2.3 KOHONEN MAP OR SELF-ORGANIZING MAP (SOM)

Another unsupervised learning method for constructing a low-dimensional (usually two-dimensional) representation of a higher-dimensional data set while preserving the topological structure of the data is the Self-Organizing Map (SOM) or Kohonen Map [62].SOM is a neural network-based method for dimension reduction in clustering [63].We can display huge datasets and identify likely clusters by using a SOM, which constantly moves a dataset's topological layout by bringing its neurons close to the data points inside the dataset.The input layer is the initial layer of a SOM, followed by the output layer, also known as the feature map, is the second layer. SOMs use competitive learning, which makes use of a neighboring function to preserve the topological properties of the input space, in contrast to other neural network models that use error-correction learning, such as Backpropagation with gradient descent [64].Sequence identification, sickness or health diagnosis, fault diagnosis, and virus or parasite attack detection are just a few of the many activities that SOM is frequently employed for [65].The main advantage of using a SOM is that it facilitates the discovery and recognition of patterns in high-dimensional data.

3.2.4 BOLTZMANN MACHINE WITH RESTRICTIONS (RBM)

A generative statistical neural network with the ability to learn a likelihood function over its inputs is the Restricted Boltzmann Machine (RBM) [66].Each node in a Boltzmann machine can be either visible or hidden and is linked to every other node.By understanding how the system functions normally, we can better comprehend anomalies. In RBMs, a subset of Boltzmann machines, there is a limit on the number of linkages between the accessible and deep layers [67].Due to this constraint, training techniques for Boltzmann machines in general can be more effective than those for Boltzmann machines, such as the gradient-based contrastive divergence algorithm [68]. Among the various uses of RBMs are data reduction, categorization, prediction, content-based- based filtering, pattern recognition, subject modelling, and many others.

3.2.5 DEEP BELIEF NETWORK (DBN)

A Deep Belief Network (DBN) [69] is a multiple-layer adaptive visuals model composed of so many unsupervised networks, such as AEs or RBMs, layered one on top of the other and using the hidden layer of each model as the input for the layer below it or connected sequentially.As a result, there are two types of DBNs: AE-DBNs, also called as stacked AE, and RBMDBNs, also known as stacked

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RBMs.The AE-DBN consists of autoencoders, whereas Boltzmann machines with constraints constitute RBM-DBN, as was already indicated. The final objective is to create a descriptive divergence-based quicker unsupervised training method for every sub-network [70].The deep structure of DBN allows it to store a hierarchical representation of incoming data.Network architectures for unsupervised feed-forward are trained using unlabeled data as the basic tenet of DBN, and the networks are subsequently fine-tuned using marked input. DBN's potentially significant benefits over traditional shallow learning networks is its capacity to identify specific patterns, with strengthens logic and the ability to distinguish between true and wrong data [71].

Thus, the strategies for generative learning that were previously explored frequently allow us to use research method to build a new set of the data. Deep generative models of these kinds with supervised or discriminative learning methods may benefit from this preparation and to guarantee model correctness because improving classifier generalization through unsupervised representation learning.

4. CHALLENGES AND OPEN ISSUES

The fundamental difficulty with Re-ID is the variance in a person's appearance across multiple cameras. Re-ID is challenging to make self-operating for a variety of reasons.Re-ID networks usually consist of two essential parts:the capture of a unique individual description and the process of comparing two models to see if they match or don't match.The capacity to automatically recognize and track individuals in photos or videos is necessary in order to develop a distinctive person description.Numerous difficulties and problems are apparent, and they will guide future research in the area of person Re-ID.

4.1 RE-ID DEPENDING ON DEPTH

Depth photos capture the bones and contours of the body.Re-ID is made possible by this, which is crucial for applications involving individualized human contact in lighting and clothing variations [88]. In [72], a paradigm based on recurrent attention is put out to solve individual identification based on depth. Convolutional and recurrent neural networks are used to locate tiny, exclusionary localized parts of the body in a reinforcement learning framework.

4.2 RE-ID USING VISIBLE-INFRARED TECHNOLOGY

Visible-Infrared Re-ID handles the cross-modality matching between the noticeable and thermal pictures [88]. Because only infrared cameras can take photographs in low-light conditions, it is essential [73].Along with the cross- modality shared embedding learning [74] also looks into the classifier level discrepancy. Recent methods [75] decrease cross-modality disparity at both the picture and feature level by creating cross-modality person photographs and applying the GAN approach.[76] models the cross-modal reconstruction using a hierarchy elements. [77] presents a dual-attentive aggregate learning strategy to identify multi-level links.

4.3 CROSS-RESOLUTION RE-ID

Taking into consideration the major resolution variations [78], Cross-Resolution Re-ID [88] compares images with different resolutions.The high-resolution human pictures are produced in a cascaded fashion using a cascaded SR-GAN [79], which also incorporates the identification data. The adversarial learning method is used by Li et al. [80] to create representations of pictures that are independent of resolution.

4.4 LABEL NOISE FOR RE-ID

It is typically hard to eliminate label noise when there is an annotation issue [88]. To prevent label overfitting problems, Zheng et al. use a label smoothing algorithm [81]. To effectively learn a Re-ID model while avoiding label noise and the consequences of data with high characteristic ambiguity should be mitigated, a Distribution Net (DNet) is described in [82] that encodes the feature uncertainty.

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For each identity, there aren't enough data for powerful Re-ID model training, unlike the generic classification issue [83]. It is also more challenging to learn the potent Re-ID model because of the unidentified new identities.

4.5 MULTI-CAMERA DYNAMIC NETWORK

The constantly updated multi-camera network [84], which necessitates model change for new cameras or probes, is another challenging issue. The Re-ID model may be updated and the representation can be tailored for different probing galleries employing an adaptive learning method with humans in the loop[85]. Active learning was a component of early research on continuous Re-ID in multi-camera networks [86]. [87] Introduces an approach for flexibility to adapt relying on the selective use of limited, non-redundant samples. Utilizing the ideal source camera theory and a geodesic flow kernel serve as the foundation for the development of a transitive inference strategy. An open-world person Re-ID system adds a number of contextual limitations (such Camera Topology) while dealing with large crowds and social interactions[88].

4.6 FEATURE LEARNING

High-level semantic representations of a person's traits, such their hair, gender, and age, can resist several environmental changes. For deep learning-based person Re-ID systems like those in [89], some research has used these traits to fill the gap between the photos and high-level conceptual data.Since it delivered on its predictions, feature recognition is one of the next possibilities.

4.7 ARCHITECTURE FOR AUTOMATED RE-ID

An advanced learning model's architectures must be manually created, which takes time, effort, and is prone to mistakes. The approach of automating architectural engineering, known as neural architecture search (NAS) [90], has recently been applied to address this issue. The study of NAS is currently receiving greater attention. Therefore, one of the essential aspects that must be considered in future research is the use of NAS for person Re-ID activities, as the majority of NAS techniques don't assure that there commended CNN is suitable for person Re-ID tasks.

4.8 ACCURACY VERSUS EFFICIENCY

Large models are typically employed to obtain the greatest accuracy, but they can be time and memory intensive, which reduces their usefulness, particularly when used to mimic real-time video monitoring systems. The majority of modern models did not consider CPU speed and memory capacity into account in order to increase accuracy. Authors who work in these fields must strike acompromise between processing speed and ranking accuracy.

4.9 LIGHTWEIGHT MODEL

The creation of a lightweight Re-ID model is another approach for dealing with the scalability problem. The issue of changing the network topology to create a light model is investigated [91, 92]. Another strategy is to employ model distillation. A system for multi-teacher customizable comparison reduction, for instance, is provided in [93], which in the absence of a primary data source, trains a user-specified lightweight student model from a number of teacher models.

5. DATASETS AND EVALUATION

Individuals' appearances vary greatly depending on Lighting, stances, view angles, scales, and camera resolutions may all vary while using different cameras. Visual ambiguities are further increased by elements like occlusions, a crowded background, and articulated figures. Therefore, it is crucial to gather data that successfully captures these aspects in order to create viable Re-ID approaches. In addition to good quality data that replicates actual circumstances, it is essential to compare and assess the Re-ID methodologies that are developed and find ways to enhance the methodology and databases.

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5.1 DATASETS BASED ON IMAGES

Person re-ID using images, there have been a variety of datasets; the most common datasets are listed below.

VIPeR[94]: It is made up of 1,264 photos for 632 people and was taken by 2 non-overlapping cameras.

CUHK01 [95]: It is made up of 3,884 photographs for 971 individuals that were recorded by two separate cameras on a university campus.

Market-1501 [96]: It is comprised of 32,643 photos for each of the 1,501 people that make up the sample, which was taken from the front of a shop using 2 to 6 separate cameras.

DukeMTMC-ReID[97]: The dataset consists of 46,261 photos for 1852 individuals that were captured by 8 non-overlapping cameras on the Duke University campus.

Kinect-REID [98]: It has 71 person sequences that were recorded at the authors' department.

RGBD-ID [99]: There are four groups with various viewpoints, each with the same 80 people. It is produced on several days and has various aesthetic variants.

RegDB[100]: 412 people are represented by 4120 RGB photos and 4120 thermal photographs. Two types of cameras are used to capture them.

SYSU-MM01 [101]: It comprises of 491 people's 15,792 infrared photographs and 287,628 RGB photos. Six cameras, including two infrared cameras and four RGB cameras, are used to capture them from the writers' section.

5.2 DATASETS BASED ON VIDEOS

PRID2011 [102]: It is constituted of 24541 photos for 934 individuals from 600 recordings taken from two separate cameras in an airport's multi-camera network.

iLIDSVID[103]: It is made up of 600 movies shot by 2 non-overlapping airport cameras and 42495 photos for 300 people.

MARS [104]: The greatest video-based individual Re-ID dataset resides in this one. It is made up of around 1191003 photos for 1261 individuals from 200 recordings that were captured by 2 to 6 non-overlapping cameras.

RPIfield [105]: It consists of 601,581 images for 112 individuals, captured by 2 separate cameras on an open field at a college.

5.3 EVALUATION METRICS

Cumulative matching characteristics (CMC) [106] and mean average precision (mAP) are the two mostly used metrics for assessing Re-ID systems [107].

6. CONCLUSION

We have discussed the subject of human re-identification, as well as difficult problems and a summary of recent research in the discipline of person recognition, in this paper. Both closed set and open set Re-ID tasks have been taken into consideration. The approaches employed have been grouped, and their advantages and disadvantages have been covered. Additionally, we have outlined the benefits and drawbacks of the various Re-ID datasets. Popular Re-ID assessment methods are briefly discussed, along with potential expansions.In conclusion, person Re-ID is a broad and difficult field with much of space for growth and research.An effort is made in this work to give a concise overview of the Re-ID problem, its limitations, and related problems.

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/18/

NEAR-LOSSLESS COMPRESSION SCHEME USING HAMMING CODES FOR NON-TEXTUAL IMPORTANT REGIONS IN DOCUMENT IMAGES

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ABSTRACT

Working at Bell Labs in 1950, irritated with error-prone punched card readers, R W Hamming began working on error-correcting codes, which became the most used error-detecting and correcting approach in the field of channel coding in the future. Using this parity-based coding, two-bit error detection and one-bit error correction was achievable. Channel coding was expanded further to correct burst errors in data. Depending upon the use of the number of data bits 'd' and parity bits 'k' the code is specified as (n, k) code, here 'n' is the total length of the code (d+k). It means that 'k' parity bits are required to protect 'd' data bits, which also means that parity bits are redundant if the code word contains no errors. Due to the framed relationship between data bits and parity bits of the valid codewords, the parity bits can be easily computed, and hence the information represented by 'n' bits can be represented by 'd' bits. By removing these unnecessary bits, it is possible to produce the optimal (i.e., shortest length) representation of the image data. This work proposes a digital image compression technique based on Hamming codes. Lossless and near-lossless compression depending upon need can be achieved using several code specifications as mentioned here. The achieved compression ratio, computational cost, and time complexity of the suggested approach with various specifications are evaluated and compared, along with the quality of decompressed images.

KEYWORDS

Hamming code, Parity, Lossless Compression, Near Lossless Compression, Compression Ratio.

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1. INTRODUCTION

Image coding and compression is used mainly for effective data storage and transmission over a network and in some cases for encryption. Image data is also coded for achieving compression to optimize the use of these resources. In digital image compression, depending upon the quality of decompressed image the compression algorithms employed are categorised in the categories like Lossless compression, Lossy compression, and Near-lossless compression. The data redundancy is a statistically quantifiable entity, it can be defined as $R\_D=1-1/CR$, where the CR is the compression ratio represents the ratio of number of bits in compressed representation to number of bits in original representation. A compression ratio of 'C' (or 'C':1) means that the original data contains 'C' information bits for every 1 bit in the compressed data. The associated redundancy of 0.5 indicates that 50% of the data in the first data set is redundant. Three primary data redundancies can be found and used in digital image compression i.e. coding redundancy, interpixel redundancy, and psychovisual redundancy. When one or more of these redundancies are considered as a key component for reduced representation of data and accordingly these redundancies are encoded with some method the compression is achieved, Sayood (2017). There is no right or wrong decision when deciding between lossless and lossy image compression techniques. Depending on what suits your application the most, you can choose. Lossy compression is a fantastic option if you don't mind sacrificing image quality in exchange for smaller image sizes. However, if you want to compress photographs without sacrificing their quality or visual appeal, you must choose lossless compression Kumar and Chandana (2009). Based on a knowledge of visual perception, the irrelevant part of the data may be neglected, a lossy compression, includes a process for averaging or eliminating this unimportant information to reduce data size. In lossy compression the image quality is compromised but a significant amount of compression is possible. When the quality of decompressed image and integrity are crucial then lossy compression shouldn't be employed Ndjiki-Nya et.al (2007). Not all images react the same way to lossy compression. Due to the constantly changing nature of photographic images, some visual elements, including slight tone variations, may result in artefacts (unintended visual effects), but these effects may largely go unnoticed. While in line graphics or text in document images will more obviously show the lossy compression artefacts than other types of images. These may build up over generations, particularly if various compression algorithms are employed, so artefacts that were undetectable in one algorithm may turn out to be significant in another. So, in this scenario one should try to bridge the consequences of lossless and lossy compression algorithms. So, the near-lossless compression algorithm should be practiced in case of document images to optimise the compression ratio and the quality of reconstruction Ansari et al. (1998). One of the very famous Error detection and correction technique used in channel coding may be used for the digital image compression Caire et.al (2004) and Hu et.al (2000). In this paper use of Hamming codes with different specifications for various compression algorithms mentioned above is done and the compression is achieved.

2. HAMMING CODES

When the channel is noisy or error-prone, the channel encoder and decoder are crucial to the overall encoding-decoding process. By adding a predetermined amount of redundancy to the source encoded data, it is possible to minimize the influence of channel noise. As the output of the source encoder is highly sensitive to transmission noise, it is based on the principle that enough controlled redundant bits must be added to the data being encoded to guarantee that a specific minimum number of bits will change during the transmission. Hamming demonstrated, showed that all single-bit errors can be detected and corrected if 3 bits of redundancy are added to a 4-bit code word, providing the Hamming distance between any two valid code words 3 bits. The 7-bit Hamming (7, 4) code word P1, P2, D3, P4, D5, D6, D7 for a 4-bit binary number $b1,b2,b3,b4 \cong D3,D5,D6,D7$ padded with parity bits $p1,p2,p3 \cong P1,P2,P4$.

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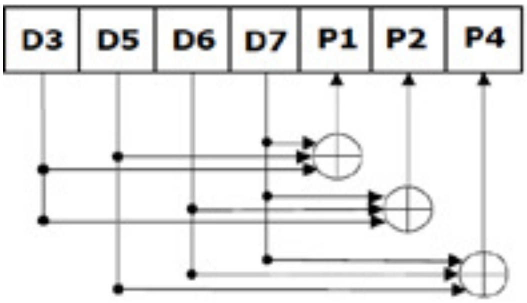

Fig. 1. Hamming Code

2.1. ERROR DETECTION

One of the most popular hamming code specifications used is (7,4); here 7 is total length of codeword and 4 is number of data bits.

Here (n-k) i.e., 7-4 =3 parity bits are used to encode 4-bit message to detect one bit error and correct one bit error.

In the following example, the first step of algorithm is to identify the position of the data bits and parity bits. All the bit positions at powers of 2 are marked as parity bits (e.g., 1, 2, 4, 8). Given below is the structure of 7-bit hamming code.

Here a data "0 1 0 1" is encoded using (7,4) even parity hamming code and an error is introduced in the fifth (D5) bit.

P1	P2	D3	P4	D5	D6	D7
1	1	0	1	1	0	1

Three parity checks and needed to be considered to determine whether there are any errors in the received hamming code.

Check 1: Here the parity of bits at position 1,3,5,7 should be verified

P1	D3	D5	D7
1	0	1	1

It is observed that the parity of above codeword is odd, it is concluded that the error is present and check1 is failed.

Check 2: Here the parity of bits at position 2,3,6,7 should be verified

P2	D3	D6	D7
1	0	0	1

It is observed that the parity of above codeword is even, then we will conclude the check2 is passed.

Check 3: Here the parity of bits at position 4,5,6,7 should be verified

P4	D5	D6	D7
1	1	0	1

It is observed that the parity of above codeword is odd, it is concluded that the error is present and check3 is failed.

So, from the above parity analysis, check1 and check3 are failed so we can clearly say that the received hamming code has errors.

2.2. ERROR CORRECTION

They must be repaired because it was discovered that the received code contains an error. Use the next few steps to fix the mistakes:

The error checks word has become:

check1	check2	check3
1	0	1

The error is in the fifth data bit, according to the decimal value of this error-checking word, which was calculated as "1 0 1" (the binary representation of 5). Simply invert the fifth data bit to make it correct.

So, the correct data will be:

P1	P2	D3	P4	D5	D6	D7
1	1	0	1	0	0	1

In the same fashion other specifications of code like (3,1), (6,3), (7,3), and so on can be implemented and error detection and error correction of messages using the parity bits can be accomplished. There has never been a way for error checking and correcting that is more effective than Hamming codes, so, it is still widely used channel encoding. It offers an effective balance between error detection and correction, in addition to that one may verify its application in data compression field, which is being discussed in the next topic.

3. PROPOSED WORK

The document image compression, if various logical regions of the document image are segmented and appropriate compression algorithm is used for those regions, then the trade-off between compression ratio and the quality of decompressed image may be resolved a bit. Lossy compression techniques for photographs, Near-lossless compression for Figures, Tables and other text-like important regions, and Lossless compression for the Text contents Shanmugasundaram *et.al* (2011).

3.1. THE EXISTING LOSSLESS ENCODING ALGORITHM

i. Reshaping: 2D to 1D conversion of image data

ii. Resizing: Divide the data in terms of 7-bit chunks

iii. Checking Hamming encodability of newly formed chunks; valid and invalid hamming codes (7,4)

iv. Lossless Encoding of data

v. beginning with ‘0’+codeword (7 bit), if the chunk is invalid codeword

vi. beginning with ‘1’+encoded codeword (4 bit), if the chunk is valid codeword

vii. If the size of the coded data is smaller than the input data, then compression is achieved.

The following algorithms are the step by step modification in the existing work of Hu *et.al* (2000). Here, the near-lossless encoding algorithm based on the idea of bit plane slicing is presented as Algorithm 1. It is presumptive that the least significant bit (LSB) in an image contains the least significant information, and that if this information is removed from a gray-level image, a little visual degradation will be caused.

3.2. ALGORITHM 1: BIT PLANE SLICING BASED LOSSY ENCODING

i. Perform bit-plane slicing: remove LSB (assuming that LSB plane consists of least information)
ii. Reshaping: 2D to 1D conversion of image data
iii. Resizing: Divide the data in terms of 7-bit chunks
iv. Checking Hamming encodability of new chunks; valid and invalid hamming codes (7,4)
v. Lossless Encoding of data
vi. beginning with '0'+codeword (7 bit), if the chunk is invalid codeword
vii. beginning with '1'+encoded codeword (4 bit), if the chunk is valid codeword
viii. If the size of the coded data is smaller than the input data, then compression is achieved.

The second approach, which also uses (8,4) Hamming codes for lossless compression, has the additional capability of detecting and correcting error in the eighth bit, which is set or reset based on the even or odd parity of the entire 7-bit codeword in (7,4) variant. This technique offers the lossless compression in addition to the additional 1-bit detection and correction capabilities.

3.3. ALGORITHM 2: LOSSLESS ENCODING USING (8,4) HAMMING CODE SPECIFICATION

i. Reshaping: 2D to 1D conversion of grey image data
ii. Checking Hamming encodability of newly formed chunks; valid and invalid hamming codes (8,4)
iii. Lossless Encoding of data
iv. beginning with '0'+codeword (8 bit), if the chunk is invalid codeword
v. beginning with '1'+encoded codeword (4 bit), if the chunk is valid codeword
vi. If the size of the coded data is smaller than the input data, then compression is achieved.

It is computationally expensive to scan the complete image to determine if the codewords (its grey levels) are valid or invalid hamming codewords. Therefore, a novel technique for eliminating this avoidable routine is being tested, which involves discovering the valid codewords beforehand and simply classifying image gray-levels as valid or invalid codewords by comparing with them. This process is similar to quantization but here the quantization levels are neither equidistant nor generated by some programming language function. Even though this quantization inevitably results in information loss, the compression ratio will be improved Li *et.al* (2002). These quantized gray-levels offers spectral compression at the same time the probabilities of the resultant gray-levels also get changed. The image with changed gray-level probabilities is further feed to probability-based coding like Huffman's coding and added compression is achieved Jasmi *et.al* (2015) and Huffman (1952). The algorithms 3 is subsequently modified in algorithm 4 and 5 using (8,4) code for lossless compression with 16 and 32 quantization levels respectively. After the successful compression and decompression, the quality of decompressed image using quantization approach is computed based on the parameters like Correlation of input output images Dhawan (2011), its Mean Squared Error (MSE), Signal to Noise Ratio (SNR), Structural Similarity Index Metric (SSIM) to compute the retainment of structural properties of the input images, Compression Ratio (CR) achieved and the Computational Time (CT).

3.4. ALGORITHM 3: FAST NEAR-LOSSLESS ENCODING USING (7,4) HAMMING CODE SPECIFICATION AND QUANTIZATION

i. Reshaping: 2D to 1D conversion of image data
ii. Perform bit-plane slicing: remove LSB (considering that it LSB plane consists of the least information) OR
Resizing: Divide the data in terms of 7-bit chunks

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iii. Identify the '16' valid codewords and enlist them in an array valcod
iv. valcod = [0 15 22 25 37 42 51 60 67 76 85 90 102 105 112 127]
v. Lossy step: Quantize the entire image pixels to grey levels in valcod variables
vi. If size of coded data is smaller than input data, then compression is achieved.

3.5. ALGORITHM 4: FAST NEAR-LOSSLESS ENCODING USING (8,4) HAMMING CODE SPECIFICATION AND 16 LEVEL QUANTIZATION

i. Reshaping: 2D to 1D conversion of image data
ii. Identify the '16' valid codewords and enlist them in an array valcod16
iii. Valcod8 = [0 15 51 60 85 90 102 105 150 153 165 170 195 204 240 255]
iv. Lossy step: Quantize the entire image pixels to grey levels in valcod16 variables
v. If size of coded data is smaller than input data, then compression is achieved.

3.6. ALGORITHM 5: FAST NEAR-LOSSLESS ENCODING USING (8,4) HAMMING CODE SPECIFICATION AND 32 LEVEL QUANTIZATION

i. Reshaping: 2D to 1D conversion of image data
ii. Identify the 'total 32' valid even, odd conjugate codewords along-with its complements and enlist them in an array valcod32
iii. valcodeoc = [0 15 23 24 36 43 51 60 66 77 85 90 102 105 119 129 136 142 150 153 165 170 178 189 195 204 212 219 231 232 240 255]
iv. Lossy step: Quantize the entire image pixels using the 32 grey-levels in valcod32 variable
v. If the size of the coded data is smaller than the input data, then compression is achieved.

4. RESULTS

The experimentation carried out on a dataset consisting of three classes of images like graphs, diagrams, and equations, which are the text-like regions in the document image. As discussed above the existing Hamming code-based algorithms is implemented and executed over the dataset. Performance of these algorithms is evaluated by means of the performance metrics like compression ratio and computation time. The algorithms are implemented using MATLABR2020b software on a 64-bit, 2.11 GHz processor with 8 GB RAM computer system. The algorithm offers compression generally but, in some times, (4/30) it failed to achieve the compression. The average compression ratio achieved is 1.2 : 1, which is not significant considering the computation cost of the algorithm. The regular scanned document takes computation time up to 10 min. So, the images in the dataset are resized to 512 X 512 pixel size and further computation time is observed. For this size of images, the computation time results 21.30 sec./ image. This time is also higher considering the size of image. To improve the CR and minimizing the computation cost the bit-plane slicing based lossy compression algorithm (Algorithm 1) is proposed, it has enhanced the CR a bit and the CT is also halved. Further to avoid information loss a (8,4) and need of image resizing, the Hamming code-based algorithm (Algorithm 2) is tested. It further enhanced the CR but the CT once again increased up to original algorithm. This performance of mentioned algorithms is presented using Table 1 below.

Table 1. Comparison of purely Hamming code-based algorithms based on CR and CT.

Sr. No.	Algorithm	CR (CR:1)	CT (sec/ image)
1	Algorithm 0	1.20	21.30
2	Algorithm 1	1.24	10.32
3	Algorithm 2	1.29	19.15

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As both the parameters CR and CT offered by these algorithms are not attractive, the quantization-based algorithms (Algorithm 3, Algorithm 4, and Algorithm 5) are proposed, avoiding traversing throughout the image for checking the validity of codewords. These algorithms have achieved the significant CR along with 100 times less CT. The quality of decompressed image using quantization approaches is computed based on the parameters like Correlation of input output images, its Mean Squared Error, Signal to Noise Ratio, Structural Similarity Index Metric to compute the retainment of structural properties of the images, Compression Ratio achieved and the Computational Time. The results are presented in Table 2.

Table 2. Comparison of quantization-based algorithms.

	Algorithm 3	Algorithm 4	Algorithm 5
Correlation (0-1)	0.9420	0.9629	0.9658
MSE (bits/image)	11.22	37.80	12.13
PSNR (Ratio)	39.19	33.22	39.17
SSIM (0-1)	0.9295	0.9308	0.9519
CR (CR:1)	3.04	3.31	3.31
CT (sec/image)	0.2575	0.1132	0.1314

It can be observed from the above table that Algorithm 5 performs better than others considering the mentioned parameters except MSE and CT; the MSE of algorithm 3, that is less 1 bit/image lesser than Algorithm 5 and CT of Algorithm 4 is least, that is about 0.02 sec lesser than Algorithm 5. The Correlation and SSIM offered by this algorithm are highest, which is very significant while proposing the near-lossless algorithm. The performance of quantization-based approach computed over mentioned three classes using the image quality metrics discussed is presented below. The average values for these classes are shown to get overall idea about performance of algorithm.

4.1. CORRELATION

The 2D correlation between the input image and the decompressed result is calculated, it ranges from 0-1, where value ‘1’ represent that both the images are identical. The values near to ‘1’ signifies the near-lossless compression. The performance is displayed using Fig. 2.

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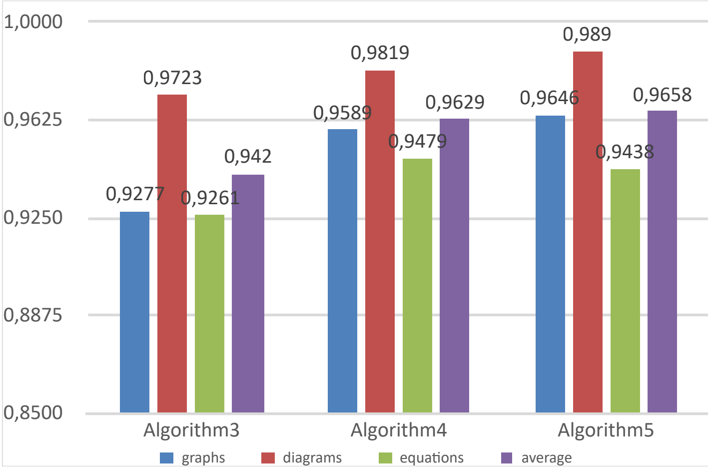

Fig. 2. Comparison of algorithms based on Correlation.

4.2. MSE

The most used estimator of image quality is MSE refers to the metric giving cumulative squared error between the original and compressed image. Near zero values are desirable. The performance is displayed using Fig. 3.

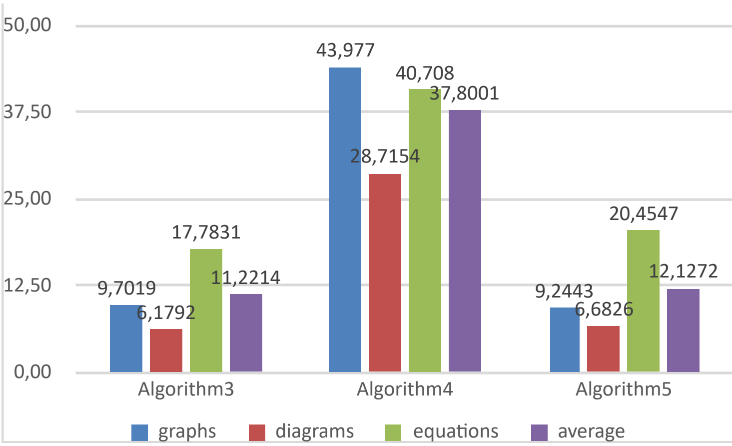

Fig. 3. Comparison of algorithms based on MSE.

4.3. PSNR

The Peak Signal to Noise Ratio is positive indicator of performance of compression algorithms. The performance is displayed using Fig. 4, here the Algorithm 5 offers the highest PSNR value, which is a good indicator of its applicability.

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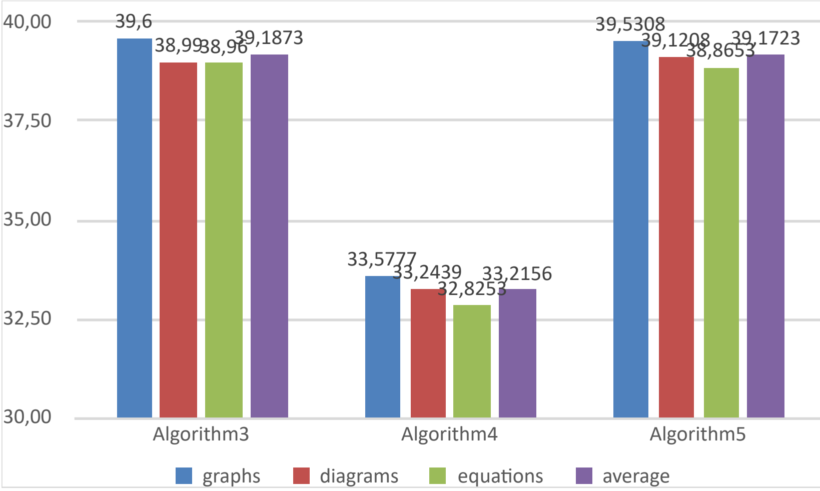

Fig. 4. Comparison of algorithms based on SNR.

4.4. SSIM

The Structural Similarity Index Measure (SSIM), a perception-based measure, considers image degradation as a perceived change in structural information. These techniques differ from others like Mean Squared Error (MSE) and Signal to Noise Ratio that include evaluate absolute errors (SNR). According to the theory behind structural information, pixels are highly interdependent, when they are spatially close to one another. These dependencies carry important details about how the elements in an image are arranged. In our experimentation Algorithm 5 has the highest SSIM, which is another good indicator of the quality of a compression algorithm. The performance is displayed using Fig. 5.

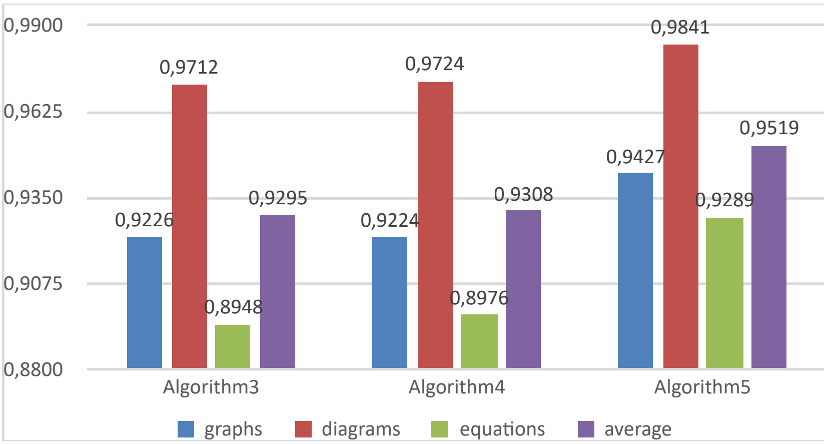

Fig. 5. Comparison of algorithms based on SSIM.

4.5. COMPRESSION RATIO

Here, the average compression ratio for the considered images, ranges from 3.04 : 1 to 3.31 : 1, algorithm 5 having the highest value and algorithm 3 having the lowest. The compression ratio is a useful indicator of decompression effectiveness, the higher value indicates a better algorithm. The performance is displayed using Fig.6.

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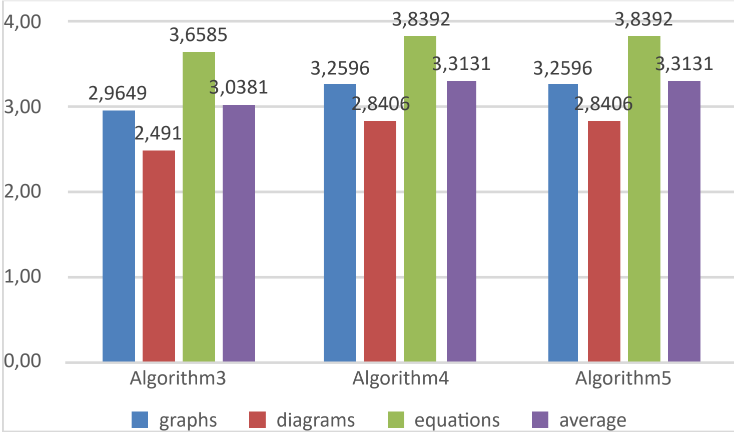

Fig. 6. Comparison based on Compression Ratio.

4.6. COMPUTATION TIME

Finally, computational time is used to study the time complexity of the algorithm; in this case, Algorithm 4 tends to be the most efficient and requires the least amount of time. The performance is displayed using Fig. 7.

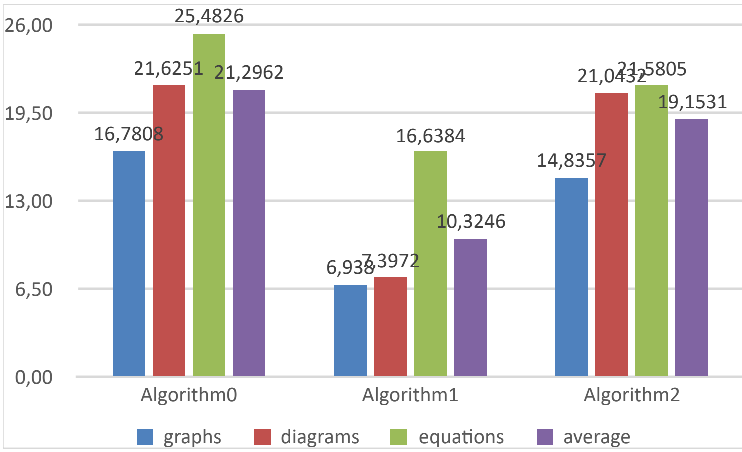

Fig. 7. Comparison based on Computation Time.

Algorithm 5 having highest CR, executes faster and performs better in terms of quality of decompressed image. So, one can conclude that the Algorithm 5 tends to the most optimized algorithm for Near-lossless compression of Document Images.

5. CONCLUSIONS

Compression offers an attractive option for efficiently storing large amounts of data. Document Images are multi-tone images, separate algorithm for different regions may improve the compression outcomes. Lossy algorithms are best for regions like photographs, lossless are necessary in case of sensitive regions like text and the non-textual but text consisting important regions like figures, plots and equations needed to be treated with near-lossless techniques. They are advantageous for maintaining the trade-off between compression ratio and quality of reconstructed image. In this work the channel coding algorithm like Huffman codes is implemented with its different specifications, the work is further optimized based on time complexity and quality of decompressed image. Considering the higher Correlation, PSNR and SSIM values obtained, one can understand that the quality of the decompressed images is really good, and the algorithms can be considered as near-lossless algorithms. It is also concluded that the (8, 4) Hamming code with even and odd conjugate complementary codes and based 32 quantization levels works best amongst the other discussed.

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6. FUTURE SCOPE

After observing the performance of various hamming code-based algorithms, one may be encouraged to practice them and do some modification. So, the first thing that may be tried is to make use of (3,1) and (4,1) specifications of Hamming code. Further, after the quantization to newly achieved gray levels is done, one can implement probability-based coding scheme like Huffman or Arithmetic codes to represent the data more effectively.

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EFFICIENT SYSTEM FOR CPU METRIC VISUALIZATION

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ABSTRACT

There are multiple metrics associated with the smooth and efficient working of a computer system. Some of the crucial parts are like the CPU, memory usage and GPUs. For different Operating Systems, they have their own System Software for managing and analysing their sessions. Like Task Manager in Windows, Nmon in Linux and Activity Monitor in Mac. In addition to it, there are few applications software's which perform similar tasks with slight modifications. In this paper, a web application is proposed that will fetch these performance metrics from the user's system and display them using dynamic charts. The proposed application is a system independent tool and can be useful for all operating system. The application also can be used to determine whether or not a game is compatible with the user's system based on the system requirements.

KEYWORDS

CPU Metrics Visualization, Operating Systems, Resource Utilization, System Monitoring, Performance measures.

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1. INTRODUCTION

Computer Systems have evolved by leaps and bounds in the last few decades stemming from the introduction of technology trends and improvements. This has paved the way for new architectures and systems with higher and better performance to evolve. It is always a good idea to keep an eye on how much of your resources are being used when working on various chores and projects while having a lot of files, folders, and tabs open on your local system to make sure they don't go over their allotment. Although higher-end computers or workstations rarely have these problems, it may be argued that most typical laptop and PC users must confirm the best use of their resources. The advances in new technology comes with proper analysis and evaluation of the performance metric of the processor. Understanding the performance measure is essential to comprehend the underlying computer organization and bring about modifications based on the different dependent factors. So, there are even various Hardware metrics which are essential and its detailed description helps in getting the insights of our system. Given are few metrics in accordance with Windows Task Manager. Few of them are:

- CPU: CPU's name and model number, speed, number of cores, and accessibility to hardware virtualization features. Additionally, it displays the amount of "uptime" the time since your system's last boot which is another useful statistic.
- Memory: How much RAM you have, how fast it is, and how many of the motherboard's RAM slots are occupied. You can also view the amount of cached material that is currently using your memory. For Windows, this is "standby." Although Windows will instantly clear the cached data and make room if your system requires more RAM for another job, this data will be available and waiting if your system needs it.
- Disc: Your disc driver's name, model number, size, and current read and write speeds.
- Wi-Fi: It displays the name of a network adapter along with its IP addresses (IPv4 and IPv6 addresses).
- GPU: The GPU pane displays different graphs for various activities, such as 3D vs. video encoding or decoding. Due to the GPU's own internal memory, it also displays GPU memory utilization. Here, you may also see the name, model number, and graphics driver version of your GPU.

There are various factors that may affect the system's performance. The instructions used, the memory hierarchy, handling of the input/output all contribute to the computer's performance. The most important parameter is time and is crucial to evaluate the performance of a computer. There are several time-based parameters such as the response time which can be defined as the time taken by the processor to respond after the execution of a particular task, or the throughput which is the total work done for a given time. One such parameter is CPU utilization which is commonly used to understand and evaluate the system performance. It describes the CPU usage for a given time interval as a percentage. It is also an excellent indicator of whether the system has been attacked by a virus or if the system does not have sufficient CPU power. This can be observed in a case when CPU utilization is high (indicating a heavy load) despite there being not many programs running in the background. A computer's RAM is extremely important when it comes to handling tasks and applications. The computer or device is observed to slow down in case there isn't sufficient RAM available. To get the maximum use out of a system, a tool for monitoring these performance parameters to generate logs and allow visualization in the form of graphs is absolutely necessary. Such an application provides other insights about the amount of memory used by a particular program, the availability of the computer's hardware resources and can also be used to force a frozen program. In this paper, another such application is proposed that will help the user view the system information of his device and view certain performance parameters such as CPU Utilization and availability of the RAM. While using applications such as games, the CPU usage (utilization for a given process) can go up as high as 50 or

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60 % which may result in overheating of the system, decrease frame rates and even cause system crashes.

Thus, the player must be made aware if the game has system requirements not compatible with his system. A good RAM specification is needed for a smooth experience while playing games. This ensures that there are fewer lags and allows for higher frame rates. RAM usage also depends on the kind of software or game that is being run on the user's device. Hence, this paper also proposes an additional feature that will help determine if the user can run a particular game on his or her device based on his system requirements.

The paper has the following structure, the 'Literature Survey' Section provides a concise overview of the existing literature and tools available for the analysis of CPU metrics, the next section – 'Comparison Table' illustrates the different tables and diagrams used to support the ideas presented in this paper, 'Proposed Work' section discusses the methodology of implementation in detail while the results and outputs are presented in the 'Results and Discussions' section. Finally the paper discusses some aspects of improvement in the future and provides a conclusion outlining the key points of the entire paper.

2.1. LITERATURE SURVEY

Stefanov and Gradskov (2016) monitored and analysed some properties of CPU usage data provided by Linux Kernel. This work analysed CPU usage data provided by the Linux kernel and how CPU load level is calculated based on these data. For every active CPU in the system, the kernel gives the amount of time, measured in 1/100th of a second, that the system spent in different modes of execution since boot. These different modes are 'user mode' (running user processes), 'user mode with low priority' (nice), 'system mode' (running kernel), 'idle', 'iowait' (idle while waiting for IO request to complete), 'irq' (processing interrupts), 'softirq' (processing software interrupts).

Formula for calculating Load Level (L): If 'T^i_m' is the time spent in 'm-th' mode at time moment 'i' then, the CPU load level 'L_m' for the mode m is represented below using Equation (1):

$$L_m = \frac{T^i_m - T^{i-1}_m}{\sum_m (T^i_m - T^{i-1}_m)} \quad (1)$$

Urriza and Clariño (2021) used Python to web scrape reviews written on 'Steam' website and classifies them into either Audio, Gameplay or Graphics. Further it also categorizes them into Positive, Negative and Neutral sentiments.
Gomes and Correia (2020) Cryptojacking infects the browsers and does CPU intensive computation to mine cryptocurrencies on behalf of cyber criminals. The paper introduces a new Cryptojacking detection mechanism based on monitoring CPU usage of visited web pages. They use machine learning along with monitoring and monitoring the precision and recall values to make a decision. In this Study they used CPU utilization for decking if certain games can be run or not on the computer, similarly here we are utilizing the CPU for detecting cryptojacking. It uses 'mpstat', a command line tool to monitor the CPU. It gives us several metrics such as CPU Utilization at every level, CPU Utilization at user level, CPU Utilization at system level, time that the CPU or CPUs were idle during which the system had an outstanding disk I/O request, time spent by the CPU or CPUs servicing hardware interrupts, time spent by the CPU or CPUs servicing software interrupts, time spent in involuntary wait by the virtual CPU or CPUs while the hypervisor was servicing another virtual processor, time spent by the CPU or CPUs running a virtual processor; time that the CPU or CPUs were idle and the system did not have an outstanding disk I/O request.

"Windows Task Manager" can optimize performance by terminating programmes and processes, changing processing priorities, and setting processor affinity as necessary. Task Manager shows fundamental performance information and visual representations of CPU, swap file, and memory utilization as a monitoring tool. Disk and network information are also included in later versions of Task Manager.

"Glances - an eye on your system" Glance-Glances is a system monitoring tool for Linux computers, it is used to monitor system resources. Views can show more system information than any other conventional monitor. This comprises disc and network I/O, temperature information—which can reveal the temperatures of the CPU and other hardware—as well as fan speeds and disc utilization by logical volume and hardware device.

"What is system explorer?,System Explorer - Keep Your System Under Control." System Explorer is not only a replacement for task manager, but it is more than that. Additionally, to aiding in process management, it has a number of characteristics that can boost system security and avert disasters. Even a portable version is available for it. It provides great features like calculating the CPU metrics history, taking snapshots and more.

"Can you Run it?" System Requirements Lab is an online platform which provides information about different games and the minimum hardware and software requirements to run them in any local system. The platform also helps you to determine whether the games can run on your current system by comparing the minimum requirements and the local systems information.

2.2. COMPARISON TABLE

Various System Software's and Application Software's with the task of evaluating metrics of a system are discussed in Table 1 below:

Table 1. Comparison of various system monitoring tools.

Sr No	System Monitoring Tool	Operating System	Type	Performance Metrics Monitored
1	Task Manager	Windows	System Software	CPU, Memory, Disk, Network
2	Glances	Linux	Application Software	CPU, Memory
3	Process Explorer	Windows	Application Software	Real Time – CPU, GPU, Memory, Disk, Network
4	System Explorer	Windows	Application Software	CPU, Memory

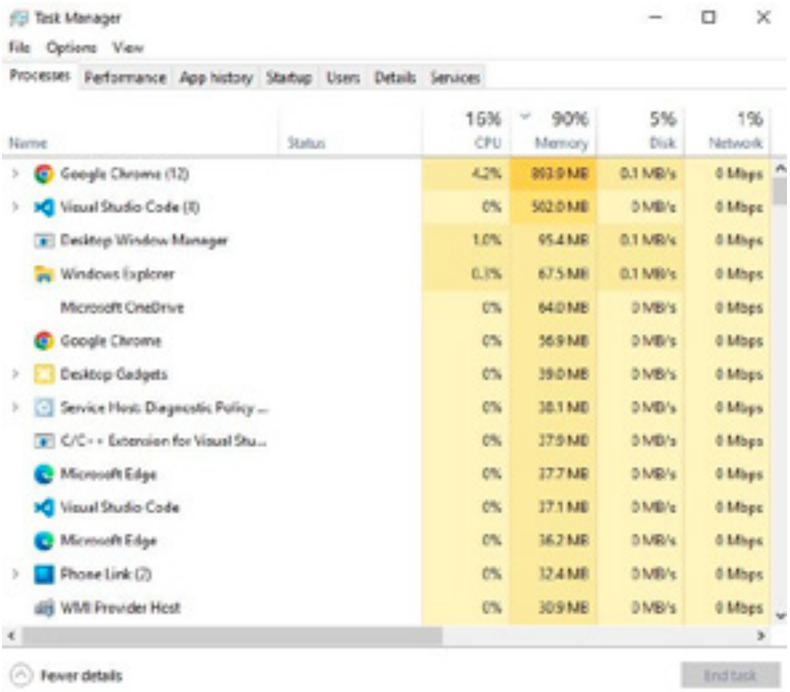

Fig. 1. Task Manager Interface.

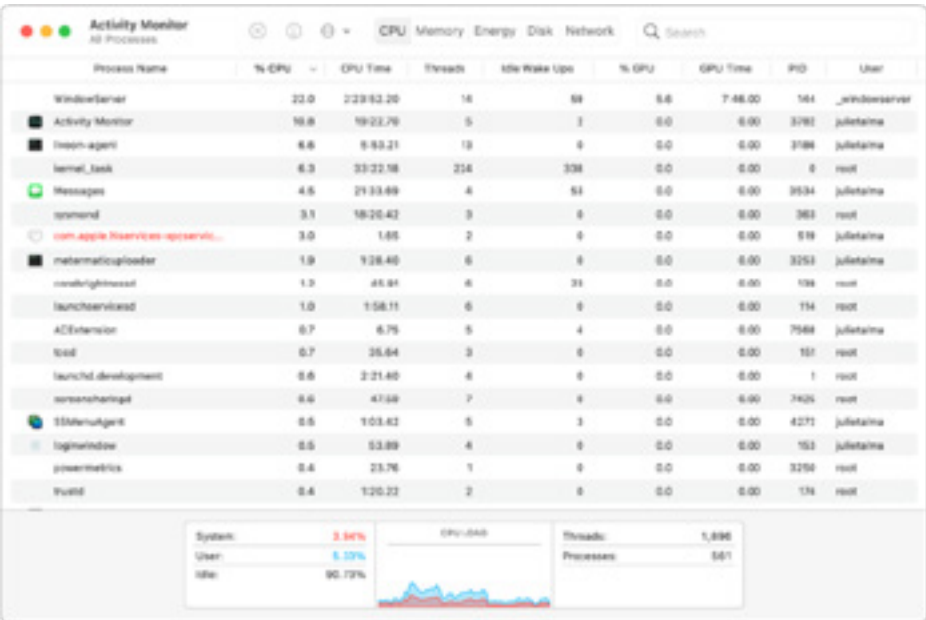

Fig. 2. Activity Monitor Interface.

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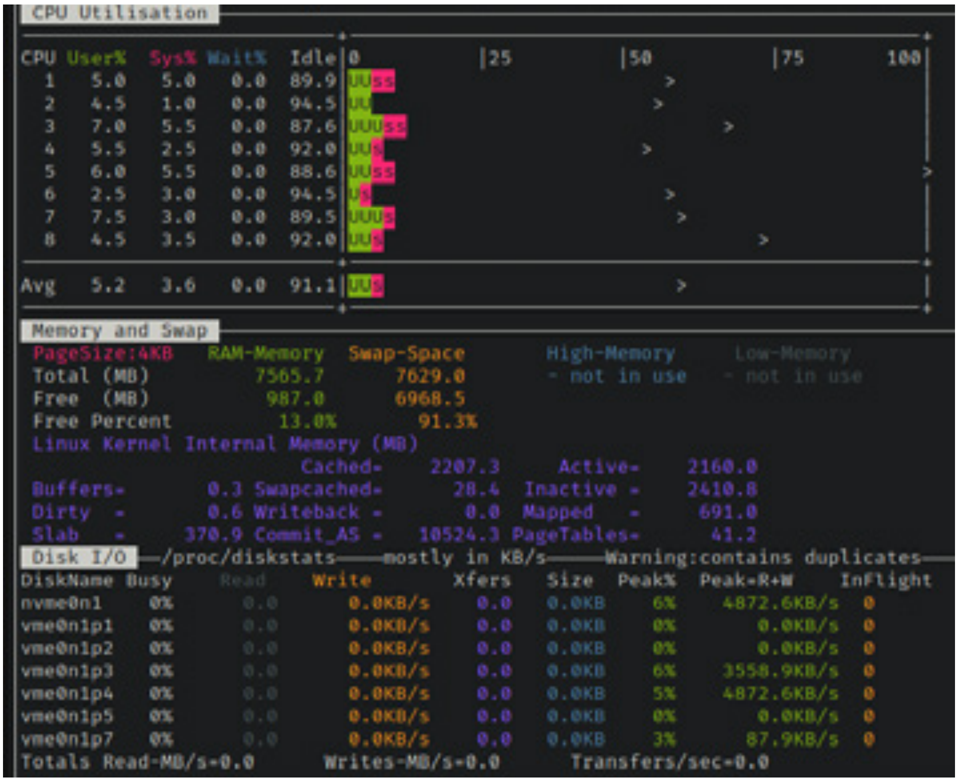

Fig. 3. Nmon Interface.

3. PROPOSED WORK

The need for maintaining and tracking different CPU Metrics is mentioned in this paper. Along with it, various tools which are used to monitor these metrics are described. Similarly, a web-based system for fetching system information and displaying it, has been developed which will be able to provide graphical insights to the performance parameters such as CPU usage, Memory, and GPU. It will be an effective light-weight data visualization tool, so that the history of the parameters can also be recorded.

The website fetches useful system information from the local system with the help of a '.exe' file which the user needs to download and run on the local system. The website displays different static system metrics and visualizes the dynamic information in the form of graphs. There is an additional feature added to the system where the user can check whether a particular application can work on their system or not by comparing the minimum requirements with the local systems information.

Figure 4. below illustrates the process flow diagram of the proposed system and describes the end to end working of the system.

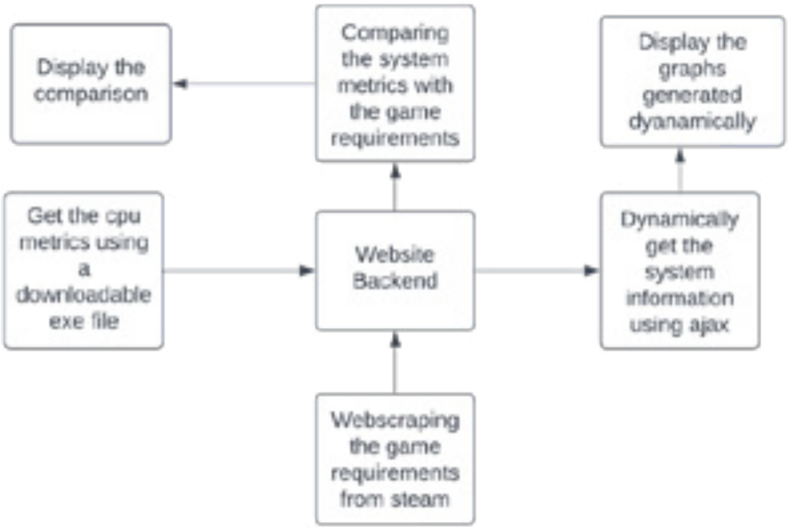

Fig. 4. Flowchart illustrating proposed work.

3.1. METRIC

There are various parameters to monitor the system performance and system resources in a computer system. Some of the most important metrics that affect the system performance are CPU utilization, Used RAM, GPU specifications, System information, etc. The website focuses on fetching the useful system information, displaying it and visualizing dynamic information that updates after every second. The metrics are fetched from the system using the 'psutil', 'wmi' and 'platform' Python libraries. The 'platform' Python library is used to fetch the operating system, node name, operating system release version and processor details. The 'psutil' library is used to retrieve the number of physical and logical cores, CPU utilization and RAM availability of the system. The 'wmi' module fetches the GPU and video RAM details. In order to get the metric from the local system, a '.exe' file is used which sends it to the web server.

3.2. VISUALIZATION

The CPU utilization and RAM availability are dynamic metrics that update after every second, so these metrics are visualized graphically as a line and pie chart. This visualization helps the user to track the systems performance and know how exactly the metrics perform. The graphs are created using the 'Chart.js' library in JavaScript and Ajax to update the graphs in real time.

3.2. SYSTEM REQUIREMENTS SATISFIED OR NOT

'Steam' is used to web-scrape minimum requirements to run games entered by the user. We compare the minimum requirements with the system metrics and warn the user if the requirements are not met by the user's system.

4.RESULTS AND DISCUSSION

Figure 5. shown below is the home page of the website of the proposed idea. It contains a navigation bar with the options – System information, Graphs and the 'Can you Run this game?' feature. The main section of the website contains the button "Download Link". A '.exe' file is downloaded on clicking the button which when executed on the user's device/computer will fetch and display the exact system information of the user's computer/laptop.

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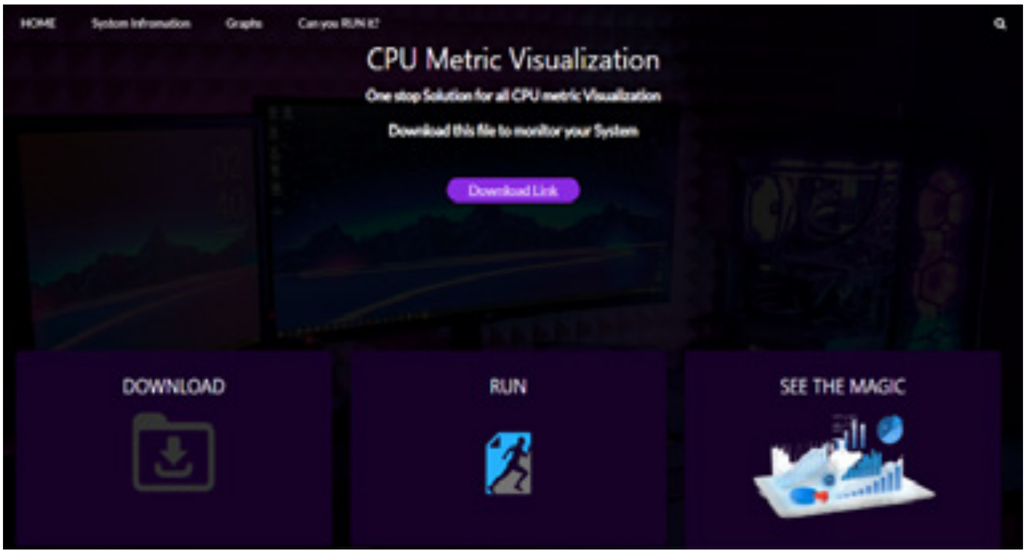

Fig. 5. Home page of the website.

Figure 6. displays the information that is fetched from the user's computer or laptop device and is displayed in a new window. The system information contains the following – NodeName, Version, Machine, Processor, LogicalCore, PhysicalCore, Operating System, Operating System Release and Version, RAM available and lastly the Total Disk Storage Space.

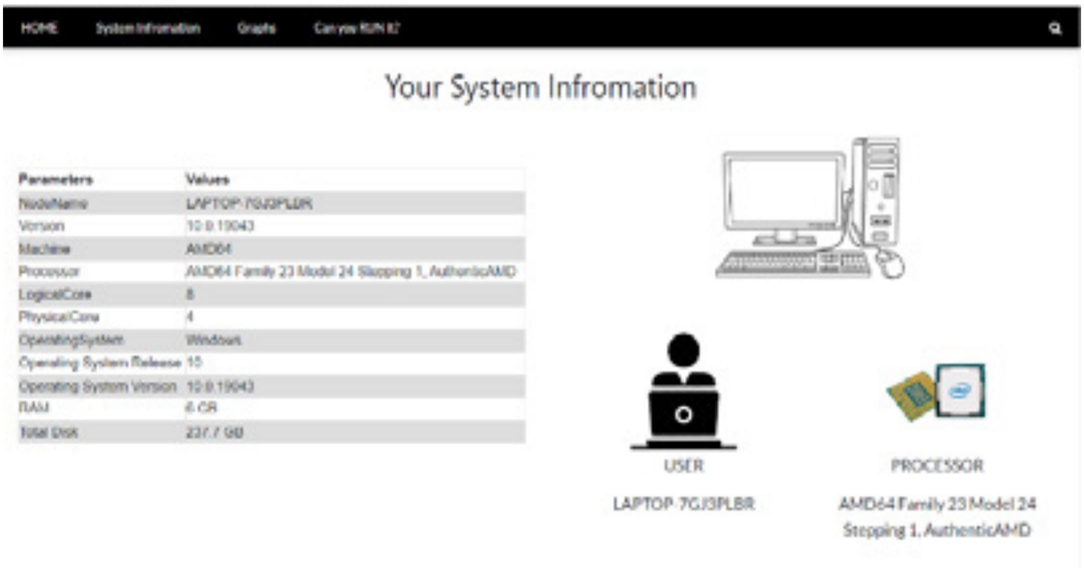

Fig. 6. System information of the user's device.

Table 2. illustrates the actual data fetched from local system in regular intervals. The table contains the CPU Utilization percentage which will be further visualized as a line chart and the RAM usage and availability which are very important in any operating system to perform well.

Table 2. CPU Metrics.

Sr No	Time (in seconds)	CPU Utilization	RAM Used	RAM Available
1	2	31.8	5.96	0.40
2	4	9.4	5.98	0.38
3	6	17.4	5.89	0.51
4	8	12.3	5.84	0.51

Figure 7 (a). and Fig 7 (b). illustrate the CPU Utilization metric and the available RAM dynamically using a line chart and a pie chart respectively. The charts are dynamic in nature and the proposed application fetches real time information about the metrics from the user's system and updates the charts with the new values every second.

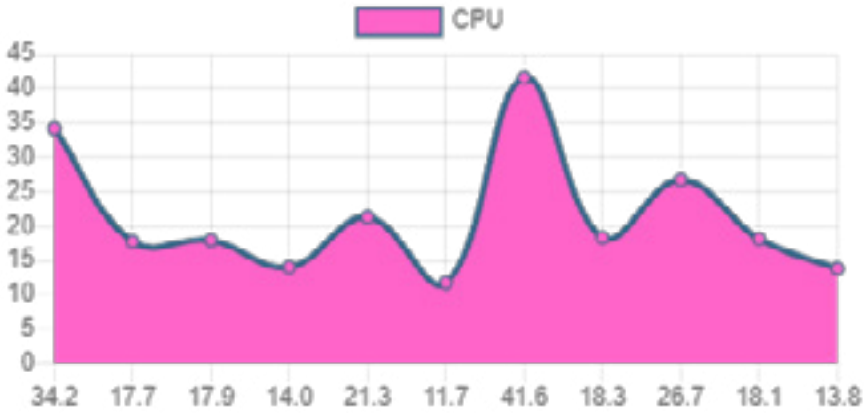

Fig. 7. (a) CPU Utilization visualized graphically.

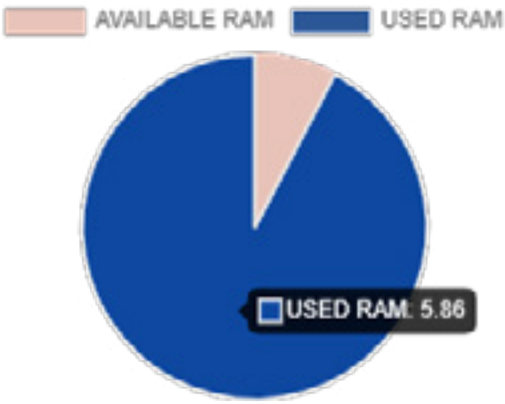

Fig. 7. (b) RAM availability visualized graphically.

Another essential feature of the proposed application allows the user to check if a game is compatible with his own system. This feature requires the user to enter the name of the game in the search bar and then displays the required system specification for the same. It also displays the user's system specification and based on the comparison between the two, it displays whether the game will be able to run on the user's system successfully or not.

Figure 8. Shows an example of the game "Watch Dogs" being entered by the user. Based on the comparison between the user's system specifications and the required system specifications for the game, a message "Yes you can run it!" specifies that the game is compatible with the user's system.

Fig. 8. Example of a game that can be run on the user's system.

Figure 9. Shows an example of the game "Red Dead Redemption 2" being entered by the user. Based on the comparison between the user's system specifications and the required system specifications for the game, a message "No you can't run it!" specifies that the game is not compatible with the user's system

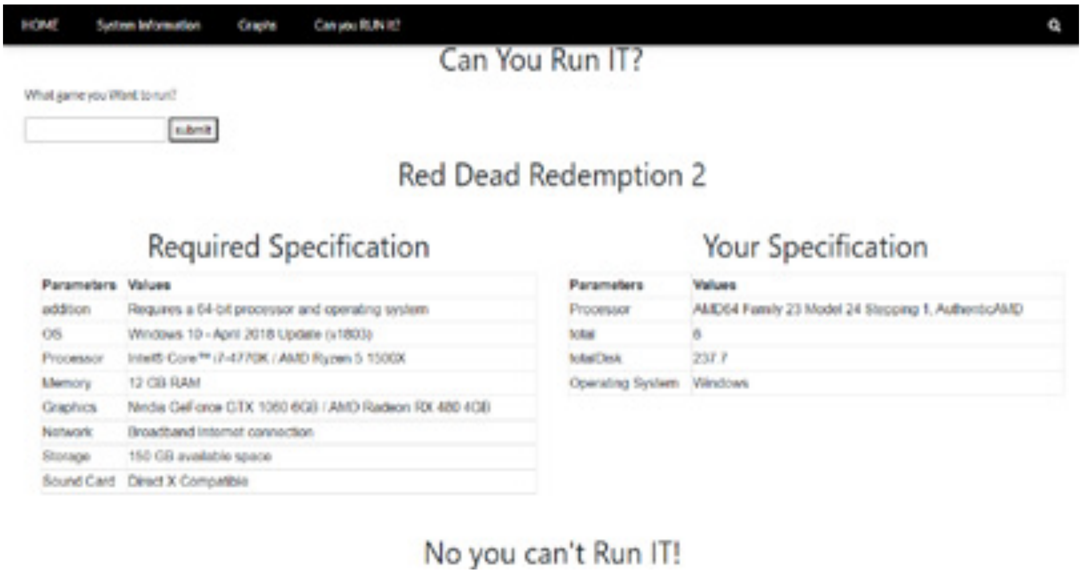

Fig. 9. Example of a game that cannot be run on the user's system.

5. FUTURE SCOPE

The proposed system provides information about whether the application will run on users' local system or not, so a recommendation system can be developed in future which will recommend necessary hardware and software to the user for better performance by analysing the system information.

6. CONCLUSION

The CPU metrics measurements are essential because they guide one's decision-making around the computer equipment which one intends to buy. It enables one to cut through the marketing spin that surrounds computer systems; without a basic understanding of how they operate, one won't be able to handle this and won't be able to make an informed decision when buying systems. Understanding performance measurements is essential for comprehending the underlying organizational motivation, or the reasons why people try to make these changes in order to increase performance for future development.

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