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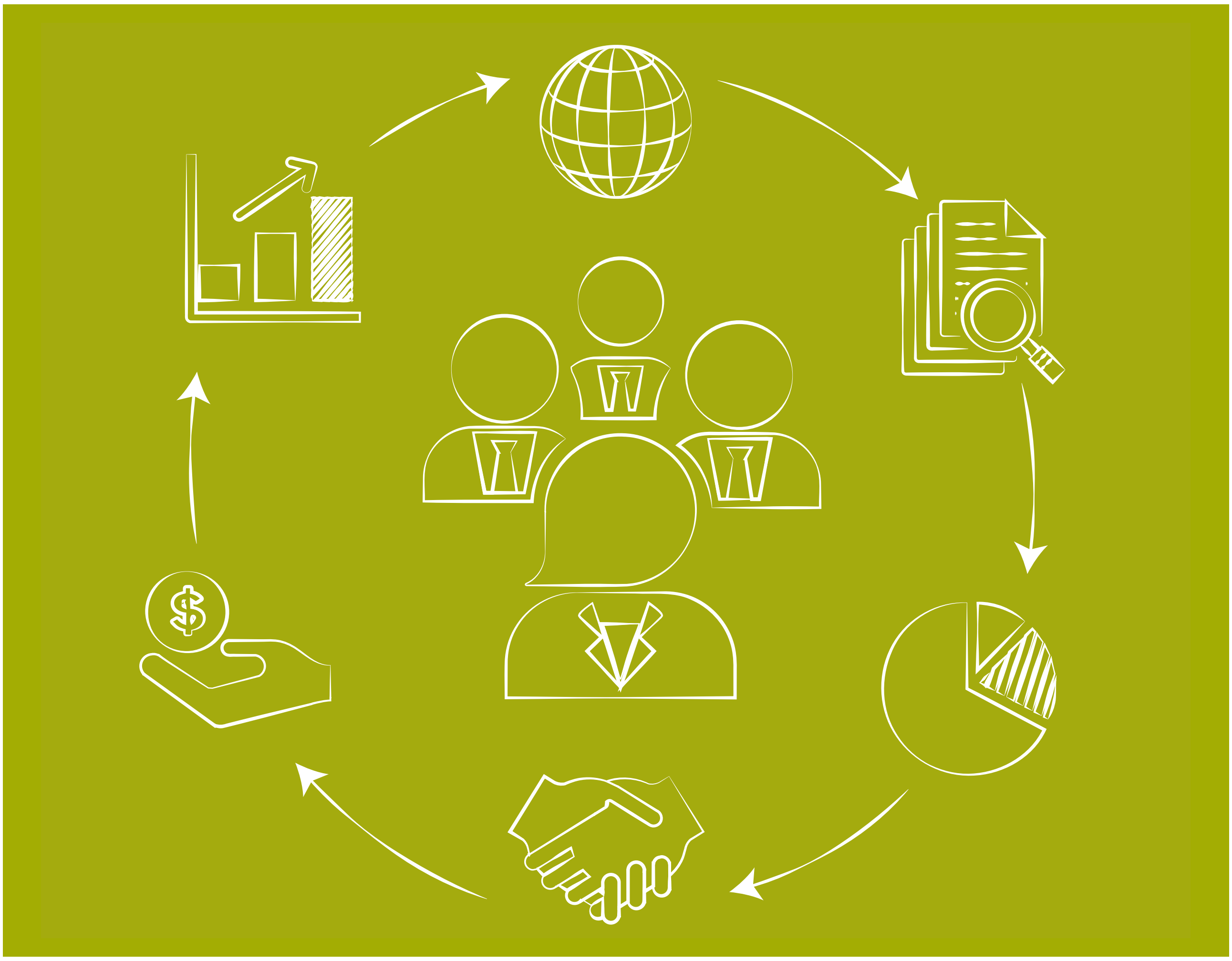

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/01/

ON A QUEUEING INVENTORY WITH COMMON LIFE TIME AND REDUCTION SALE CONSEQUENT TO INCREASE IN AGE

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ABSTRACT

Consider an inventoried item for which reduction in sales price is declared as the age of the item increases. Decision to maintain sales price at the same level/reduce, is taken at stages $2, \cdots, k-1, k$. *On the items attaining CLT, they are sold at scrap value, provided items are still left in stock. Customer arrival forms a non-homogenous Poisson process, with rate increasing with each sales price reduction. Service time follows exponential distribution.The items are replenished according to* (S, s) *policy with positive lead time. Each stage of CLT is iid which follows a Phase type distribution with representation*(α, S) *of order* m. *The k-fold convolution of this distribution is the CLT of the inventoried items. The problem is modelled as a queueing-inventory problem which is a continuous time Markov chain (CTMC). The stationary distribution of this CTMC is computed and various performance measures are discussed. A cost function is constructed to compute the optimal order quantity and reorder level.The model is compared with queueing inventory model in which the CLT follows Erlang Distribution of order k.*

KEYWORDS

Inventory, Lead time, Common Life Time, Reduction Sale

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1 INTRODUCTION

The quality of inventory items, especially perishable decreases gradually as their age increases.Seasonal items also have this property.However for that price can be very high during the time which is not the season for that item. It is very common to declare reduction in sales price while the quality of the item decreases. Textile items are real life examples for this phenomenon. The quality of textile items decreases gradually and finally it becomes unusable. Price reduction is an effective strategy to increase the sales volume and to settle the items before they perish.
In this paper we consider a queueing inventory system with S items which has a common life time of k stages. In each stage it is phase type distributed. When it is absorbed from one stage the items reach the next stage and finally reach the k^th stage. When it is absorbed from k^th stage the items attain CLT and become scrapped. The arriving customer gets the inventory in the present stage. The arrival rate of customers is assumed to be increasing in each stage due price reduction. The customers are not allowed in the absence of the inventory. Here we seek answers to two questions. Firstly, how much stock we need to maximize profits, and secondly, when to place an order to deliver items on time to waiting customers.
The common life time inventory models were first introduced by Lian, Z and Neuts, M.F (2005). The common life time was assumed to be of discrete phase type distribution. A perishable queueing inventory system with (Q, r) policy, Poisson demands, identical life times, constant lead times and lost sales with fixed ordering costs was studied by Berke, E and Gurler, L (2008). A cost function is used to find the optimal (Q, r) policy. Krishnamoorthy *et. al* (2006) describes a queueing inventory with reservation, cancellation, common life-time and retrial. Broekmeulen, R and Donselaar, K (2009) discuss a perishable queueing inventory with batch ordering, positive lead time and time varying demand. The replenishment is done using EWA policy (Estimated Withdrawal and Aging) in which the replenishment is done considering the items that will be out-dated within the lead time. Williams, C and Patuwo, B (1999) deal with a perishable inventory queueing system with positive lead time. The optimal reorder quantity is found using numerical computations. Schwarz, M. (2006) derive stationary distributions of joint queue length and inventory processes in explicit product form for various M/M/1-systems with inventory under continuous review and different inventory management policies, and with lost sales. Krishnamoorthy *et al* (2021) provides an overview of the work done in mathematical inventory models with positive service time. Kirubhashankar.*et. al* (2021) analyzes a production inventory model for deteriorating products with constant demand and backlogged shortages which depends on the length of the waiting time for the next order level. Optimum shortage time and total cycle length with the objective of optimizing the total cost are found.

Mehrez, A and Ben-Arieh, D (1991) describes a model that combines several dichotomies, into a single model. It has the objective of deciding the optimal order quantities for a multi-item inventory system over a finite horizon. The demand is probabilistic with service level constraints, and there is an all-unit price break, for orders that exceed a given size. Amirthakodi *et. al* (2015) consider a continuous review perishable inventory system with service facility consisting of finite waiting hall and a single server. The items are replenished based on variable ordering policy. The lead time is assumed to have phase type distribution. develop a deterministic inventory model with quantity discount, pricing and partial back ordering when the product in stock deteriorates with time. Wee, H. M (1999) investigate a base-stock perishable inventory system with Markovian demand and general lead-time and lifetime distributions. Janssen, L *et. al* (2016) presents a review of literature of deteriorating inventory models.

The rest of the paper is distributed as follows. In the second section model description is given. The steady state analysis, stability condition and various performance measures are discussed in the third section. The numerical illustration is given in the fourth section.

2 MODEL DESCRIPTION

Consider a queuing-inventory which has S items. Initially all items are in stage 1. As the items getting older the stages are changed to $2, \cdots, k-1, k$ and finally the CLT of the items is reached and the

 https://doi.org/10.17993/3cemp.2022.110250.15-31

items remain if any, are scrapped. In each stage the CLT is iid that has a phase type distribution with representation (α, S) of order$m$. In the ith stage the distribution is the i-fold convolution of the PH distribution. The inventory is replenished with (S,s) policy with positive lead time which follows an exponential distribution with parameter β.The replenishment order is placed when either the inventory level is dropped to s or CLT of the stocked items attains the level q where $1 \leq q \leq k$. The customers arrive to the system according to a Poisson process of rate $\lambda_i, i = 1, 2, \cdots, k$. A reduction in price is declared when the stage is changed from $1 \to 2, 2 \to 3, 3 \to 4, \ldots\ldots..k-1 \to k$. The service of customers is exponentially distributed with rate $\mu_i, i = 1, 2, \cdots, k$. The service of customers is exponentially distributed with rate $\mu_i, i = 1, 2, \cdots, k$. Customers are not allowed to enterwhen there is no inventory in the system.

Let $N(t)$ be the number of customers, $I(t)$ be the inventory level,$C(t)$, the stage of CLT and $J(t)$ is the phase of CLT at time t.
Then the process $\{(N(t), I(t), C(t), J(t)) : t \geq 0\}$ is a continuous time Markov chain with state space $\{(n, 0) : n \geq 0\}U\{(n, i, r, j) : n \geq 0, 1 \leq i \leq S, 1 \leq r \leq k, 1 \leq j \leq m\}U\{\hat{0}\}$
Transition due to arrival:
$(n, i, r, j) \to (n+1, i, r, j)$ with rate $\lambda_r, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq r \leq k, \quad 1 \leq j \leq m$
Transition due to service:
$(n, 1, r, j) \to (n-1, 0)$ with rate $\mu_r, \quad n \geq 1, \quad 1 \leq r \leq k, \quad 1 \leq j \leq m$
$(n, i, r, j) \to (n-1, i-1, r, j)$ with rate $\mu_r, \quad n \geq 1, \quad 2 \leq i \leq S, \quad 1 \leq r \leq k, \quad 1 \leq j \leq m$
Transition due to replenishment:
$(n, 0) \to (n, S, 1, j)$ with rate $\beta\alpha_j, \quad n \geq 0, \quad 1 \leq j \leq m$
$(n, i, r, j) \to (n, S, 1, j)$ with rate $\beta, \quad n \geq 0, \quad 1 \leq i \leq s, \quad 1 \leq r \leq k, \quad 1 \leq j \leq m$
$(n, i, r, j) \to (n, S, 1, j)$ with rate $\beta, \quad n \geq 0, \quad s+1 \leq i \leq S, \quad c \leq r \leq k, \quad 1 \leq j \leq m$
Transition due to phase change of Common Life Time:
$(n, i, k, j) \to (n, 0)$ with rate $t_j, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq j \leq m$
$(n, i, r, j) \to (n, i, r, p)$ with rate $t_{jp}, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq r \leq k-1, \quad 1 \leq j, p \leq m$
Transition due to stage change of Common Life Time:
$(n, i, r, j) \to (n, i, r+1, p)$ with rate $\alpha_p t_j, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq r \leq k-1, \quad 1 \leq j, p \leq m$
$(n, i, k, j) \to (n, 0)$ with rate $t_j, \quad n \geq 0, \quad 1 \leq i \leq S, \quad 1 \leq j \leq m$
The infinitesimal generator matrix of the process is given by,

$$Q = \begin{bmatrix} B_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

3 STEADY STATE ANALYSIS

Let $A = A_0 + A_1 + A_2$ and let $\Pi = (\pi_0, \pi_1, ..., \pi_S)$ be the steady state probability vector where $\pi_i = ((\pi_i)^1, (\pi_i)^2, \cdots, (\pi_i)^k)$. Here $(\pi_i)^r = (\pi_i(r, 1), \pi_i(r, 2), ...\pi_i(r, m)), 1 \leq i \leq S, 1 \leq r \leq k$
The condtions

 https://doi.org/10.17993/3cemp.2022.110250.15-31

$\pi A = 0$ and $\pi e = 1$, can be re-written as

$$-\pi_0\beta + \sum_{r=1}^{k}(\pi_1)^r\mu_r e_m + \sum_{i=1}^{S}(\pi_i)^k T^0 = 0$$
$$(\pi_i)^{r-1}(T^0\otimes\alpha)(1-\delta_{ir}) + (\pi_i)^{r-1}[(T-(\mu_r+\beta)I] + (\pi_{i+1})^r\mu_r = 0, 1\le i\le s, 1\le r\le k$$
$$(\pi_i)^{r-1}(T^0\otimes\alpha)(1-\delta_{ir}) + (\pi_i)^{r-1}(T-\mu_r I) + (\pi_{i+1})^r\mu_r = 0, s+1\le i\le S-1, 1\le r\le q-1$$
$$(\pi_i)^{r-1}(T^0\otimes\alpha)(1-\delta_{ir}) + (\pi_i)^{r-1}(T-(\mu_r+\beta)I) + (\pi_{i+1})^r\mu_r = 0, s+1\le i\le S-1, q\le r\le S$$
$$\pi_0\alpha + (\sum_{i=1}^{S}\sum_{r=1}^{k}(\pi_i)^r)T = 0$$
$$\sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j) = 1$$

Simplifying the last equation, $\pi_0^{-1} = 1 - \alpha T^{-1}e$
Solving the equations we get, π_i, for each $i = 0, 1, 2, ..., S$.

3.1 STABILITY CONDITION

Theorem:The system is stable if and only if

$$\text{ie, }\sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j)\lambda_r < \sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j)\mu_r$$

Proof The system is stable if $\pi A_0 e < \pi A_2 e$

$$\text{ie, }\sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j)\lambda_r < \sum_{i=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m}\pi_i(r,j)\mu_r$$

3.2 STATIONARY PROBABILITIES

Let $x = (x_0, x_1, ...)$be the stationary probability vector of the system where
$x_i = \{x_i(0) : i \ge 0\}U\{x_i(p,r,j) : i \ge 0, 1 \le p \le S, 1 \le r \le k, 1 \le j \le m\}$
Then $xQ = 0$, $\quad xe = 1$ which results in the following equations.

$$x_0 B_1 + x_1 A_2 = 0$$
$$x_{i-1}A_0 + x_i A_1 + x_{i+1}A_2 = 0, \quad i \ge 1$$

Let $x_i = x_0 R^i$, $\quad i \ge 1$. Substituting in $\quad x_{i-1}A_0 + x_i A_1 + x_{i+1}A_2 = 0$, we get,

$$R^2 A_2 + R A_1 + A_0 = 0$$

Let $R(0) = 0$ be the initial solution.
From the recurrence relation

$$R(n) = (R^2(n-1)A_2 + A_0)(-A_1^{-1}), \quad n \ge 1$$

we find R .

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3.3 PERFORMANCE MEASURES

1. Expected number of customers in the system $=\sum_{i=0}^{\infty}\sum_{p=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m} ix_i(p,r,j)$
2. Expected number of items in the stage r
   $E_r=\sum_{i=0}^{\infty}\sum_{p=1}^{S}\sum_{j=1}^{m} px_i(p,r,j), r=1,2,\cdots,k$
3. Expected number of scrapped items $E_s=\sum_{i=0}^{\infty}\sum_{p=1}^{S}\sum_{j=1}^{m} px_i(p,k,j)t_j$.
4. Probability that a newly joining customer is served instantaneously
   $=\sum_{p=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m} x_0(p,r,j)$.
5. Probability that the server is busy$=\sum_{i=1}^{\infty}\sum_{p=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m} x_i(p,r,j)$
   $=1-x_0e-\sum_{i=1}^{\infty} x_i(0)$
6. Probability that all items are sold before the realization of CLT $=\sum_{i=0}^{\infty} x_i(0)$
7. Probability that all items in a cycle are scrapped $=\sum_{i=0}^{\infty}\sum_{j=1}^{m} x_i(S,k,j)t_j$
8. Probability that there are no customers in the system $=\sum_{p=1}^{S}\sum_{r=1}^{k}\sum_{j=1}^{m} x_i(p,r,j)$

3.4 DISTRIDUTION OF WAITING TIME OF A TAGGED CUSTOMER

Consider a customer who joins the system as n^{th} tagged customerand waiting for service where $n>0$. The rank of the customer reduces to $n-1, n-2,\cdots,2,1$ as the customers in the queue join for service. Let $M(t)$ be the rank of the customer, $I(t)$ be the inventory level, C(t) the stage of CLT and J(t) the phase of CLT at time t. Then the process $\{(M(t),I(t),C(t),J(t)):t\geq 0\}$ is a Markov chain with state space
$\{n,n-1,\cdots,2,1\}\times\{0\}\cup\{n,n-1,\cdots,2,1\}\times\{1,2,\cdots,S\}\times\{1,2,\cdots,k\}\times\{1,2,\cdots,m\}\cup\{\hat{0}\}$
where $\hat{0}$ is the absorbing state of the Markov Chain which denotes the service epoch of the n^{th} tagged customer.

Let $q_r=Pr\{N(t)=r,I(t)=i,C(t)=p,J(t)=j\}=x_r(0)+\sum_{i=1}^{S}\sum_{p=1}^{k}\sum_{j=1}^{m} x_r(i,p,j)$
Let $\omega_r=\frac{q_r}{\sum_{r=1}^{\infty} q_r}$ Then $\omega=(\omega_r,\omega_{r-1},\ldots\ldots..\omega_1)$ is the initial probability vector of the distribution.

The infinitesimal generator matrix of the process is,

$$\hat{P}=\begin{bmatrix} W & W^0\\ \mathbf{0} & 0\end{bmatrix}$$

where

$$W=\begin{bmatrix} W_{n,n} & W_{n,n-1} & & \\ & W_{n-1,n-1} & W_{n-1,n-2} & \\ & & & \\ & & & W_{2,1}\\ & & & W_{1,1}\end{bmatrix}$$

$W_{i,i}=A_0+A_1, i=1,\cdots r$ and $W_{i,i-1}=A_2, i=2,\cdots,r$

and

$$W^0=\begin{bmatrix}\mathbf{0}\\ \\ A_2\end{bmatrix}$$

The waiting time distribution is $w(t)=\omega e^{Wx}W^0$ The average waiting time of a tagged customer$=-\omega(W^{-}1)e$

https://doi.org/10.17993/3cemp.2022.110250.15-31

3.5 DISTRIBUTION OF THE DURATION OF A CYCLE

The expected cycle length is the duration of time elapsed between two consecutive replenishment orders.Starting from any system state, with the number of items in inventory S (an order has been placed and got materialised), we look at the evolution until the next replenishment takes place, either through realization of CLT or by reaching the inventory level to s. Consider the Markov chain $\{(I(t), N(t), S(t), J(t)) : t \geq 0\}$ with the state space $\{(0, r) : r \geq 0\} \cup \{(r, i, p, j) : 1 \leq r \leq S, 0 \leq i \leq H, 1 \leq p \leq k, 1 \leq j \leq m\} \cup \{\Delta\}$, where$\{\Delta\}$ is the absorbing state of the Markov chain which denotes the realization of the next replenishment by completing one cycle.

Let $p_r = Pr\{N(t) = i, I(t) = r, S(t) = 1\}$
$i\ e$:$p_r = \sum_{i=1}^{H} \sum_{j=1}^{m} x_i(r, 1, j)$
Let $\gamma_r = p_r / \sum_{r=0}^{S} p_r$ Then $\gamma = (\gamma_S, \gamma_{S-1}, \ldots\ldots..\gamma_0)$ is the initial probability vector of the distribution. The infinitesimal generator matrix of the process is,

$$\hat{Q} = \begin{bmatrix} \mathbf{U} & U^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$U = \begin{bmatrix} U_{S,S} & U_{S,S-1} & \mathbf{0} & \cdots & \mathbf{0} & U_{S,0} \\ U_{S-1,S} & U_{S-1,S-1} & U_{S-1,S-2} & \mathbf{0} & \cdots & U_{S-1,0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ U_{s,S} & \mathbf{0} & \cdots & U_{s,s} & \ddots & U_{s,0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ U_{0,S} & \mathbf{0} & \cdots & \cdots & \cdots & -\beta I_{S+1} \end{bmatrix}_{(S+1)\times(S+1)}$$

Here,

for $s + 1 \leq i \leq S$

$$U_{i,i} = \begin{bmatrix} K_{0,0} & K_{0,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & K_{1,1} & K_{0,1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & K_{1,1} & K_{0,1} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & K_{1,1} \end{bmatrix}_{(S+1)\times(S+1)}$$

and for $1 \leq i \leq s$

$$U_{i,i} = \begin{bmatrix} L_{0,0} & K_{0,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & L_{1,1} & K_{0,1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & L_{1,1} & K_{0,1} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & L_{1,1} \end{bmatrix}_{(S+1)\times(S+1)}$$

$$K_{0,0} = \begin{bmatrix} L_1 & M_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & L_2 & M_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & L_{k-1} & M_1 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & L_k \end{bmatrix}_{k\times k}$$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

$$K_{1,1} = \begin{bmatrix} P_1 & M_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & P_2 & M_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & P_{k-1} & M_1 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & P_k \end{bmatrix}_{k\times k}$$

$$K_{0,1} = \begin{bmatrix} N_1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & N_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & N_{k-1} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & N_k \end{bmatrix}_{k\times k}$$

$$L_{0,0} = \begin{bmatrix} U_1 & M_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & U_2 & M_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & U_{k-1} & M_1 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & U_k \end{bmatrix}_{k\times k}$$

$$L_{1,1} = \begin{bmatrix} V_1 & M_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & V_2 & M_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & V_{k-1} & M_1 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & V_k \end{bmatrix}_{k\times k}$$

For $1 \leq i \leq q-1, L_i = T - \lambda_i I_m, U_i = T - (\lambda_i + \mu_i)I_m$
For $q \leq i \leq k, L_i = T - (\lambda_i + \beta)I_m, U_i = T - (\lambda_i + \beta + \mu_i)I_m$
For $1 \leq i \leq k, P_i = T - (\lambda_i + \beta)I_m,\ V_i = T - (\lambda_i + \beta + \mu_i)I_m,\ N_i = \lambda_i I_m$ and $M_1 = T^0 \otimes \alpha$.

For $1 \leq i \leq S$

$$U_{i,i-1} = \begin{bmatrix} \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ J_{1,0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & J_{1,0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & J_{1,0} & \mathbf{0} \end{bmatrix}_{(S+1)\times(S+1)}$$

where

$$J_{1,0} = \begin{bmatrix} H_1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & H_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & H_{k-1} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & H_k \end{bmatrix}_{k\times k}$$

For $1 \leq i \leq k,\ H_i = \mu_i I_m$
For $s+1 \leq i \leq S,\ W_{i,S} = \mathbf{0}$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

$$U_{0,S} = \begin{bmatrix} M_{0,0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & M_{0,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & M_{0,0} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & M_{0,0} \end{bmatrix}_{(S+1)\times(S+1)}$$

$$M_{0,0} = \begin{bmatrix} \beta\alpha & \mathbf{0} & \cdots & \cdots & \mathbf{0} \end{bmatrix}_{1\times k}$$

For $1 \leq i \leq s$

$$U_{i,S} = \begin{bmatrix} M_{1,1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & M_{1,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & M_{1,1} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & M_{1,1} \end{bmatrix}_{(S+1)\times(S+1)}$$

$$M_{1,1} = \begin{bmatrix} \beta T & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \beta T & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \beta T & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \beta T \end{bmatrix}_{k\times k}$$

For $1 \leq i \leq S$

$$U_{i,0} = \begin{bmatrix} G_{1,1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & G_{1,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & G_{1,1} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & G_{1,1} \end{bmatrix}_{(S+1)\times(S+1)}$$

$$G_{1,1} = \begin{bmatrix} \mathbf{0} \\ \cdots \\ \mathbf{0} \\ T^0 \end{bmatrix}_{k\times 1}$$

The probability density function of the distribution is, $f(x) = \gamma e^{Ux}U^0$
So the mean of the distribution is the first raw moment, $(\mu_1)^{'} = -\gamma U^{-1}e$
So expected cycle length= $-\gamma U^{-1}e$

4 SPECIAL CASE: ERLANG CLT DISTRIBUTION

Consider an inventory item that has a common life time(CLT) with k stages. At the beginning of stage 1,the age of the item is zero. As the item gets older the age of the item changes to $2, 3, ..., \cdots, k-1, k$.Each stage duration is iid exponention random variable. After k^{th} stage the items remaining, if any, are scrapped.The CLT is modelled as an Erlang Distribution of order k. Customers arrive according to a Poisson process with rates depending on the stage of the CLT, $\lambda_i, i = 1, 2, \cdots, k$. The service rate in each stage is exponential with rate $\mu_i, i = 1, 2, \cdots, k$. A reduction in price is declared when the stage is changed from $1 \to 2, 2 \to 3, 3 \to 4, \ldots\ldots..k-1 \to k$. The stages are changed according to exponential

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distribution with rate α. When the inventory level drops to s or the item reaches the stage $q, 1 \leq q \leq k$ an order is placed.he inventory is replenished with positive lead time which is exponentially distributed with rate β.The customers are blocked in the absence of inventory to reduce the system cost.
Let $N(t)$ = Number of customers in the system, $I(t)$ = Inventory level, $C(t)$ is the stage of CLT at time t. Then $\{X(t) : t \geq 0\}$= $\{(N(t), I(t), S(t)) : t \geq 0 \}$ is a Markov process with state space $\{(0,0)\}U\{0,1,2,3,\ldots\} \times \{1,2,\ldots,S\} \times \{1,2,\cdots,k\}$.

4.1 INFINITESIMAL GENERATOR MATRIX

$$Q = \begin{bmatrix} B_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}$$

Where, each block has order $(kS+1) \times (kS+1)$ and,

$$B_1 = \begin{bmatrix} -\beta & \mathbf{0} & & B_{0,S} \\ B_{1,0} & B_{1,1} & & B_{1,S} \\ & & & \\ B_{s,0} & & B_{s,s} & B_{s,S} \\ & & & \\ B_{S,0} & \mathbf{0} & & B_{S,S} \end{bmatrix}$$

where, for $1 \leq i \leq s$

$$B_{i,i} = \begin{bmatrix} -(\lambda_1+\beta+\alpha) & \alpha & & \\ & -(\lambda_2+\beta+\alpha) & \alpha & \\ & \ddots & \ddots & \\ & & & -(\lambda_k+\beta+\alpha) \end{bmatrix}$$

, for $s+1 \leq i \leq S$

$$B_{i,i} = \begin{bmatrix} -(\lambda_1+\alpha) & \alpha & & & \\ & \ddots & \ddots & & \\ & & -(\lambda_q+\beta+\alpha) & \alpha & \\ & & & & -(\lambda_k+\beta+\alpha) \end{bmatrix}$$

, for $1 \leq i \leq S$,

$$B_{i,0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha \end{bmatrix}$$

,

$$B_{0,S} = \begin{bmatrix} \beta & 0 & \cdots & 0 \end{bmatrix}$$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

for $1 \leq i \leq s$

$$B_{i,S} = \begin{bmatrix} \beta & 0 & \cdots & 0 \\ \beta & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ \beta & 0 & \cdots & 0 \end{bmatrix}$$

,

and for $s+1 \leq i \leq S$

$$B_{i,S} = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ \beta & 0 & \cdots & & 0 \\ \vdots & \cdots & \ddots & & \vdots \\ \beta & 0 & \cdots & & 0 \end{bmatrix}$$

,

$$A_0 = \begin{bmatrix} 0 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & L & \ddots & \cdots & \mathbf{0} \\ \vdots & \ddots & L & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & L \end{bmatrix}$$

$$L = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_k \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ M_1 & \mathbf{0} & \ddots & \cdots & \vdots \\ \mathbf{0} & M & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & M & \mathbf{0} \end{bmatrix}$$

where,

$$M_1 = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}$$

,

$$M = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mu_k \end{bmatrix}$$

https://doi.org/10.17993/3cemp.2022.110250.15-31

$$A_1 = \begin{bmatrix} -\beta & \mathbf{0} & \mathbf{0} & \cdots & \cdots & A_{0,S} \\ A_{1,0} & A_{1,1} & \mathbf{0} & \ddots & \cdots & A_{1,S} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ A_{s,0} & \cdots & \mathbf{0} & A_{s,s} & \ddots & A_{s,S} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ A_{S,0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_{S,S} \end{bmatrix}$$

where, for $1 \leq i \leq s$

$$A_{i,i} = \begin{bmatrix} -(\lambda_1+\mu_1+\beta+\alpha) & \alpha & \cdots & 0 \\ 0 & -(\lambda_2+\mu_2+\beta+\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha \\ 0 & \cdots & 0 & -(\lambda_k+\mu_k+\beta+\alpha) \end{bmatrix}$$

, for $s+1 \leq i \leq S$

$$A_{i,i} = \begin{bmatrix} -(\lambda_1+\mu_1+\alpha) & \alpha & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & -(\lambda_q+\mu_q+\beta+\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -(\lambda_k+\mu_k+\beta+\alpha) \end{bmatrix}$$

, for $1 \leq i \leq S$,

$$A_{i,0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha \end{bmatrix}$$

,

$$A_{0,S} = \begin{bmatrix} \beta & 0 & \cdots & 0 \end{bmatrix}$$

for $1 \leq i \leq s$

$$A_{i,S} = \begin{bmatrix} \beta & 0 & \cdots & 0 \\ \beta & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ \beta & 0 & \cdots & 0 \end{bmatrix}$$

and

$$A_{i,S} = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ \beta & 0 & \cdots & & 0 \\ \vdots & \cdots & \ddots & & \vdots \\ \beta & 0 & \cdots & & 0 \end{bmatrix}$$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

4.2 STEADY STATE ANALYSIS

Let $A = A_0 + A_1 + A_2$ and $\pi = (\pi_0, \pi_1, \pi_2, ..., \pi_S)$ be the steady state probability distribution of $\{(I(t), S(t)) : t \geq 0\}$, where $\pi_i = (\pi_{i1}, \pi_{i2}, ..., \pi_{ik}), \quad 1 \leq i \leq S.$
Then we have,
$-\pi_0\beta + \pi_1(M_1 + A_{1,0}) + \cdots + \pi_S A_{1,0} = 0$
$\pi_i(A_{1,1} + L) + \pi_{i+1}M = 0, \quad 1 \leq i \leq s$
$\pi_i(A_{S,S} + L) + \pi_{i+1}M = 0, \quad s+1 \leq i \leq S-1$ and
$\pi_0 A_{0,S} + (\pi_1+\pi_2+\cdots+\pi_{S-1})A_{1,S} + \pi_S(A_{S,S}+L) = 0$. Simplifying, $\pi_i = \pi_1[(A_{1,1}+L)M^{-1}]^i, \quad 1 \leq i \leq s$
$\pi_i = \pi_1[(A_{1,1} + L)M^{-1}]^s[(A_{S,S} + L)M^{-1}]^{i-s-1}, \quad s+1 \leq i \leq S$
Here π_1 and $\pi_0$can be obtained from the normalizing condition and the first equation.

4.3 STABILITY CONDITION

Theorem: The system is stable if and only if,
$\sum_{i=1}^{S}\sum_{j=1}^{k}\pi_{ij}\lambda_j < \sum_{i=1}^{S}\sum_{j=1}^{k}\pi_{ij}\mu_j$
Proof: The system is stable if and only if

$$\pi A_0 e < \pi A_2 e$$

Substituting we get,

$$\begin{array}{l}[0 \quad \pi_1 L \quad \pi_2 L \cdots \quad \pi_S L]e < \\ [0 \quad \pi_1 M_1 \quad \pi_2 M \cdots \quad \pi_S M \quad 0]e\end{array}$$

Simplifying, $\sum_{i=1}^{S}\sum_{j=1}^{k}\pi_{ij}\lambda_j < \sum_{i=1}^{S}\sum_{j=1}^{k}\pi_{ij}\mu_j$

Stationary Distribution

Let $x = (x_0, x_1, x_2, \cdots)$ be the steady state probability vector of the Makov process $X(t) = (N(t), I(t), S(t)), t \geq 0$. Then, $x_i = x_{i-1}R^i$, where R is the minimal non negative solution of the matrix equation,

$$R^2 A_2 + RA_1 + A_0 = 0$$

Here, $x_i = x_i(j, m), 1 \leq j \leq S, 1 \leq m \leq k$.

4.4 PERFORMANCE MEASURES

1. Expected number of customers in the queue, $E_c = \sum_{i=0}^{\infty} ix_i(0) + \sum_{i=0}^{\infty}\sum_{j=1}^{S}\sum_{m=0}^{k} ix_i(j, m)$.
2. Expected number of items in the in the stage m, $E_m = \sum_{i=0}^{\infty}\sum_{j=1}^{S} jx_i(j, m)$.
3. Expected number of items scrapped, $E_s = \sum_{i=0}^{\infty}\sum_{j=1}^{S}\alpha jx_i(j, k)$.
4. Probability that all items are sold in the first stage, $E_s = \sum_{i=1}^{\infty}\mu_1 x_i(j, 1)$.
5. Probability that all items are scrapped, $x_0(S, k)$.

4.5 DISTRIBUTION OF THE DURATION OF A CYCLE

The expected cycle length is the duration of time elapsed between two consecutive replenishment orders. To compute this we proceed as follows.

Choose an ϵ and a $H(\epsilon)$ such that the probability of the number of customers in system exceeding H has probability $< \epsilon$. Take ϵ to be quite small (of the order 10^{-5}). Accordingly H will be large. Yet we have a finite system. We have the corresponding system state probability vector; the sum of these probabilities is very close to 1. Starting from any system state, with the number of items in inventory S

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(an order has been placed and got materialised), we look at the evolution until the next replenishment takes place, either through realization of CLT or by the sales of the last item left in the inventory in that cycle.
Consider the Markov chain $X = \{X(t) : t \geq 0\} = \{(N(t), I(t), S(t)) : t \geq 0\}$

with the state space $\{(i, j, l) : 0 \leq i \leq H, 1 \leq j \leq S, 1 \leq l \leq k\}U\{\Delta\}$, where$\{\Delta\}$ is the absorbing state of the Markov chain which denotes the realization of CLT.

Let $q_i = Pr\{N(t) = i, I(t) = S, S(t) = 1\}$
and $\eta_i = q_i / \sum_{i=0}^{H} q_i$ Then $\eta = (\eta_0, \eta_1, \ldots\ldots..\eta_H)$ is the initial probability vector of the distribution. The infinitesimal generator of the process is

$$\hat{U} = \begin{bmatrix} \mathbf{V} & V^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where,

$$V = \begin{bmatrix} V_{S,S} & V_{S,S-1} & \mathbf{0} & \cdots & \mathbf{0} & V_{S,0} \\ \mathbf{0} & V_{S-1,S-1} & V_{S-1,S-2} & \mathbf{0} & \cdots & V_{S-1,0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & V_{s,s} & V_{s,s-1} & \ddots & V_{s,0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \ddots & V_{1,1} & V_{1,0} \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & V_{0,0} \end{bmatrix}_{(S+1)\times(S+1)}$$

Here,

for $s + 1 \leq i \leq S$

$$V_{i,i} = \begin{bmatrix} B_{S,S} & L & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_{S,S} & L & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & A_{S,S} & L \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_{S,S} \end{bmatrix}_{(H+1)\times(H+1)}$$

and for $1 \leq i \leq s$

$$V_{i,i} = \begin{bmatrix} B_{1,1} & L & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_{1,1} & L & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & A_{1,1} & L \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_{1,1} \end{bmatrix}_{(H+1)\times(H+1)}$$

For $2 \leq i \leq S$

$$V_{i,i-1} = \begin{bmatrix} \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ M & \ddots & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & M & \mathbf{0} \end{bmatrix}_{(H+1)\times(H+1)}$$

 https://doi.org/10.17993/3cemp.2022.110250.15-31

$$V_{1,0} = \begin{bmatrix} A_{1,0} & \mathbf{0} & \cdots & \mathbf{0} \\ M_1 & A_{1,0} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & M_1 & A_{1,0} \end{bmatrix}_{(H+1)\times(H+1)}$$

For $2 \leq i \leq S$

$$V_{i,0} = \begin{bmatrix} A_{1,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_{1,0} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & A_{1,0} \end{bmatrix}_{(H+1)\times(H+1)}$$

$V_{0,0} = -\beta I_{H+1}$ and

$$V^0 = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ B \\ \vdots \\ B \end{bmatrix}_{(S+1)\times 1}$$

$$B = \begin{bmatrix} \beta \\ \vdots \\ \beta \end{bmatrix}_{k\times 1}$$

The probability density function of the distribution is, $f(x) = \eta e^{Vx} V^0$

So the mean of the distribution is the first raw moment, $\mu_1' = -\eta V^{-1} e$
Therefore expected cycle length$= -\eta V^{-1} e$

5 OPTIMIZATION PROBLEM

Based on the above performance measures we construct a cost function to find the profit of the retailer. Let C_r be the price per unit of the item in the rth stage for$r = 1, \cdots, k$. Assume that the holding cost increases with the age of the stocked item and let H_r be the holding cost per unit of the item in the$r^t h$ stage. Let K be the fixed ordering cost while placing a replenishment order and c be the variable procurement cost (per unit) of the item to the store.
The total revenue $= \sum_{r=1}^{k} (C_r - H_r)E_r + C_s crap E_s crap - (K + cS)/cycle\ length$, where $C_s crap$ is the tagged price for scrapped item and $E_s crap$ is the expected number of scrapped items.

6 NUMERICAL ILLUSTRATION

In this section we provide numerical illustration of the profit function with variation in the values of s and S in both Phase Type and Erlang cases.
Table 1: **Effect of S and s on Profit in Phase-Type Case** We fixthe parameters as $k = 4, m =$

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$3, \lambda_1 = 4, \lambda_2 = 4.5, \lambda_3 = 5, \lambda_4 = 6, \mu_1 = 7, \mu_2 = 7.5, \mu_3 = 8, \mu_4 = 8.5, \beta = 5, \alpha = [0.4, 0.4, 0.2]$

$$T = \begin{bmatrix} -15 & 4 & 6 \\ 1 & -13 & 5 \\ 3 & 4 & -17 \end{bmatrix}$$

and costs as $C_1 = 10, C_2 = 8, C - 3 = 6, C_4 = 2, C_scrap = 1.5, H_1 = 2, H_2 = 1.5, H_3 = 1, H_4 = 1, c = 5, K = 2$

$S \downarrow s \rightarrow$	1	2	3	4	5	6	7
6	105.91	104.67	103.04	99.02	96.26		
7	112.82	112.40	111.67	110.60	107.32	105.32	
8	121.60	121.38	121.01	120.32	119.20	116.07	113.18
9	131.24	131.09	130.86	130.50	129.80	128.57	125.47
10	141.34	141.22	141.06	140.84	140.47	139.77	138.50
11	151.73	151.62	151.49	151.33	151.10	150.72	149.98
12	162.32	162.21	162.01	161.97	161.80	161.57	161.17
13	173.06	172.94	172.95	172.71	172.58	172.42	172.19
14	183.89	183.78	183.67	183.55	183.44	183.31	183.15
15	194.81	194.69	194.58	194.46	194.35	194.23	194.11

Table 2: **Effect of S and s on Profit in Erlang Case** We fix the parameters as $k = 4, m = 3, \lambda_1 = 4, \lambda_2 = 4.5, \lambda_3 = 5, \lambda_4 = 6, \mu_1 = 7, \mu_2 = 7.5, \mu_3 = 8, \mu_4 = 8.5, \beta = 5, \alpha = 5$
and the costs as $C_1 = 10, C_2 = 8, C - 3 = 6, C_4 = 2, C_scrap = 1.5, H_1 = 2, H_2 = 1.5, H_3 = 1, H_4 = 1, c = 5, K = 2$

$S \downarrow s \rightarrow$	1	2	3	4	5	6	7
6	75.09	64.82	53.67	41.60	29.13		
7	87.14	77.71	67.49	56.25	43.83	30.73	
8	98.51	89.98	80.77	70.57	59.13	46.30	32.52
9	109.19	101.58	93.35	84.23	73.92	62.19	48.88
10	119.28	112.55	105.31	97.24	88.08	77.58	65.51
11	128.74	122.86	116.52	109.42	101.34	92.04	81.27
12	137.66	132.57	127.05	120.86	113.78	105.60	96.10
13	146.07	141.69	136.94	131.58	125.42	118.27	109.93
14	154.08	150.34	146.27	141.66	146.35	130.14	122.88
15	161.66	158.49	155.03	151.10	146.53	141.17	134.89

From the table it is clear that the profit increses as S increases but it decreases as s increases. The change of profit when s is changed is very sensitive in Erlang case compared to Phase Type case.

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TWO-COMMODITY PERISHABLE INVENTORY SYSTEM WITH PARTIAL BACKLOG DEMANDS

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ABSTRACT

This article examines a two-commodity continuous review perishable inventory system. The demands are arrived at for each product by independent Markovian arrival processes (MAP). Lifetimes follow an exponential distribution. The commodities are assumed to be substitutable. If both commodities have reached zero, demand is backlogged up to predetermined levels. This article's novelty has been a local purchase, which is made to clear the backlog instantaneously when demand reaches a predetermined level. In the steady-state, the joint probability distribution of inventory levels of both commodities is obtained. Several metrics of system performance in steady-state are derived and also provided as numerical examples to explain the optimum values of the system's parameters.

KEYWORDS

Two-commodity Inventory system, Substitutable items, Joint ordering Policy, Markov arrival demands, Partial backlog.

1 INTRODUCTION

Over the last few decades, researchers have been fascinated with the study of a two-commodity inventory system. It has more importance because these systems are more sophisticated than single commodity inventory systems due to the large number of items held and their coordinated behaviors. Also, many organisations have increasingly used multi-commodity inventory systems. However, the correlation of the reorder points for each item is the major challenge in multi-product inventory systems. Unlike systems that deal with a single commodity, the reordering methods in these systems are more complicated. So, replenishment orders for groups of products must be well coordinated. Initially, research focused on inventory models with independently defined reorder points. The individual ordering policy includes calculating the best order quantity and reordering duration for each item. This ordering policy implementation provides the system with significant flexibility in picking the appropriate inventory models for each item and separately modifying the policy. However, joint ordering policies are preferred over individual ordering policies when the products share the same storage space and transportation facilities. Joint replenishment has several advantages because the joint ordering policy allows for the simultaneous replenishment of several commodities, quantity discounts, and significant savings in ordering and purchasing expenses. The joint replenishment was proposed by Balintfy and developed by Silver. More details about joint replenishment can be seen in Anbazhagan et al. (2012, 2015), Senthil Kumar, and Sivakumar. Various models with two-commodity readers can read in Anbazhagan and Arivarignan, Benny et al., Krishnamoorthy et al., and Ozkar et al.

In the earlier literature on inventory systems, it has generally been recognized that inventory models built under the presumption of a product's lifetime being indefinite until its storage, i.e., an item once placed in a storeroom stays unmodified and entirely functional for supplying future demand. However, this is not the case. When constructing inventory models, one aspect for consideration is an item's perishability, as commodities do not necessarily retain their properties when held for future use. In general, perishability is the outcome of stock depletion, which consists of obsolescence, breakage, decay, losing usefulness, and many other factors. Some examples of perishable objects are meals, evaporative fluids, chemicals, drugs, and radioactive substances. For more details about perishable product readers can refer Karthikeyan and Sudesh, Nahmias, Sivakumar et al., Smrutirekha Debataa et al., Umay and Bahar, Yadavalli et al. (2010, 2015), Zhang et al.

Several research articles examine inventory systems in which required products are directly provided from stock if the item is available. Demand that appears during stock-out times results in either lost sales or a backlog (demand satisfied immediately after the arrival of ordered items). Initially, it is believed that there is a total backlog of unfilled demand. In actuality, many customers are willing to wait until the end of the shortage period to pick up their orders, while others are not. As a result, it is presumed that any predefined quantity of demand (partial backlog) that appeared during the stock-out time is satisfied. For more details about backlog concept readers an refer Adak Sudip and Mahapatra, Cárdenas-Barrón Leopoldo et al., Khan et al., Kurt et al., San José et al., Stanley et al. and Tai et al. Generally, customer satisfaction generates a lot of profit for the system. So the shopkeeper does the maximum amount of work to satisfy the customers. In a practical situation, the local purchase is made by the shopkeeper when the shop runs out of stock and that item's replenishment has been delayed. We can see this act in clothing stores, supermarkets, and all the retailers' shops.

In this article we assume that demands during the stock-out periods are backlogged. We further assume that when the number of backlogged demands reaches a prefixed level a local purchase is made to clear the backlog instantaneously so that the inventory level of the corresponding commodity becomes zero. In the following sections, We have obtained the joint probability distribution for the inventory levels of both commodities in the steady state case in section 3. Various system performance measures in the steady state are derived in section 4 and the cost analysis and the results are illustrated numerically in section 5 and 6.

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2 THE MODEL

We consider a two-commodity inventory system with the maximum capacity S_i units for $i-$th commodity $(i = 1, 2)$. The demands for $i-$th commodity is of unit size. The demands for commodity-1 arrive according to a Markovian arrival process *(MAP)* with representation (D_0, D_1) where D's are of order $m_1 \times m_1$. The underlying Markov chain $J_1(t)$ of the MAP has the generator $D(= D_0 + D_1)$ and a stationary row vector λ_1 of length m_1. Independently of this process, demands for commodity-2 arrive according to a MAP with representation (F_0, F_1) where F's are of order $m_2 \times m_2$. The underlying Markov chain $J_2(t)$ of this MAP has the generator $F(= F_0 + F_1)$ and a stationary row vector λ_2 of length m_2. The items are perishable in nature. The life time of each commodity is assumed to be distributed as exponential with parameter $\gamma_i, (i = 1, 2)$. The two-commodities serve as substitute for each other, that is, a demand for a commodity that is sold out, is satisfied with the other commodity when still in stock. If both the commodities are out of stock, any arriving demands are backlogged. The backlog is allowed up to the level $N_i(< \infty)$ for the $i-$th commodity $(i = 1, 2)$. Whenever the backlog level reaches $N_i, (i = 1, 2)$ an order for N_i items is placed which is replenished instantaneously. The reorder level for the i-th commodity is fixed at $s_i(1 \leq s_i \leq S_i)$ with an ordering quantity for the i-th commodity is $Q_i(= S_i - s_i > s_i + N_i + 1)$ items when both inventory levels are less than or equal to their respective reorder levels. The requirement $S_i - s_i > s_i + N_i + 1$ ensures that after the replenishment the inventory levels of both commodities will be always above the respective reorder levels; otherwise it may not be possible to place reorder (according to this policy) which leads to perpetual shortage. More explicitly if $L_i(t)$ represents inventory level of i-th commodity at time t, then a reorder for both commodities is made when $L_1(t) \leq s_1$ and $L_2(t) \leq s_2$. The lead time is assumed to be distributed as negative exponential with parameter $\beta(> 0)$.

Notations

$[A]_{ij}$	: The element/submatrix at (i, j)-th position of A.
$\mathbf{0}$	: Zero matrix.
I	: An identity matrix.
I_k	: An identity matrix of order k.
$A \otimes B$	: Kronecker product of matrices A and B.
$A \oplus B$	: Kronecker sum of matrices A and B.
$\boldsymbol{e}$	: A column vector of $1'$s of appropriate dimension.

3 ANALYSIS

From the assumptions made on the input and output processes it can be shown that the quadruple $(L_1, L_2, J_1, J_2) = \{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ is a Markov process with state space given by

$$\begin{aligned} E &= \{(i, k, j_1, j_2) | i = 1, 2, \ldots, S_1, k = 0, 1, \ldots, S_2, j_1 = 1, 2, \ldots, m_1, j_2 = 1, 2, \ldots, m_2\} \\ &\cup \{(i, k, j_1, j_2) | i = 0, k = -(N_2 - 1), -(N_2 - 2), \ldots, S_2, j_1 = 1, 2, \ldots, m_1, j_2 = 1, 2, \ldots, m_2\} \\ &\cup \{(i, k, j_1, j_2) | i = -(N_1 - 1), -(N_1 - 2), \ldots, -1, k = -(N_2 - 1), -(N_2 - 2), \ldots, 0, \\ &j_1 = 1, 2, \ldots, m_1, j_2 = 1, 2, \ldots, m_2\}. \end{aligned}$$

Define the following ordered sets :

$$\begin{aligned} \mathbf{i} &= (\ (i, 0), (i, 1), \ldots, (i, S_2)\ )\,, \\ < i > &= (\ (i, -N_2 + 1), (i, -N_2 + 2), \ldots, (i, S_2)\ )\,, \\ [i] &= (\ (i, -N_2 + 1), (i, -N_2 + 2), \ldots, (i, 0)\ )\,, \\ (i, j) &= (\ (i, j, 1), (i, j, 2), \ldots, (i, j, m_1)\ )\,, \\ (i, j, k) &= (\ (i, j, k, 1), (i, j, k, 2), \ldots, (i, j, k, m_2)\ )\,, \end{aligned}$$

Then the state space is ordered as $([-N_1+1],[-N_1+2],\ldots,[-1],<0>,\mathbf{1},$ $\mathbf{2},\ldots,\mathbf{S_1})$. The infinitesimal generator of P of the Markov process (L_1,L_2,J_1,J_2) has the following block partitioned form :

$$[P]_{ij}=\begin{cases} B_i, & j=i-1, & i=0,1,\ldots,S_1,\\ \widehat{B}, & j=i-1, & i=-(N_1-2),-(N_1-3),\ldots,-1,\\ \widetilde{B}, & j=i+(N_1-1), & i=-(N_1-1),\\ C, & j=i+Q_1, & i=1,2,\ldots,s_1,\\ \widehat{C}, & j=i+Q_1, & i=0,\\ \widetilde{C}, & j=i+Q_1, & i=-(N_1-1),-(N_1-2),\ldots,-1,\\ A_i, & j=i, & i=0,1,\ldots,S,\\ \widehat{A}, & j=i, & i=-(N_1-1),-(N_1-2),\ldots,-1,\\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C]_{kl}=\begin{cases} \beta I_{m_1}\otimes I_{m_2}, & l=k+Q_2, \quad k=0,1,\ldots,s_2,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

$$[\widehat{C}]_{kl}=\begin{cases} \beta I_{m_1}\otimes I_{m_2} & l=k+Q_2, \quad k=-(N_2-1),-(N_2-2),\ldots,s_2,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

$$[\widetilde{C}]_{kl}=\begin{cases} \beta I_{m_1}\otimes I_{m_2}, & l=k+Q_2, \quad k=-(N_2-1),-(N_2-2),\ldots,0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

For $i=2,3,\ldots,S_1,$

$$[B_i]_{kl}=\begin{cases} D_1\otimes I_{m_2}+i\gamma_1 I_{m_1}\otimes I_{m_2}, & l=k, & k=1,2,\ldots,S_2,\\ D_1\oplus F_1+i\gamma_1 I_{m_1}\otimes I_{m_2}, & l=k, & k=0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

For $i=1,$

$$[B_i]_{kl}=\begin{cases} D_1\otimes I_{m_2}+i\gamma_1 I_{m_1}\otimes I_{m_2}, & l=k, & k=1,2,\ldots,S_2,\\ D_1\oplus F_1+i\gamma_1 I_{m_1}\otimes I_{m_2}, & l=k, & k=0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

For $i=0,$

$$[B_i]_{kl}=\begin{cases} D_1\otimes I_{m_2}, & l=k, \quad k=-(N_2-1),(N_2-2),\ldots,0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

$$\left[\hat{B}\right]_{kl}=\begin{cases} D_1\otimes I_{m_2}, & l=k, \quad k=-(N_2-1),(N_2-2),\ldots,0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

$$\left[\tilde{B}\right]_{kl}=\begin{cases} D_1\otimes I_{m_2}, & l=k, \quad k=-(N_2-1),(N_2-2),\ldots,0,\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

For $i=1,2,\ldots s_1,$

$$[A_i]_{kl}=\begin{cases} I_{m_1}\otimes F_1+k\gamma_2 I_{m_1}\otimes I_{m_2}, & l=k-1, & k=1,2,\ldots,S_2,\\ D_0\oplus F_0-(i\gamma_1+\beta)I_{m_1}\otimes I_{m_2}, & l=k, & k=0,\\ D_0\oplus F_0-(i\gamma_1+\beta+k\gamma_2)I_{m_1}\otimes I_{m_2}, & l=k, & k=1,2,\ldots,s_2\\ D_0\oplus F_0-(i\gamma_1+k\gamma_2)I_{m_1}\otimes I_{m_2}, & l=k, & k=s_2+1,s_2+2,\ldots,S_2\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

https://doi.org/10.17993/3cemp.2022.110250.33-48

For $i = s_1+1, s_1+2, \ldots S_1$,

$$[A_i]_{kl} = \begin{cases} I_{m_1} \otimes F_1 + k\gamma_2 I_{m_1} \otimes I_{m_2}, & l = k-1, \quad k = 1,2,\ldots,S_2, \\ D_0 \oplus F_0 - i\gamma_1 I_{m_1} \otimes I_{m_2}, & l = k, \quad k = 0, \\ D_0 \oplus F_0 - (i\gamma_1 + k\gamma_2) I_{m_1} \otimes I_{m_2}, & l = k, \quad k = 1,2,\ldots,S_2 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $i = 0$,

$$[A_i]_{kl} = \begin{cases} D_1 \oplus F_1 + k\gamma_2 I_{m_1} \otimes I_{m_2}, & l = k-1, \quad k = 1,2,\ldots,S_2, \\ I_{m_1} \otimes F_1, & l = k-1, \quad k = -(N_2-2), -(N_2-3), \ldots, -1, 0, \\ & (or) \\ & l = k + N_2 - 1, \quad k = -(N_2-1), \\ D_0 \oplus F_0 - \beta I_{m_1} \otimes I_{m_2}, & l = k, \quad k = -(N_2-1), -(N_2-2), \ldots, 0, \\ D_0 \oplus F_0 - (\beta + k\gamma_2) I_{m_1} \otimes I_{m_2}, & l = k, \quad k = 1,2,\ldots,s_2, \\ D_0 \oplus F_0 - k\gamma_2 I_{m_1} \otimes I_{m_2}, & l = k, \quad k = s_2+1, s_2+2, \ldots, S_2, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$\left[\widehat{A}\right]_{kl} = \begin{cases} I_{m_1} \otimes F_1, & l = k-1, \quad k = -(N_2-2), -(N_2-3), \ldots, -1, 0, \\ & (or) \\ & l = k + N_2 - 1, \quad k = -(N_2-1), \\ D_0 \oplus F_0 - \beta I_{m_1} \otimes I_{m_2}, & l = k, \quad k = -(N_2-1), -(N_2-2), \ldots, 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

It may be noted that the matrices $A_i, i = 1,2,\ldots,S_1, B_i, i = 2,3,\ldots,S_1$ and C are of size $(S_2+1)m_1m_2 \times (S_2+1)m_1m_2$, B_1 is of size $(S_2+1)m_1m_2 \times (S_1+N_2)m_1m_2$, B_0 is of size $(S_2+N_2)m_1m_2 \times N_2m_1m_2$, $\widehat{B}$ is of size $N_2m_1m_2 \times N_2m_1m_2$, $\widetilde{B}$ is of size $N_2m_1m_2 \times (S_2+N_2)m_1m_2$, $\widehat{C}$ is of size $(S_2+N_2)m_1m_2 \times (S_2+1)m_1m_2$, $\widetilde{C}$ is of size $N_2m_1m_2 \times (S_2+1)m_1m_2$, A_0 is of size $(S_2+N_2)m_1m_2 \times (S_2+N_2)m_1m_2$ and $\widehat{A}$ is of size $N_2m_1m_2 \times N_2m_1m_2$.

3.1 STEADY STATE ANALYSIS

It can be seen from the structure of P that the homogeneous Markov process $\{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ on the finite state space E is irreducible. Hence the limiting distribution $\phi_{(i,k,j_1,j_2)} =$

$$\lim_{t\to\infty} Pr\left[L_1(t) = i, L_2(t) = k, J_1(t) = j_1, J_2(t) = j_2 | L_1(0), L_2(0), J_1(0), J_2(0)\right]$$

exists. Let

$$\begin{aligned} \phi_{(i,k,j_1)} &= (\phi_{(i,k,j_1,1)}, \phi_{(i,k,j_1,2)}, \ldots, \phi_{(i,k,j_1,m_2)}), j_1 = 1,2,\ldots,m_1, \\ \phi_{(i,k)} &= \left(\phi_{(i,k,1)}, \phi_{(i,k,2)}, \ldots, \phi_{(i,k,m_1)}\right), k = -N_2+1, -N_2+2, \ldots, S_2, \\ \boldsymbol{\phi^{(i)}} &= \begin{cases} (\phi_{(i,0)}, \phi_{(i,1)}, \ldots, \phi_{(i,S_2)}), & \text{if } i = 1,2,\ldots,S_1, \\ (\phi_{(i,-N_2+1)}, \phi_{(i,-N_2+2)}, \ldots, \phi_{(i,S_2)}), & \text{if } i = 0, \\ (\phi_{(i,-N_2+1)}, \phi_{(i,-N_2+2)}, \ldots, \phi_{(i,0)}), & \text{if } i = -N_1+1, -N_1+2, \ldots, -1. \end{cases} \\ \text{and} & \\ \boldsymbol{\Phi} &= (\boldsymbol{\phi^{(-N_1+1)}}, \boldsymbol{\phi^{(-N_1+2)}}, \ldots, \boldsymbol{\phi^{(S_1-1)}}, \boldsymbol{\phi^{(S_1)}}). \end{aligned}$$

Then the vector of limiting probabilities $\boldsymbol{\Phi}$ satisfies

$$\boldsymbol{\Phi} P = \mathbf{0} \quad \text{and} \quad \boldsymbol{\Phi}\mathbf{e} = 1. \tag{1}$$

The first equation of the above yields the following set of equations:

https://doi.org/10.17993/3cemp.2022.110250.33-48

$$
\begin{aligned}
\boldsymbol{\phi}^{(i+1)}\widehat{B}+\boldsymbol{\phi}^{(i)}\widehat{A}&=\mathbf{0}, && i=-N_1+1,-N_1+2,\dots,-2,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}\widehat{A}&=\mathbf{0}, && i=-1,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-N_1+1)}\widetilde{B}&=\mathbf{0}, && i=0,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i&=\mathbf{0}, && i=1,2,\dots,Q_1-N_1,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-Q_1)}\widetilde{C}&=\mathbf{0}, && i=Q_1-N_1+1,Q_1-N_1+2,\dots,Q_1-1,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-Q_1)}\widehat{C}&=\mathbf{0}, && i=Q_1,\\
\boldsymbol{\phi}^{(i+1)}B_{i+1}+\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-Q_1)}C&=\mathbf{0}, && i=Q_1+1,Q_1+2\dots,S_1-1,\\
\boldsymbol{\phi}^{(i)}A_i+\boldsymbol{\phi}^{(i-Q_1)}C&=\mathbf{0}, && i=S_1.
\end{aligned}
\tag{2}
$$

The equations (except (2)) can be recursively solved to get

$$\boldsymbol{\phi}^{(i)}=\boldsymbol{\phi}^{(Q_1)}\theta_i,\quad i=-N_1+1,-N_1+2,\dots,S_1,$$

where

$$
\theta_i=\begin{cases}
-\theta_{i+1}\widehat{B}\widehat{A}^{-1}, & i=-(N_1-1),-(N_1-2),\dots,-2,\\
-\theta_{i+1}B_0\widehat{A}^{-1}, & i=-1,\\
-\left(\theta_{i+1}B_{i+1}+\theta_{i-N_1+1}\widetilde{B}\right)A_i^{-1}, & i=0,\\
-\theta_{i+1}B_{i+1}A_i^{-1}, & i=1,2,\dots,Q_1-N_1,\\
-\left(\theta_{i+1}B_{i+1}+\theta_{i-Q_1}\widetilde{C}\right)A_i^{-1}, & i=Q_1-N_1+1,Q_1-N_1+2,\dots,Q_1-1,\\
I, & i=Q_1,\\
-\left(\theta_{i+1}B_{i+1}+\theta_{i-Q_1}C\right)A_i^{-1}, & i=Q_1+1,Q_1+2,\dots,S_1-1,\\
-\theta_{i-Q_1}CA_i^{-1}, & i=S_1.
\end{cases}
$$

Substituting the values of θ_i in equation (2) and in the normalizing condition we get the value of $\boldsymbol{\phi}^{(Q_1)}$.

4 SYSTEM PERFORMANCE MEASURES

In this section we derive some stationary performance measures of the system. Using these measures, we can construct the total expected cost per unit time.

4.1 MEAN INVENTORY LEVEL

Let η_{I_i} denote the mean inventory level of $i-$th commodity in the steady state $(i=1,2)$. Since $\phi_{(i,j)}$ is the steady state probability vector for inventory level of first commodity is i and the second commodity is j, we have

$$\eta_{I_1}=\sum_{i=1}^{S_1}\sum_{k=0}^{S_2} i\phi_{(i,k)}\mathbf{e}.$$

and

$$\eta_{I_2}=\sum_{i=0}^{S_1}\sum_{k=1}^{S_2} k\phi_{(i,k)}\mathbf{e}.$$

4.2 MEAN REORDER RATE

A reorder for both commodities is made when the joint inventory level, drops to either (s_1,s_2) or $(s_1,j),j<s_2$ or $(i,s_2),i<s_1$. Let ζ_R denote the mean joint reorder rate for both commodities in the

 https://doi.org/10.17993/3cemp.2022.110250.33-48

steady state and it is given by

$$\begin{aligned}\eta_R &= \frac{1}{\lambda_1}\sum_{k=0}^{s_2}\phi_{(s_1+1,k)}\left(D_1\otimes I_{m_2}\right)\mathbf{e}+\frac{1}{\lambda_2}\sum_{i=0}^{s_1}\phi_{(i,s_2+1)}\left(I_{m_1}\otimes F_1\right)\mathbf{e}\\ &+\frac{1}{\lambda_1}\phi_{(0,s_2+1)}\left(I_{m_1}\otimes F_1\right)\mathbf{e}+\frac{1}{\lambda_2}\phi_{(s_1+1,0)}\left(D_1\otimes I_{m_2}\right)\mathbf{e}\\ &+(s_1+1)\gamma_1\sum_{k=0}^{s_2}\phi_{(s_1+1,k)}\mathbf{e}+(s_2+1)\gamma_2\sum_{i=0}^{s_1}\phi_{(i,s_2+1)}\mathbf{e}.\end{aligned}$$

Let η_{R_i} denote the mean individual reorder rate for commodity-i in the steady state ($i=1,2$). When the inventory level of commodity-1 is $-(N_1-1)$, a demand for commodity-1 will trigger the individual reorder for commodity-1. Hence we get

$$\eta_{R_1} = \frac{1}{\lambda_1}\sum_{k=-N_2+1}^{0}\phi_{(-N_1+1,k)}\left(D_1\otimes I_{m_2}\right)\mathbf{e}.$$

Similar arguments lead to

$$\eta_{R_2} = \frac{1}{\lambda_2}\sum_{i=-N_1+1}^{0}\phi_{(i,-N_2+1)}\left(I_{m_1}\otimes F_1\right)\mathbf{e}.$$

4.3 AVERAGE BACKLOG

Let η_{B_i} denote the mean backlog of commodity-i in the steady state ($i=1,2$). Then we have

$$\eta_{B_1} = \sum_{i=-N_1+1}^{-1}\sum_{k=-N_2+1}^{0}|i|\phi_{(i,k)}\mathbf{e}.$$

and

$$\eta_{B_2} = \sum_{i=-N_1+1}^{0}\sum_{k=-N_2+1}^{-1}|k|\phi_{(i,k)}\mathbf{e}.$$

4.4 MEAN PERISHABLE RATE

Let the mean perishable rate of commodity-i in the steady state de denoted by $\zeta_{F_i}, (i=1,2)$. Then we have

$$\eta_{F_1} = \sum_{i=1}^{S_1}\sum_{k=0}^{S_2}i\gamma_1\phi_{(i,k)}\mathbf{e}.$$

and

$$\eta_{F_2} = \sum_{i=0}^{S_1}\sum_{k=1}^{S_2}k\gamma_2\phi_{(i,k)}\mathbf{e}.$$

5 COST ANALYSIS

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

$$\begin{aligned}TC(S_1,S_2,s_1,s_2,N_1,N_2) &= c_{h_1}\eta_{I_1}+c_{h_2}\eta_{I_2}+c_r\eta_R+c_{r_1}\eta_{R_1}+c_{r_2}\eta_{R_2}\\ &+c_{b_1}\eta_{B_1}+c_{b_2}\eta_{B_2}+c_{p_1}\eta_{F_1}+c_{p_2}\eta_{F_2},\end{aligned}$$

where

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c_r : Setup cost per order.
c_{r_i} : Setup cost for the i-th commodity under local purchase (i=1,2).
c_{h_i} : Holding cost for the i-th commodity per unit time, $i = 1, 2$.
c_{p_i} : Perishable cost per unit item per unit time of i-th commodity (i=1,2).
c_{b_i} : Cost per unit backlog for the i-th commodity per unit time, $i = 1, 2$.

By substituting the values for η's we can compute the value of $TC(S_1, S_2, s_1, s_2, N_1, N_2)$.

Since the evaluation of the ϕ's involve recursive computations, it is quite difficult to show the convexity of the total expected cost rate. However we present the following example to demonstrate the computability of the results derived in our work, and to illustrate the existence of local optima when the total cost function is treated as a function of only two variables.

6 NUMERICAL ILLUSTRATION

We consider the following numerical example : The demand for first commodity is given by (D_0, D_1) where

$$D_0 = \begin{pmatrix} -50 & 0 \\ 0 & -5 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 39 & 11 \\ 3.9 & 1.1 \end{pmatrix}.$$

The demand for second commodity is given by (F_0, F_1) where

$$F_0 = \begin{pmatrix} -20 & 0 \\ 0 & -2 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 19 & 1 \\ 1.9 & 0.1 \end{pmatrix}.$$

In the following tables, the optimal cost for each row is shown in underlined and the optimal cost for each column is shown in bold.
Let $\gamma_1 = 1, \gamma_2 = 1, \beta = 25, s_1 = 2, s_2 = 2, N_1 = 3, N_2 = 3, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 75, c_{r_1} = 2, c_{r_2} = 2, c_{b_1} = 1, c_{b_2} = 1, c_{p_1} = 2, c_{p_2} = 1$.

$$\text{Let} \qquad \overline{TC}(S_1, S_2) = TC(S_1, S_2, 2, 2, 3, 3).$$

From table 1, the numerical values shows that $\overline{TC}(S_1, S_2)$ is a convex function in (S_1, S_2) and the

Table 1 – Total Expected Cost Rate of S_1 and S_2

S_1 \ S_2	10	11	12	13	14
13	9.872429	9.630808	<u>9.596855</u>	9.743616	9.775500
14	9.709404	9.520833	<u>9.501561</u>	9.633814	9.684700
15	9.594168	9.451205	<u>9.446881</u>	9.569767	9.634135
16	9.517610	<u>9.413976</u>	**9.424364**	**9.541746**	**9.616449**
17	9.472896	<u>**9.403231**</u>	9.427757	9.542487	9.625701
18	**9.454763**	<u>9.414471</u>	9.452342	9.566473	9.657093
19	9.459070	<u>9.444205</u>	9.494506	9.609474	9.706772

(possibly local) optimum occurs at $(S_1, S_2) = (17, 11)$.
Let $\gamma_1 = 0.01, \gamma_2 = 0.8, \beta = 18, S_2 = 20, s_2 = 3, N_1 = 3, N_2 = 3, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 0.55, c_{r_1} = 0.45, c_{r_2} = 0.5, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.4$.

$$\text{Let} \qquad \overline{TC}(S_1, s_1) = TC(S_1, 20, s_1, 3, 3, 3).$$

From table 2, the numerical values shows that $\overline{TC}(S_1, s_1)$ is a convex function in (S_1, s_1) and the

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Table 2 – Total Expected Cost Rate of S_1 and s_1

S_1 \ s_1	4	5	6	7	8
49	5.076500	5.075467	5.080127	5.088899	5.100611
50	**5.076459**	5.075080	5.079444	5.087956	5.099435
51	5.076589	5.074872	5.078945	5.087203	5.098452
52	5.076888	**5.074837**	5.078626	5.086635	5.097661
53	5.077351	5.074974	**5.078484**	5.086248	5.097055
54	5.077976	5.075278	5.078514	5.086039	5.096631
55	5.078760	5.075747	5.078714	**5.086004**	5.096385
56	5.079700	5.076377	5.079080	5.086140	**5.096314**
57	5.080793	5.077165	5.079609	5.086443	5.096414

(possibly local) optimum occurs at $(S_1, s_1) = (52, 5)$.

let $\gamma_1 = 0.01, \gamma_2 = 0.9, \beta = 10, S_2 = 20, s_2 = 2, S_1 = 20, s_1 = 2, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 21, c_{r_1} = 15, c_{r_2} = 18, c_{b_1} = 5, c_{b_2} = 5, c_{p_1} = 0.8, c_{p_2} = 0.75$.

$$\text{Let} \qquad \overline{TC}(N_1, N_2) = TC(20, 20, 2, 2, N_1, N_2).$$

From table 3, the numerical values shows that $\overline{TC}(N_1, N_2)$ is a convex function in (N_1, N_2) and the

Table 3 – Total Expected Cost Rate of N_1 and N_2

N_1 \ N_2	3	4	5	6	7
4	10.565306	10.543407	10.533310	10.531192	10.535652
5	10.509154	10.491453	10.486547	10.490709	10.502146
6	10.476386	10.465856	10.468392	**10.481047**	**10.501967**
7	10.456879	10.454703	**10.465625**	10.487919	10.520371
8	**10.453162**	**10.449303**	10.468540	10.500486	10.545501
9	10.461411	10.459465	10.492293	10.541267	10.609235

(possibly local) optimum occurs at $(N_1, N_2) = (8, 4)$.

Let $\gamma_1 = 0.1, \gamma_2 = 0.8, \beta = 18, S_1 = 20, s_1 = 3, s_2 = 2, N_1 = 3; c_{h_1} = 0.1, c_{h_2} = 0.1, c_r = 0.11, c_{r_1} = 0.1, c_{r_2} = 0.1, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.1$.

$$\text{Let} \qquad \overline{TC}(S_2, N_2) = TC(20, S_2, 3, 2, 3, N_2).$$

From table 4, the numerical values shows that $\overline{TC}(S_2, N_2)$ is a convex function in (S_2, N_2) and the

Table 4 – Total Expected Cost Rate of S_2 and N_2

S_2 \ N_2	5	6	7	8	9
39	8.654678	8.653849	8.654060	8.654295	8.654545
40	**8.517849**	**8.517004**	**8.517197**	**8.517414**	**8.517645**
41	8.689863	8.680004	8.680181	8.680380	8.680593
42	8.843738	8.842865	8.843027	8.843210	8.843407
43	9.005788	9.005604	9.005752	9.005920	9.006102

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(possibly local) optimum occurs at $(S_2, N_2) = (40, 6)$.

Let $\gamma_1 = 0.1, \gamma_2 = 0.8, \beta = 18, S_1 = 20, s_1 = 3, N_2 = 3, N_1 = 3; c_{h_1} = 0.1, c_{h_2} = 0.1, c_r = 0.11, c_{r_1} = 0.1, c_{r_2} = 0.1, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.1$.

$$\text{Let} \quad \overline{TC}(S_2, s_2) = TC(20, S_2, 3, s_2, 3, 3).$$

From table 5, the numerical values shows that $\overline{TC}(S_2, s_2)$ is a convex function in (S_2, s_2) and the

Table 5 – Total Expected Cost Rate of S_2 and s_2

s_2 / S_2	2	3	4	5	6
39	8.656617	8.655266	8.656616	8.657680	8.658492
40	**8.519803**	**8.518349**	**8.519626**	**8.520642**	**8.521425**
41	8.699830	8.681279	8.682486	8.683454	8.684207
42	8.846717	8.844073	8.845214	8.846135	8.846858
43	9.007478	9.006747	9.007824	9.008700	9.009393

(possibly local) optimum occurs at $(S_2, s_2) = (40, 3)$.
Let $\gamma_1 = 0.01, \gamma_2 = 0.8, \beta = 18, S_2 = 20, s_1 = 2, s_2 = 3, N_2 = 3; c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 0.55, c_{r_1} = 0.45, c_{r_2} = 0.5, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.4$.

$$\text{Let} \quad \overline{TC}(S_1, N_1) = TC(S_1, 20, 2, 3, N_1, 3).$$

From table 6, the numerical values shows that $\overline{TC}(S_1, N_1)$ is a convex function in (S_1, N_1) and the

Table 6 – Total Expected Cost Rate of S_1 and N_1

N_1 / S_1	5	6	7	8	9
49	5.034028	5.023524	5.021157	5.023724	5.029325
50	**5.033895**	5.023100	5.020510	5.022892	5.028327
51	5.033935	5.022856	5.020048	5.022249	5.027524
52	5.034146	**5.022789**	5.019767	5.021791	5.026910
53	5.034523	5.022894	**5.019662**	5.021515	5.026482
54	5.035065	5.023168	5.019732	**5.021415**	5.026235
55	5.035768	5.023608	5.019971	5.021490	**5.026165**
56	5.036629	5.024210	5.020377	5.021734	5.026268

(possibly local) optimum occurs at $(S_1, N_1) = (53, 7)$.
The Figure 1 grants the impact of the perishable rate γ_1, on the total expected cost rate TC via four curves which relate to β = 18.5,18.6,18.7,18.8. Since figure 1, we perceive that the total cost value decreases when the perishable rate γ_1 and the replenishment rate β increases.

The Figure 2 grants the impact of the perishable rate γ_2, on the total expected cost rate TC via three curves which relate to β = 19,20 and 21. Since figure 2, we perceive that the total cost value decreases when the perishable rate γ_2 and the replenishment rate β increases.
In tables 7 and 8, we show that the impact of the cost values on the optimal values (S_1^*, s_1^*) and the corresponding total expected cost rate. Towards this end, we first fix the parameters and cost value as $S_2 = 20, s_2 = 3, N_1 = 3, N_2 = 3, \beta = 18, \gamma_1 = 0.01, \gamma_2 = 0.8, c_{r_1} = 0.45, c_{r_2} = 0.5$.

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Table 7 – Impact of Cost Values

C_{h_2}				0.01					
C_{p_2}				0.4			0.5		
C_{b_2}				0.09	0.1	0.11	0.09	0.1	0.11
C_r	C_{h_1}	C_{p_1}	C_{b_1}						
0.4	0.01	0.10	0.09	55 6 5.0730	55 6 5.0733	55 6 5.0736	55 6 6.1763	55 6 6.1765	55 6 6.1768
			0.10	55 6 5.0734	55 6 5.0737	55 6 5.0740	55 6 6.1767	55 6 6.1770	55 6 6.1773
			0.11	55 6 5.0735	55 6 5.0738	55 6 5.0741	55 6 6.1768	55 6 6.1771	55 6 6.1774
		0.20	0.09	55 6 5.1261	55 6 5.1264	55 6 5.1267	55 6 6.2202	55 6 6.2205	55 6 6.2208
			0.10	55 6 5.1266	54 6 5.1269	54 6 5.1272	54 6 6.2206	54 6 6.2209	54 6 6.2212
			0.11	54 6 5.1271	54 6 5.1273	54 6 5.1276	54 6 6.2211	54 6 6.2214	54 6 6.2217
	0.02	0.10	0.09	54 6 5.5517	54 6 5.5520	54 6 5.5523	53 6 6.6793	53 6 6.6795	53 6 6.6798
			0.10	54 6 5.5522	54 6 5.5524	54 6 5.5527	53 6 6.6797	53 6 6.6800	53 6 6.6873
			0.11	54 6 5.5526	54 5 5.5529	54 5 5.5532	53 5 6.6817	53 5 6.6819	53 5 6.6893
		0.20	0.09	54 6 5.6281	53 5 5.6284	53 5 5.6287	53 5 6.6822	53 5 6.6835	53 5 6.6900
			0.10	53 6 5.6296	53 5 5.6299	53 5 5.6309	53 5 6.6826	53 5 6.6837	53 5 6.6912
			0.11	53 6 5.6371	53 5 5.6373	53 5 5.6376	53 5 6.6829	53 5 6.6838	52 5 6.6915
0.5	0.01	0.10	0.09	55 6 5.6462	55 6 5.6468	55 6 5.6471	55 6 6.6983	55 6 6.6985	55 6 6.6988
			0.10	55 6 5.6771	55 6 5.6777	55 6 5.6780	55 6 6.7067	55 6 6.7079	55 6 6.7083
			0.11	55 6 5.6784	55 6 5.6787	55 6 5.6790	55 6 6.7177	55 6 6.7178	55 6 6.7179
		0.20	0.09	55 6 5.6861	55 6 5.6884	55 6 5.6887	55 6 6.7202	55 6 6.7265	55 6 6.7268
			0.10	55 6 5.6886	54 6 5.6889	54 6 5.6890	54 6 6.7566	54 6 6.7569	54 6 6.7570
			0.11	54 6 5.6887	54 6 5.6890	54 6 5.6891	54 6 6.7571	54 6 6.7574	54 6 6.7577
	0.02	0.10	0.09	54 6 5.6892	54 6 5.6894	54 6 5.6896	53 6 6.7692	53 6 6.7698	53 6 6.7791
			0.10	54 6 5.6893	54 6 5.6902	54 6 5.6907	53 6 6.7857	53 6 6.7860	53 6 6.7893
			0.11	54 6 5.6896	54 5 5.6909	54 5 5.6912	53 5 6.7919	53 5 6.7942	53 5 6.7958
		0.20	0.09	54 5 5.7281	53 5 5.7284	53 5 5.7287	53 5 6.8012	53 5 6.8015	53 5 6.8018
			0.10	53 6 5.7296	53 5 5.7299	53 5 5.7309	53 5 6.8026	53 5 6.8029	53 5 6.8032
			0.11	53 6 5.8371	53 5 5.8373	53 5 5.8376	53 5 6.8036	53 5 6.8044	52 5 6.8047

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Table 8 – Impact of Cost Value

C_{h_2}				0.02					
C_{p_2}				0.4			0.5		
C_{b_2}				0.09	0.1	0.11	0.09	0.1	0.11
C_r	C_{h_1}	C_{p_1}	C_{b_1}						
0.4	0.01	0.10	0.09	54 6 5.2189	54 6 5.2191	54 6 5.2194	54 6 6.2197	54 6 6.2213	54 6 6.2232
			0.10	54 6 5.2196	54 6 5.2199	54 6 5.3101	54 6 6.2367	54 6 6.2370	54 6 6.2373
			0.11	54 6 5.2234	54 6 5.2237	54 6 5.3240	54 6 6.2767	54 6 6.2770	54 6 6.2773
		0.20	0.09	54 6 5.2281	54 6 5.2284	54 6 5.3287	54 6 6.3902	54 6 6.3913	54 6 6.3956
			0.10	53 6 5.3296	53 6 5.3299	53 6 5.3302	53 6 6.4106	53 6 6.4109	53 6 6.4112
			0.11	53 6 5.3311	53 6 5.3323	53 6 5.3346	53 6 6.4152	53 6 6.4163	53 6 6.4177
	0.02	0.10	0.09	53 6 5.6517	53 6 5.6520	53 6 5.6523	52 6 6.7793	52 6 6.7795	52 6 6.7798
			0.10	53 6 5.6522	53 6 5.6524	53 6 5.6527	52 6 6.7797	52 6 6.7800	52 6 6.7873
			0.11	53 5 5.6526	53 5 5.6529	53 5 5.6532	52 5 6.7817	52 5 6.7819	52 5 6.7883
		0.20	0.09	53 5 5.7281	53 5 5.7284	53 5 5.7287	52 5 6.7912	52 5 6.7915	52 5 6.7918
			0.10	52 5 5.7296	52 5 5.7299	52 5 5.7309	52 5 6.8026	52 5 6.8029	52 5 6.8112
			0.11	52 5 5.7371	52 5 5.7373	52 5 5.7376	52 5 6.8116	52 5 6.8134	52 5 6.8147
0.5	0.01	0.10	0.09	54 6 5.8762	54 6 5.8768	54 6 5.8771	54 6 6.8783	54 6 6.8785	54 6 6.8788
			0.10	54 6 5.8771	54 6 5.8777	54 6 5.8780	54 6 6.9767	54 6 6.9790	54 6 6.9793
			0.11	54 6 5.8784	54 6 5.8787	54 6 5.8790	54 6 6.9777	54 6 6.9870	54 6 6.9873
		0.20	0.09	54 6 6.1761	54 6 6.1784	54 6 6.1787	54 6 7.2202	54 6 7.2565	54 6 7.2568
			0.10	53 6 6.1786	53 6 6.1789	53 6 6.1792	53 6 7.2566	53 6 7.2569	53 6 7.2572
			0.11	53 6 6.1791	53 6 6.1793	53 6 6.1796	53 6 7.2671	53 6 7.2674	53 6 7.2677
	0.02	0.10	0.09	53 6 6.5627	53 6 6.5670	53 6 6.5823	52 6 7.6797	52 6 7.6894	52 6 7.6898
			0.10	53 6 6.5792	53 6 6.5802	53 6 6.5907	52 6 7.6857	52 6 7.6920	52 6 7.6923
			0.11	53 5 6.5906	53 5 6.5929	53 5 6.5942	52 5 7.6919	52 5 7.6942	52 5 7.6958
		0.20	0.09	53 5 6.7281	53 5 6.7284	53 5 6.7287	52 5 7.7312	52 5 7.7315	52 5 7.7318
			0.10	52 5 6.7296	52 5 6.7299	52 5 6.7309	52 5 7.7426	52 5 7.7429	52 5 7.7512
			0.11	52 5 6.8371	52 5 6.8373	52 5 6.8376	52 5 7.7516	52 5 7.7534	52 5 7.7547

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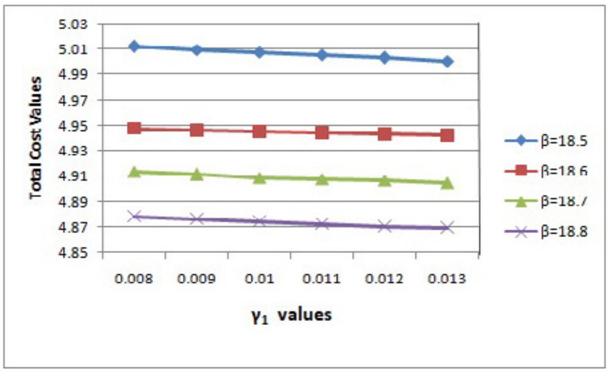

Figure 1 – TC versus γ_1

$\gamma_2 = 0.8, S_1 = 52, s1 = 5, S_2 = 20, s_2 = 3, N_1 = 3, N_2 = 3, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 0.55, c_{r_1} = 0.45, c_{r_2} = 0.5, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.4.$.

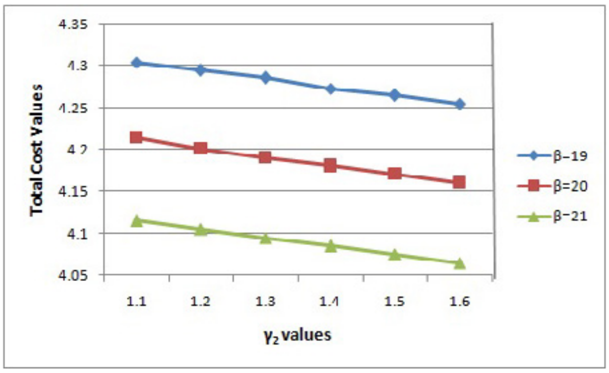

Figure 2 – TC versus γ_2

$\gamma_1 = 0.01, S_1 = 52, s_1 = 5, S_2 = 20, s_2 = 3, N_1 = 3, N_2 = 3, c_{h_1} = 0.01, c_{h_2} = 0.01, c_r = 0.55, c_{r_1} = 0.45, c_{r_2} = 0.5, c_{b_1} = 0.1, c_{b_2} = 0.1, c_{p_1} = 0.1, c_{p_2} = 0.4.$

7 CONCLUSION

In this article, we examined the substitutable perishable inventory system. Specifically, we analyzed the structure of the system performance that takes place when a local purchase is made to clear the backlog instantaneously if both commodities have reached zero and demand is backlogged up to predetermined levels. Arriving customers follow a Markovian arrival process. The commodities are assumed to be substitutable. If both commodities have reached zero, demand is backlogged up to predetermined levels. Graphical results of perishable rates and replenishment rates had been presented. This shows that if the perishable and replenished rate increases then the total cost would increases. The results of the contribution were illustrated using numerical patterns to estimate the convexity of the overall cost rate of this system. The impact of cost values on total expected cost rate were shown. In the future, our proposed model can be expanded by various reordering policies and described by real data values.

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ANALYSIS OF A DISCRETE TIME QUEUEING-INVENTORY MODEL WITH BACK-ORDER OF ITEMS

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ABSTRACT

This paper analyses a discrete-time (s, S) queueing inventory model with service time and back-order in inventory. The arrival of customers is assumed to be the Bernoulli process. Service time follows a geometric distribution. As soon as the inventory level reaches a pre-assigned level due to demands, an order for replenishment is placed. Replenishment time also follows a geometric distribution. When the inventory level reduces to zero due to the service of customers or non-replenishment of items, a maximum of k customers are allowed in the system and the remaining customers are assumed to be completely lost till the replenishment. Matrix-Analytic Method (MAM) is used to analyze the model. Stability conditions, various performance measures of the system, waiting-time distribution and reorder-time distribution are obtained. Numerical experiments are also incorporated.

KEYWORDS

Discrete Time Queueing Inventory, Back-order, Cost Analysis, Matrix Analytic Method

1 INTRODUCTION

The first reported work on discrete-time queue was done by Meisling (1958). In that work, the researcher analyzed a single-server queueing system in which interarrival time and service time are geometrically distributed and as a limiting process, the results in a continuous system are derived. In queueing inventory, stock out generates penalty costs due to the loss and disappointment of customers. This will create perturbed demand and is considered in Schwartz (1966). Gross and Harris (1973) described the progress of an (s, S) inventory system with complete back-ordering and state-dependent lead times. The arrival is assumed to be a Poisson process and the lead time depends on the count of outstanding orders. Archibald (1981) discussed the continuous review (s, S) Policies which minimize the average stationary cost in an inventory system with constant lead time, fixed order cost, linear holding cost per unit time, linear penalty cost per unit short, discrete compound Poisson demand, lost sales and back-ordering. Krishnamoorthy and Islam (2004) analyzed an (s, S) Inventory model where demands form a Poisson process. When the inventory level approaches zero due to service, upcoming arrivals are transferred to a pool of finite capacity. Deepak et al. (2004) considered a queueing system in which work gets postponed due to the finiteness of the buffer. When the buffer of finite capacity is full, further demands are shifted to a pool of customers. A potential customer discovers the full buffer, and will opt for the pool with some probability, or else it will be lost forever.

Manuel (2007) analysed a continuous perishable (s, S) inventory model in which the arrival is under a Markovian arrival process. The expiry of items in the stock and the lead time follow independent exponential distributions. Demands that arrive during a period when the items are out of stock enter either a pool of finite capacity or are lost forever. Sivakumar (2007) studied a continuous perishable inventory model in which the demand is following a Markovian arrival process. The inventoried items have lifetimes that are assumed to follow an exponential distribution. The demands that occur during stock-out periods either enter a pool that has a finite capacity or leaves the system. Any demand that arrives when the pool is full and the inventory level is zero, is also assumed to be lost.

Sivakumar (2009) studied a continuous review perishable (s, S) inventory model in which the arrival is in accordance with a Markovian arrival process. First In First Out discipline is used for the selection of customers from the pool when the inventory level is above a pre-assigned positive value N, which is at most the reorder level. The combined probability distribution of inventory level and the number of demands, the system characteristics and expected total cost are obtained in the steady-state. Sivakumar (2012) also considered a discrete-time inventory system in which demands follow a Markovian arrival process. The replenishment of inventory is according to an (s, S) policy. The lead time follows a phase-type distribution. The demands that take place in stock-out periods either enter a pool or go away from the system with a pre-assigned probability. When the pool has no space and the inventory level is dry, further demands that occur are considered to be lost. For a discussion of discrete-time queueing models, one can refer to Alfa (2002, 2001) and Meisling (1958). The present paper generalizes a work reported in the Ph.D. thesis of Deepthi (2013). The work in this paper is analysed by using the Matrix-Analytic Method discussed in Neuts (1994).

The model in this paper has many applications in real-life situations. For instance, consider an automobile showroom that accepts orders and delivers the vehicles whenever there are vehicles in stock. Here the stock of vehicles can be considered inventory. If the items are exhausted due to service and non-replenishment, orders of at most k are accepted and remaining demands are assumed to be lost.

The rest of the paper is organized as follows. Section 2 provides mathematical modeling and analysis. The stability condition is derived in section 3. Steady-state probability vector and algorithmic analysis are discussed in sections 4 and 5 respectively. Some relevant performance measures are included in section 6. Section 7 analyses the waiting-time distribution of the potential customer. Reorder time distribution is incorporated in section 8. Section 9 illustrates numerical experiments.

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2 Mathematical Modeling and Analysis

The following are the assumptions and notations used in this model.

Assumptions

(i) Inter-arrival times follow a geometric distribution with parameter p

(ii) Service time follows a geometric distribution with parameter q

(iii) Up to k customers are allowed in the system when the inventory level is zero

(iv) Lead time is geometrically distributed with parameter r

Notations

$$N(n) : \text{Number of customers in queue at an epoch } n$$
$$I(n) : \text{Inventory level at the epoch } n$$

Then $\{(N(n), I(n)); n = 0, 1, 2, 3, ..\}$ is a Discrete Time Markov Chain(DTMC) with state space $\{(i,j); i \geq 0, 0 < j \leq S\} \cup \{(i,0) : 0 \leq i \leq k,\}$. Now, the transition probability matrix of the process has the form

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ k-1 \\ k \\ k+1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \dots & k-1 & k & k+1 & k+2 & \dots \end{array} \\ \begin{pmatrix} C_1 & C_0 & 0 & & & & & & & \\ B_2 & B_1 & B_0 & & & & & & & \\ 0 & B_2 & B_1 & B_0 & & & & & & \\ & & & & & & & & & \\ & \vdots & \vdots & \ddots & \ddots & \ddots & & & & \\ & & & & B_2 & B_1 & B_0 & & & \\ & & & & . & B_2 & D_1 & D_0 & & \\ & & & & & & D_2 & A_1 & A_0 & \\ & & & & & & K & A_2 & A_1 & A_0 \\ & & & & & & K & & A_2 & A_1 & A_0 \\ & & & & & & K & & & A_2 & A_1 & A_0 \\ & & & & & & \vdots & & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

where the blocks $C_0, C_1, B_0, B_1, B_2, D_0, D_1, D_2, K, A_0, A_1,$ and A_2 are given by

$$C_0 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{pmatrix} p\bar{r} & & & & & & pr \\ & p\bar{r} & & & & & pr \\ & & \ddots & & & & \\ & & & p\bar{r} & & & pr \\ & & & & p & & \\ & & & & & \ddots & \\ & & & & & & p \end{pmatrix} \end{array}$$

$$C_1 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{pmatrix} \bar{p}\bar{r} & & & & & & \bar{p}r \\ & \bar{p}\bar{r} & & & & & \bar{p}r \\ & & \ddots & & & & \\ & & & \bar{p}\bar{r} & & & \bar{p}r \\ & & & & \bar{p} & & \\ & & & & & \ddots & \\ & & & & & & \bar{p} \end{pmatrix} \end{array}$$

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$$B_0 = \begin{array}{c|ccccccc} & 0 & 1 & \dots & s & s+1 & \dots & S \\ \hline 0 & p\bar{r} & & & & & & pr \\ 1 & & p\bar{q}\bar{r} & & & & & p\bar{q}r \\ \vdots & & & \ddots & & & & \\ s & & & & p\bar{q}\bar{r} & & & p\bar{q}r \\ s+1 & & & & & p\bar{q} & & \\ \vdots & & & & & & \ddots & \\ S & & & & & & & p\bar{q} \end{array}$$

$$B_1 = \begin{array}{c|ccccccc} & 0 & 1 & \dots & s & s+1 & \dots & S \\ \hline 0 & \bar{p}\bar{r} & & & & & & \bar{p}r \\ 1 & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & & & \bar{p}\bar{q}r + pqr \\ \vdots & & & \ddots & & & & \\ s & & & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & \bar{p}\bar{q}r + pqr \\ s+1 & & & & pq & \bar{p}\bar{q} & & \\ \vdots & & & & & & \ddots & \\ S & & & & & & pq & \bar{p}\bar{q} \end{array}$$

$$B_2 = \begin{array}{c|ccccccc} & 0 & 1 & \dots & s & s+1 & \dots & S \\ \hline 0 & 0 & & & & & & \\ 1 & \bar{p}q\bar{r} & & & & & & \bar{p}qr \\ \vdots & & \ddots & & & & & \\ s & & & \bar{p}q\bar{r} & & & & \bar{p}qr \\ s+1 & & & & \bar{p}q & & & \\ \vdots & & & & & \ddots & & \\ S & & & & & & \bar{p}q & 0 \end{array}$$

$$D_0 = \begin{array}{c|ccccccc} & 1 & 2 & \dots & s & s+1 & \dots & S \\ \hline 0 & 0 & & & & & & 0 \\ 1 & p\bar{q}\bar{r} & & & & & & p\bar{q}r \\ \vdots & & & \ddots & & & & \\ s & & & & p\bar{q}\bar{r} & & & p\bar{q}r \\ s+1 & & & & & p\bar{q} & & \\ \vdots & & & & & & \ddots & \\ S & & & & & & & p\bar{q} \end{array}$$

$$D_1 = \begin{array}{c|ccccccc} & 0 & 1 & \dots & s & s+1 & \dots & S \\ \hline 0 & \bar{r} & & & & & & r \\ 1 & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & & & \bar{p}\bar{q}r + pqr \\ \vdots & & & \ddots & & & & \\ s & & & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & \bar{p}\bar{q}r + pqr \\ s+1 & & & & pq & \bar{p}\bar{q} & & \\ \vdots & & & & & & \ddots & \\ S & & & & & & pq & \bar{p}\bar{q} \end{array}$$

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$$D_2 = \begin{array}{c|cccccccc} & 0 & 1 & \dots & s & s+1 & \dots & & S \\ \hline 1 & q\bar{r} & & & & & & & qr \\ 2 & & \bar{p}q\bar{r} & & & & & & \bar{p}qr \\ \vdots & & & \ddots & & & & & \\ s+1 & & & & \bar{p}q\bar{r} & & & & \bar{p}qr \\ s+2 & & & & & \bar{p}q & & & \\ \vdots & & & & & & \ddots & & \\ S & & & & & & & \bar{p}q & 0 \end{array}$$

$$K = \begin{array}{c|ccccccc} & 0 & 1 & \dots & s & s+1 & \dots & S \\ \hline 1 & q\bar{r} & & & & & & qr \\ 2 & & & & & & & \\ \vdots & & & & & & & \\ s & & & & & & & \\ s+1 & & & & & & & \\ \vdots & & & & & & & \\ S & & & & & & & \end{array}$$

$$A_0 = \begin{array}{c|ccccccc} & 1 & 2 & \dots & s & s+1 & \dots & S \\ \hline 1 & p\bar{q}\bar{r} & & & & & & p\bar{q}r \\ 2 & & p\bar{q}\bar{r} & & & & & p\bar{q}r \\ \vdots & & & \ddots & & & & \\ s & & & & p\bar{q}\bar{r} & & & p\bar{q}r \\ s+1 & & & & & p\bar{q} & & \\ \vdots & & & & & & \ddots & \\ S & & & & & & & p\bar{q} \end{array}$$

$$A_1 = \begin{array}{c|ccccccc} & 1 & 2 & \dots & s & s+1 & \dots & S \\ \hline 1 & \bar{p}\bar{q}\bar{r} & & & & & & \bar{p}\bar{q}r \\ 2 & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & & & \bar{p}\bar{q}r + pqr \\ \vdots & & \ddots & \ddots & & & & \vdots \\ s & & & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & \bar{p}\bar{q}r + pqr \\ s+1 & & & & pq & \bar{p}\bar{q} & & \\ \vdots & & & & & \ddots & \ddots & \\ S & & & & & & pq & \bar{p}\bar{q} \end{array}$$

$$A_2 = \begin{array}{c|ccccccc} & 1 & 2 & \dots & s & s+1 & \dots & S \\ \hline 1 & 0 & & & & & & \\ 2 & \bar{p}q\bar{r} & & & & & & \bar{p}qr \\ \vdots & & \ddots & & & & & \vdots \\ s & & & \bar{p}q\bar{r} & & & & \bar{p}qr \\ s+1 & & & & \bar{p}q & & & \\ \vdots & & & & & \ddots & & \\ S & & & & & & \bar{p}q & 0 \end{array}$$

3 STABILITY AND STEADY-STATE ANAYSIS

Theorem 1. *The above Markov chain is stable if and only if* $p < q$

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Proof. Consider the matrix $A = A_0 + A_1 + A_2$. Then

$$A = \begin{array}{c} \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & \dots & s & s+1 & \dots & S \end{array} \\ \begin{pmatrix} q\bar{r} & \bar{q}\bar{r} & & & & r \\ & \ddots & \ddots & & & \\ & & q\bar{r} & \bar{q}\bar{r} & & r \\ & & & q & \bar{q} & \\ & & & & \ddots & \ddots \\ & & & & q & \bar{q} \end{pmatrix} \end{array}$$

Let πbe the steady state probability vector of A.Then the given Markov chain is stable if and only if $\boldsymbol{\pi}(A_1 + 2A_2)\mathbf{e} > 1$,where $\mathbf{e}$ is the column vector of ones of order S . On simplification, we get the condition $p < q$.

STEADY STATE PROBABILITY VECTOR

Let $\mathbf{x} = (x_0, x_1, \dots, x_{k-1}, x_k, \dots)$ be the steady state probability vector of P, where $x_i = \begin{cases} x_{i,j} & 0 \le i \le k, 0 \le j \le S \\ x_{i,j} & i > k, 1 \le j \le S \end{cases}$

Under the stability condition, x_i, for $i \ge k+1$, is given by

$$x_{k+1+r} = x_{k+1}R^r, \ r \ge 0$$

where R is the least non-negative root of the equation

$$R^2A_2 + RA_1 + A_0 = R$$

with a value less than one for spectral radius. The vectors $x_0, x_1, \dots, x_{k+1}$ are given by solving

$$\left.\begin{array}{l} x_0C_1 + x_1B_2 = x_0 \\ x_{J-1}B_0 + x_jB_1 + x_{j+1}B_2 = x_j; (1 \le j \le k-1) \\ x_{k-1}B_0 + x_KD_1 + x_{k+1}[D_2 + RK(I-R)^{-1}] = x_k \\ x_kD_0 + x_{k+1}(A_1 + RA_2) = x_{k+1} \end{array}\right\} \tag{1}$$

subject to the normalizing condition

$$\left[\sum_{i=0}^{k} x_i + x_{k+1}(I-R)^{-1}\right]\mathbf{e} = 1 \tag{2}$$

EVALUATION OF THE TRUNCATION MATRIX R

The rate matrix R is given by $R = \lim_{n\to\infty} R_n$, where $R_{n+1} = \left(R_n^2A_2 + A_0\right)(I - A_1)^{-1}$ and $R_0 = 0$. The iteration is usually stopped when $|\ (R_{n+1} - R_n)\ |_{ij} < \epsilon, \forall$ i,j

COMPUTATION OF BOUNDARY PROBABILITIES

Now the system (1) can be solved using the block Gauss-Seidel iterative method. The vectors $x_0, x_1, \dots, x_{k+1s}$ in the $(n+1)$th iteration are given by

$$\begin{array}{l} x_0(n+1) = x_1(n)B_2(I-C_1)^{-1} \\ x_i(n+1) = [x_{i+1}(n)B_2 + x_{i-1}(n+1)B_0](I-B_1)^{-1}; (1 \le i \le k-1) \\ x_k(n+1) = (x_{k-1}(n+1)B_0 + x_{k+1}(n)[D_2 + RK(I-R)^{-1}])[I-D_1]^{-1} \\ x_{k+1}(n+1) = x_k(n+1)D_0(I - A_1 - RA_2)^{-1} \end{array}$$

Each iteration is subject to the normalizing condition (2).

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4 SYSTEM PERFORMANCE MEASURES

In order to consider some performance measures of the system under steady state, we take $x_{i,0} = 0$ for $i > k$

(i) Expected level of inventory, ELI, is given by

$$ELI = \sum_{j=1}^{S} \sum_{i=0}^{\infty} j x_{i,j}$$

(ii) Expected number of customers, EC, is obtained by

$$EC = \sum_{j=1}^{S} \sum_{i=0}^{\infty} i x_{i,j}$$

(iii) Expected departure after completing the service, EDS is given by

$$EDS = q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{i,j}$$

(iv) The Expected reorder rate, ERR, is given by

$$ERR = q \sum_{i=1}^{\infty} x_{i,s+1}$$

(v) Expected replenishment rate

$$ERR = r \sum_{j=0}^{s} \sum_{i=0}^{\infty} x_{i,j}$$

(vi) Probability that the inventory level zero, PI_0, is given by

$$PI_0 = \sum_{i=0}^{\infty} x_{i,0}$$

(vii) Expected loss rate of customers

$$ELR = p x_{k0} + q\bar{r} \sum_{i=k+1}^{\infty} (i-k) x_{i+1,1}$$

(viii) Expected number of demands waiting in the system during stock out period, EW_0 is given by

$$EW_0 = \sum_{i=1}^{k} i x_{i,0}$$

(ix) Expected reordering quantity, ERQ is given by

$$ERQ = \sum_{j=0}^{s} (S-j) y_j, \text{where } y_j \text{ is the probability that inventory level is j when}$$

replenishment takes place

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4.1 WAITING TIME DISTRIBUTION

Here we assume the queue discipline as First In First Out (FIFO). Let V_n be the number of customers in the queue ahead of the arriving customer. Then $((V_n, I(n))n = 1, 2, \dots)$ is a discrete-time Markov process with state space
$\{(i,j), 0 \leq i < \infty, 0 \leq j \leq S\}$, under the assumption that k is large. The transition probability matrix is given by

$$P(W_q) = \begin{bmatrix} 1 & 0 & & & \\ t & T_1 & 0 & & \\ & T_2 & T_1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \end{bmatrix}, \quad T_1 = \begin{bmatrix} \bar{r} & & & & & & r \\ & \bar{q}\bar{r} & & & & & \bar{q}r \\ & & \ddots & & & & \vdots \\ & & & \bar{q}\bar{r} & & & \bar{q}r \\ & & & & \bar{q} & & \\ & & & & & \ddots & \\ & & & & & & \bar{q} \end{bmatrix}_{(S+1)\times(S+1)},$$

$$T_2 = \begin{bmatrix} 0 & & & & & & 0 \\ \bar{q}\bar{r} & & & & & & \bar{q}r \\ & \ddots & & & & & \vdots \\ & & \bar{q}\bar{r} & & & & \bar{q}r \\ & & & \bar{q} & & & \\ & & & & \ddots & & \\ & & & & & \bar{q} & 0 \end{bmatrix}_{(S+1)\times(S+1)} \quad t = \begin{bmatrix} 0 \\ q \\ \vdots \\ q \end{bmatrix}_{(S+1)\times 1}$$

Suppose there are i customers in the system in front of an arriving customer and j number of items in the inventory is available to the arriving customer. Let T be the waiting time in queue. Then the upper limit for T with a certain probability is calculated as follows.

Consider $z_n = e_{ij}[P(W_q)]^n$, where e_{ij} is infinite row vector whose $(1 + i + j + Si)^{th}$ element is one remaining entries are zeros. Then $P(T \leq n) = z_n(1)$,the first entry of z_n

4.2 REORDER TIME DISTRIBUTION

In this section, we calculate the average time taken to fall inventory from S to s. If the current inventory level is $s + i$, then re-order will takes place only when the service of i customers is completed. Note that the distribution of time taken to reach inventory level to s from S is a discrete phase-type distribution having $(S - s)(S - s + 3)/2$ phases and the transition probability matrix given by

$$P^* = \begin{bmatrix} 1 & 0 \\ t & T \end{bmatrix}$$

$$\text{T}= \begin{array}{c} \\ s+1 \\ s+2 \\ \vdots \\ S-N-1 \\ S-1 \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} s+1 & s+2 & \dots & S-N-1 & S-1 & S \end{array} \\ \begin{pmatrix} H_1^1 & & & & & \\ H_2^2 & H_1^2 & & & & \\ & \ddots & \ddots & & & \\ & & & & & \\ & & & H_2^{S-s-1} & H_1^{S-s-1} & \\ & & & & H_2^{S-s} & H_1^{S-s} \end{pmatrix} \end{array}$$

$$\text{where } H_1^i = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ i-1 \\ i \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & \dots & & i-1 & i \end{array} \\ \begin{pmatrix} \bar{p} & p & & & & \\ & \bar{p}\bar{q} & p\bar{q} & & & \\ & & \ddots & \ddots & & \\ & & & & p\bar{q} & p\bar{q} \\ & & & & & q \end{pmatrix}_{(i+1)\times(i+1)} \end{array}$$

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$$H_2^i = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ i-1 \\ i \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & \dots & & i-1 & \end{array} \\ \begin{pmatrix} 0 & 0 & & & & \\ \bar{p}q & pq & & & & \\ & & \ddots & \ddots & & \\ & & & & \bar{p}q & pq \\ & & & & & q \end{pmatrix}_{(i+1)\times i} \end{array} \quad \text{and} \quad t_{=} \begin{bmatrix} 0 \\ q_1 \\ \vdots \\ 0 \end{bmatrix}_{[(S-s)(S-s+3)/2]\times 1}$$

Therefore expected time to fall the inventory level from S to s due to service is $\alpha(I-T)^{-1}\mathbf{1}$, where $\mathbf{1}$ is the column vector of 1's of order $[(S-s)(S-s+3)/2]$. If α is the $[(S-s)(S-s+3)/2]$ row vector whose $[((S-s)*(S-s+1)+2)/2]^{th}$ entry is 1 and remaining entries are zero, then expected time of reorder is given by

$$ETR = \alpha(I-T)^{-1}\mathbf{1}$$

5 NUMERICAL EXPERIMENTS

5.1 COST FUNCTION

Define the expected total cost of the system per unit time as

$$ETC = C_0 ERO + C_1(ERQ)(ERR) + C_2(ELI) + C_3(EW_0) + C_4(ELC)$$

where,

C_0 : The setup cost/order
C_1 : Procurement cost/unit
C_2 : Holding cost of inventory/unit/unit time
C_3 : Customers holding cost when inventory level is zero/customer/unit time
C_4 : Cost due to loss of customers/unit/unit time

5.2 GRAPHICAL ILLUSTRATIONS

In this paper, we obtained various performance measures. The change in the parameters such as arrival rate, service rate, replenishment rate, number of back-order, etc. may affect these performance measures.

Figures 1, 2, and 3 illustrate the variation of ETC with p, q and r by keeping all other parameters constant as indicated in the figure. The optimum value of ETC is obtained at $p = 0.5665$ in figure 1, $q = 0.885$ in figure 2 and $r = 0.1175$ in figure 3, corresponding optimum values of the expected total cost are 13.5276, 13.4706 and 14.6625 and which are indicated in the figures 1, 2 and 3 respectively.

In figure 4, we analyzed the variation of ETW using the expression for expected waiting time(ETW) that we derived in the above section. By varying q and r, we analysed the variations of ETW, assuming that arriving customers find 10 customers ahead of him and the level of inventory 2. We can see that ETW decreases with the increase of q and r.

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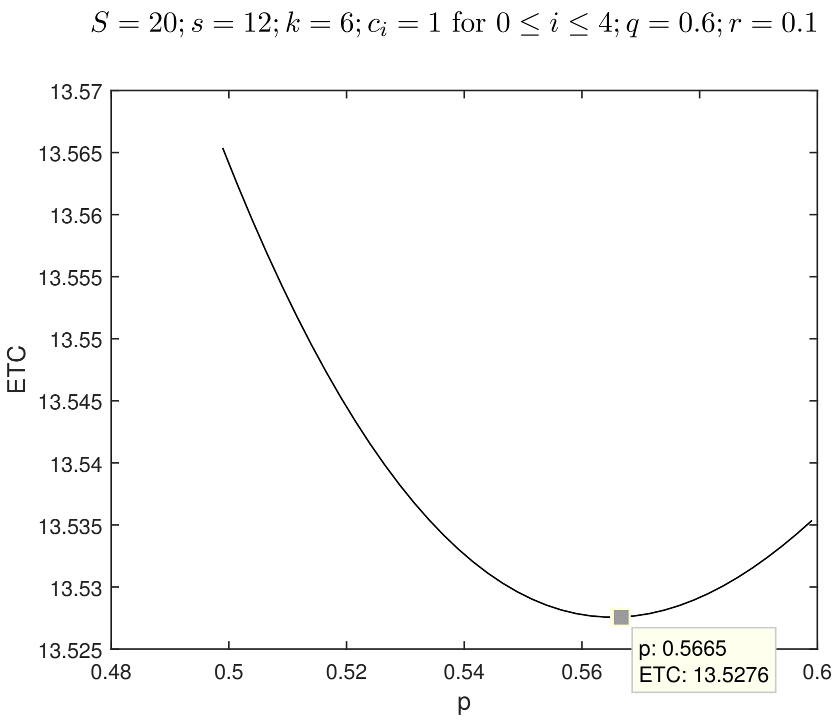

Figure 1 – Variation of expected total cost with p

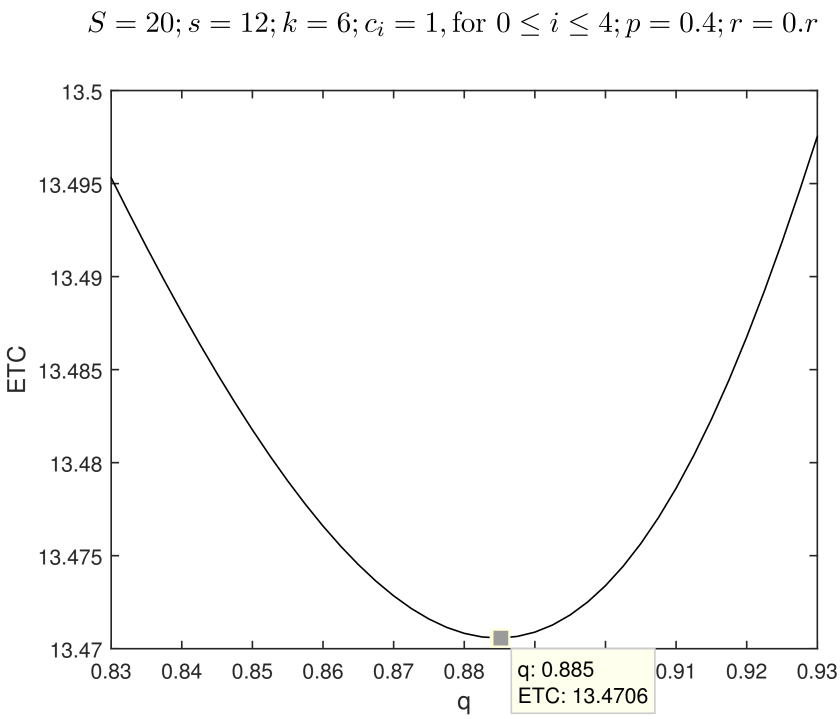

Figure 2 – Variation of expected total cost with q

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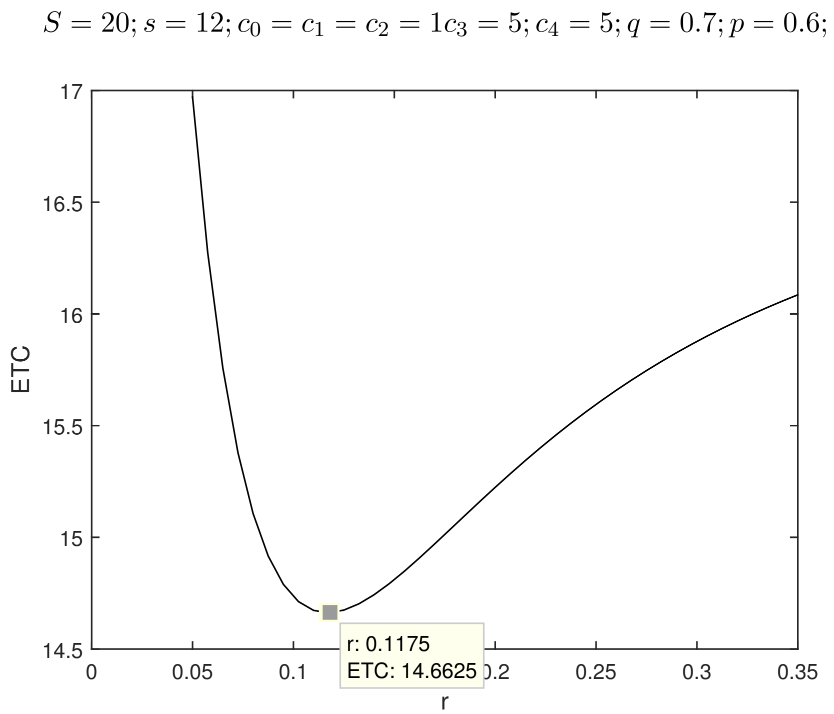

Figure 3 – Variation of expected total cost with r

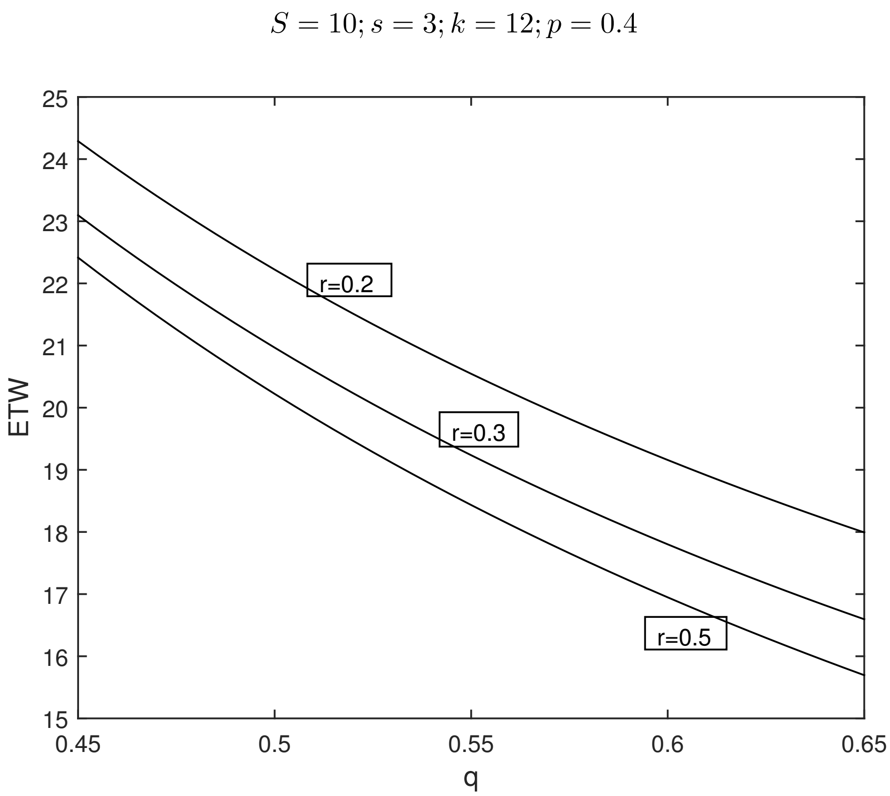

Figure 4 – Variation of expected waiting time with q

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CONCLUSIONS

In this paper, the attempt was to analyze a discrete-time inventory model with service time and back-order in inventory. Stability condition, waiting time distribution and reorder time distribution are analyzed. Numerical experiments are incorporated into the model to highlight the effect of variation in system parameters. The work can be further extended by considering the Discrete Markov Arrival Process(DMAP) and discrete phase type service distribution.

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FIXED POINT THEOREMS IN THE GENERALIZED RATIONAL TYPE OF C-CLASS FUNCTIONS IN B-METRIC SPACES WITH APPLICATION TO INTEGRAL EQUATION

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ABSTRACT

In this paper, we study some results of existence and uniqueness of fixed points for a C-class of mappings satisfying an inequality of rational type in b-metric spaces. After definition of C-class functions covering a large class of contractive conditions by Ansari [2]. Our results extend very recent results in the literature; as well as Khan in [14] and later Fisher in [9] gave a revised improved version of Khan's result and Piri in [17] a new generalization of Khan's Theorem. At the end, we present an example of finding solutions for an integral equation.

KEYWORDS

Metric space; fixed point, C-function.

1 INTRODUCTION

In 1989, Bakhtin [4] introduced b-metric spaces as a generalization of metric spaces. Since then, many articles have been published in the field of fixed point theory and Banach generalization. The contraction principle in such spaces, known as Banach's contraction principle, states that every self-contracting mapping in a complete metric space has a unique fixed point. This principle has been generalized and expanded in several ways.

In this paper, we study certain results of existence and uniqueness of fixed points for a C-class of mappings satisfying an inequality of rational type in b-metric spaces. The C-class functions cover a large class of contractive conditions which applyed by Ansari [2]. Our results develop very recent results in the literature; as well as Khan in [14] and later Fisher in [9] gave a revised improved version of Khan's result and Piri in [17] a new generalization of Khan's Theorem. At the end of the paper, we present examples of finding solutions for integral equations. For more detail and recent papers refer to [3, 7, 8, 11, 16, 18].

Definition 1 ([10]). *Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric on X if the following conditions hold for all $x, y, z \in X$:*

(i) $d(x, y) = 0$ *if and only if* $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$ *(b-triangular inequality).*

Then, the pair (X, d) is called a b-metric space with parameter s.

Example 1 ([15]). *Let (X, d) be a metric space and let $\beta > 1, \lambda \geq 0$ and $\mu > 0$. For $x, y \in X$, set $\rho(x, y) = \lambda d(x, y) + \mu d(x, y)^{\beta}$. Then (X, ρ) is a b-metric space with the parameter $s = 2^{\beta-1}$ and not a metric space on X.*

Definition 2 ([5]). *Let (X, d) be a $b-$metric space. Then a sequence $\{x_n\}$ in X is called:*

(i) $b-$convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A $b-$Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Lemma 1 ([1]). *Let (x, d) be a b-metric space with s. If sequences $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y in X. Then*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

Specially if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$. Further if $z \in X$ we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z).$$

Definition 3 ([6]). *Let Ψ denote all functions $\psi : [0, \infty) \to [0, \infty)$ satisfied:*

(i) ψ *is strictly increasing and continuous,*

(ii) $\psi(t) = 0$ *if and only if* $t = 0$.

We let Ψ denote the class of the altering distance functions.

Definition 4 ([2]). *An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$ for $t > 0$.*

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We let Φ denote the class of the ultra altering distance functions.

Zoran Kadelburg and et al. in [19], and Imdad and Ali [12, 13] defined and studied implicit functions and utilized same to prove several fixed point results for rational type condition. Also, in 2014 Ansari in [2] introduced C-type functions as follows:

Definition 5 ([2]). *A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:*

1. $F(s, t) \leq s$;
2. $F(s, t) = s$ *implies that either* $s = 0$ *or* $t = 0$.

It's clear that $F(0, 0) = 0$. We denote C-class functions by $\mathcal{C}$.

Let

$$\Omega := \{\varphi \in C([0, \infty), [0, \infty)) : \varphi^{-1}(0) = 0\}.$$

Example 2 ([2]). *The following functions $F : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:*

1. $F(s, t) = s - t$.
2. $F(s, t) = \frac{s}{(1+t)^r}$; $r \in (0, \infty)$.
3. $F(s, t) = s - \varphi(s)$ *for* φ *where* $\varphi \in \Omega$

2 RESULTS

We improve the recent results by C-class functions to generalization of contractions in the cases of fractional contraction.

Theorem 1. *Let (X, d) a b-metric space with s and $T : X \to X$ a self mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \begin{cases} g(x, y), & A_{xy} \neq 0; \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

for $x, y \in X$ where $x \neq y$ and

$$g(x, y) := F(\psi(m(x, y)), \varphi(m(x, y)))$$

$$m(x, y) := \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(Tx, y)},$$

$$A_{xy} := \max\{d(x, Ty), d(Tx, y)\}$$

and $F \in \mathcal{C}, \varphi \in \Phi, \psi \in \Psi$. Then T has a unique fixed point.

Proof.

Let $x_0 \in X$. For $n \in \mathbb{N}$ put $x_n := Tx_{n-1}$. It's clear when for some $n \in \mathbb{N}$ we have $x_n = x_{n+1}$. So we assume that

$$x_n \neq x_{n+1} \quad \forall n \in \mathbb{N}.$$

Put

$$A_{n,n-1} = \max\{d(x_n, Tx_{n-1}), d(Tx_n, x_{n-1})\} \neq 0.$$

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by (1) we get

$$\begin{aligned}&\psi(d(x_n, x_{n+1}))\\&= \psi(d(Tx_{n-1}, Tx_n))\\&\leq F(\psi(m(x_{n-1}, x_n)), \varphi(m(x_{n-1}, x_n)))\\&\leq F(\psi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n)))\\&\leq \psi(d(x_{n-1}, x_n)).\end{aligned} \tag{2}$$

Now by increasing ψ and relation (2)

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

Since the $\{d(x_{n-1}, x_n)\}$ in decreasing therefore for some r

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = r. \tag{3}$$

Again by (2)

$$\psi(r) \leq F(\psi(r), \varphi(r)) \leq \psi(r).$$

So $\psi(r) = 0$ or $\varphi(r) = 0$ hence

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = r = 0. \tag{4}$$

It's time to show that $\{x_n\}$ is a Cauchy sequence. If not, then for $\varepsilon > 0$ there exit $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ in $\{x_n\}$ and $m(k) > n(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, d(x_{m(k)}, x_{n(k)-1}) < \varepsilon$$

and

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \lim_{n\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \varepsilon. \tag{5}$$

$$\exists N \in \mathbb{N}\ \forall k \geq N \quad \max\{d(x_{m(k)}, x_{n(k)+1}), d(x_{m(k)+1}, x_{n(k)}) \geq \frac{\varepsilon}{2} > 0.$$

By take a look at (1), for each $k \geq N$

$$\begin{aligned}\psi(d(x_{m(k)+1}, x_{n(k)+1})) &= \psi(d(Tx_{m(k)}, Tx_{n(k)}))\\&\leq F(\psi(m(x_{m(k)}, x_{n(k)})), \varphi(m(x_{m(k)}, x_{n(k)})))\\&\leq \psi(d(x_{n-1}, x_n))\end{aligned} \tag{6}$$

even by the relations (3) and (5)

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)) \leq \psi(\varepsilon),$$

so $\varepsilon = 0$ which is not. Thus $\{x_n\}$ is a Cauchy sequence in complete b-metric space. Therefore it is convergent to some $x^* \in X$.

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{7}$$

If $x_{n+1} = Tx^*$ for some n, then

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx^* = Tx^*.$$

So

$$\exists N \in \mathbb{N}\ \forall n \geq N \quad d(x_n, Tx^*) > 0,$$

and also

$$\max\{d(x_n, Tx^*), d(x^*, Tx_n) > 0.$$

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From (1)

$$\begin{aligned}\psi(d(x_{n+1},Tx^*)) &= \psi(d(Tx_n,Tx^*))\\ &\leq F(\psi(m(x_n,x^*)).\varphi(m(x_n,x^*)))\\ &\leq \psi(d(x_{n-1},x_n))\end{aligned} \tag{8}$$

where

$$m(x_n,x^*) = \frac{d(x_n,Tx_n)d(x_n,Tx^*)+d(x^*,Tx^*)d(x^*,Tx_n)}{d(x_n,Tx^*)+d(Tx_n,x^*)}$$

and or

$$m(x_n,x^*) = \frac{d(x_n,x_{n+1})d(x_n,Tx^*)+d(x^*,Tx^*)d(x^*,x_{n+1})}{d(x_n,Tx^*)+d(x_{n+1},x^*)}$$

by taking lim inf and lim sup from (8) and relation (4) and (7) we have:

$$\limsup_{n\to\infty} m(x_n,x^*) \leq 0\times sd(x^*,Tx^*)+sd(x^*,Tx^*)\times 0 = 0$$

thus

$$\psi(\frac{1}{s}d(x^*,Tx^*)) \leq F(\psi(0),\varphi(0)) \leq \psi(0)=0$$

and consequently $x^* = Tx^*$.
For an other condition may be to happen: for some $n\in\mathbb{N}$

$$A_{n,n-1} = \max\{d(x_n,Tx_{n-1}),d(Tx_n,x_{n-1})\}=0.$$

So (1) thus $d(Tx_{n-1},Tx_n)=0$ $Tx_{n-1}=Tx_n$ means that $x_n=Tx_n$.
For uniqueness, let x^* and y^* be two fixed point of T such that $d(x^*,y^*)>0$. By (1)

$$\begin{aligned}\psi(d(x^*,y^*)) &= \psi(d(Tx^*,Ty^*))\\ &\leq F(\psi(m(x^*,y^*)),\varphi(m(x^*,y^*)))\\ &\leq F(\psi(0),\varphi(0)) \leq \psi(0)=0,\end{aligned}$$

it should be $d(x^*,y^*)=0$.

For the next result, it's enough $F(s,t)=\frac{s}{2+t}$.

Corollary 1. *Let (X,d) a b-metric space with s and $T: X\to X$ a selfmapping satisfying*

$$\psi(d(Tx,Ty)) \leq \begin{cases} g(x,y), & A_{xy}\neq 0;\\ 0, & otherwise\end{cases}$$

for $x,y\in X$ where $x\neq y$ and

$$g(x,y) := \frac{\psi(m(x,y))}{+\varphi(m(x,y))}$$

$$m(x,y) := \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)},$$

$$A_{xy} := \max\{d(x,Ty),d(Tx,y)\}$$

and $\varphi\in\Phi,\psi\in\Psi$. Then T has a unique fixed point.

And if we put $F(s,t)=s-t$:

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Corollary 2. *Let* (X,d) *a b-metric space with* s *and* $T:X\to X$ *a selfmapping satisfying*

$$\psi(d(Tx,Ty)) \leq \begin{cases} g(x,y), & A_{xy}\neq 0\ ; \\ 0, & otherwise \end{cases}$$

for $x,y\in X$ *where* $x\neq y$ *and*

$$g(x,y) := \psi(m(x,y)) - \varphi(m(x,y))$$

$$m(x,y) := \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)},$$

$$A_{xy} := \max\{d(x,Ty),d(Tx,y)\}$$

and $\varphi\in\Phi,\psi\in\Psi$. *Then* T *has a unique fixed point.*

And also we put $F(s,t)=ks$, $\psi(t)=t$ and $s=1$, hence:

Corollary 3 ([14]). *Let* (X,d) *a b-metric space with* s *and* $T:X\to X$ *a selfmapping satisfying*

$$\psi(d(Tx,Ty)) \leq \begin{cases} g(x,y), & A_{xy}\neq 0; \\ 0, & otherwise \end{cases} \tag{9}$$

for $x,y\in X$ *where* $x\neq y$ *and*

$$g(x,y) := km(x,y)$$

$$m(x,y) := \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)},$$

$$A_{xy} := \max\{d(x,Ty),d(Tx,y)\}.$$

Then T *has a unique fixed point.*

Our results improve the following Corollary in [17] which it is extension of the Khan's Theorem with $s=1$.

Corollary 4 ([17]). *Let* (X,d) *a b-metric space with* s *and* $T:X\to X$ *a self mapping satisfying*

$$\psi(d(Tx,Ty)) \leq \begin{cases} g(x,y), & A_{xy}\neq 0; \\ 0, & otherwise \end{cases}$$

for $x,y\in X$ *where* $x\neq y$ *and*

$$g(x,y) := m(x,y) - \varphi(m(x,y))$$

$$m(x,y) := \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)},$$

$$A_{xy} := \max\{d(x,Ty),d(Tx,y)\}.$$

where $\varphi\in\Phi$. *Then* T *has a unique fixed point.*

3 Application to integral equation

In this section, we looking for the solutions of the following integral equation:

$$u(t) = g(t,u(t)) + \int_0^t G(t,s,u(s))ds, \quad t\in[0,\infty), \tag{10}$$

where $G:[0,\infty)\times[0,\infty)\times\mathbb{R}\to\mathbb{R}$ $g:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ are continuous.

Let X be complete space $BC([0,\infty))$ containing of bounded, continuous and real valued functions on $[0,\infty)$ such that

$$\|u\| = \sup\{|u(t)| : t\in[0,\infty)\}.$$

 https://doi.org/10.17993/3cemp.2022.110250.64-74

Example 3. *We show that the following integral equation:*

$$(Tu)(t) = g(t, u(t)) + \int_0^t G(t, s, u(s))ds, \quad t \in [0, \infty), \tag{11}$$

has unique solution, if we have the under following conditions.

Let $d(u, v) = \|u - v\|^2$. So d is a b-metric with $s = 2$.

1. $\psi(t) = \varphi(t) = t$
2. $F(s, t) = s$
3. $\|u - v\| \leq \min\{\|u - Tu\|, \|v - Tv\|\}$
4. $|g(t, u(t)) - g(t, v(t))| \leq \frac{|u(t)-v(t)|}{a} \quad t \in [0, \infty)$
5. for $t \in [0, \infty)$
$$|\int_0^t \big(G(t, s, u(s)) - G(t, s, v(s))\big)ds| \leq \frac{\|u - v\|}{b}$$
6. $a, b > 0$, $c > 2$, $c = \min\{a, b\}$ and $k = \frac{4}{c^2}$.

At first we observe that

$$\begin{aligned} \|u - v\|^2(\|u - Tv\|^2 + \|Tu - v\|^2) &= \|u - v\|^2\|u - Tv\|^2 + \|u - v\|^2\|Tu - v\|^2 \\ &\leq \|u - Tu\|^2\|u - Tv\|^2 + \|Tv - v\|^2\|Tu - v\|^2, \end{aligned}$$

so

$$d(u, v) = \|u - v\|^2 \leq \frac{\|u - Tu\|^2\|u - Tv\|^2 + \|Tv - v\|^2\|Tu - v\|^2}{\|u - Tv\|^2 + \|Tu - v\|^2} = m(u, v).$$

And by $(a + b)^2 \leq 2(a^2 + b^2)$

$$|(Tu)(t) - (Tv)(t)| \leq |g(t, u(t)) - g(t, v(t))| + \int_0^t |G(t, s, u(s)) - G(t, s, v(s))|ds$$

$$\begin{aligned} |(Tu)(t) - (Tv)(t)|^2 &\leq (|g(t, u(t)) - g(t, v(t))| + \int_0^t |G(t, s, u(s)) - G(t, s, v(s))|ds)^2 \\ &\leq 2(|g(t, u(t)) - g(t, v(t))|^2 + (\int_0^t |G(t, s, u(s)) - G(t, s, v(s))|ds)^2) \\ &\leq 2\left(\frac{|u(t) - v(t)|}{a})^2 + (\frac{\|u - v\|}{b})^2\right) \leq 4\left(\frac{\|u - v\|}{c}\right)^2. \end{aligned}$$

So

$$\begin{aligned} d(Tu, Tv) &= \|Tu - Tv\|^2 \\ &\leq 4\left(\frac{\|u - v\|}{c}\right)^2 = kd(u, v) \leq km(u, v) \\ &\leq F(\psi(m(u, v)), \varphi(m(u, v))). \end{aligned}$$

Therefore integral equation (11) by the Theorem 1 has a unique answer.

Example 4. *Let*

$$Tu(t) = \frac{u(t)}{2} + \int_0^t e^{-(t-s)}\frac{u(s)}{3}ds. \tag{12}$$

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So

$$Tu(t) = \frac{u(t)}{2} + e^{-t} * \frac{u(t)}{3} = u * \left(\frac{\delta(t)}{2} + \frac{e^{-t}}{3}\right), \quad g(t, u(t)) = \frac{u(t)}{2},$$

$$G(t, s, u(s)) = e^{-(t-s)}\frac{u(s)}{3}, \quad a = 2, \quad b = 3,$$

where $*$ *is convolution of* u *and* v*; i.e.,*

$$u(t) * v(t) = \int_0^t u(t-s)v(s)ds.$$

We see that

$$u(t) - Tu(t) = u(t) * \left(\frac{\delta(t)}{2} - \frac{e^{-t}}{3}\right), \quad v(t) - Tv(t) = v(t) * \left(\frac{\delta(t)}{2} - \frac{e^{-t}}{3}\right)$$

and

$$\begin{aligned} \left|\int_0^t \big(G(t,s,u(s)) - G(t,s,v(s))\big)ds\right| &= \left|\int_0^t \left(e^{-(t-s)}\frac{u(s)}{3} - e^{-(t-s)}\frac{v(s)}{3}\right)ds\right| \\ &\leq \int_0^t e^{-(t-s)}\frac{\|u-v\|}{3}ds \\ &\leq \frac{\|u-v\|}{3}(1-e^{-t}) \\ &\leq \frac{\|u-v\|}{3} \end{aligned}$$

where $\delta(t)$ *is Dirichlet function with Laplace transformation* $L(\delta) = 1$.

$$\begin{aligned} |Tu(t) - Tv(t)| &= \left|u(t) * \left(\frac{\delta(t)}{2} + \frac{e^{-t}}{3}\right) - v(t) * \left(\frac{\delta(t)}{2} + \frac{e^{-t}}{3}\right)\right| \\ &= \left|(u(t) - v(t)) * \left(\frac{\delta(t)}{2} + \frac{e^{-t}}{3}\right)\right| \\ &\leq \frac{5}{6}|u(t) - v(t)|, \end{aligned}$$

without of loss of generality, let $|u(t)| \leq |v(t)|$ *for all* $t \in [0, \infty)$.

$$\left|u(t) * \left(\frac{\delta(t)}{2} - \frac{e^{-t}}{3}\right)\right| \leq \left|v(t) * \left(\frac{\delta(t)}{2} - \frac{e^{-t}}{3}\right)\right|$$

therefore $|Tu(t) - u(t)| \leq |Tv(t) - v(t)|$.

$$\begin{aligned} |u(t) - v(t)| &\leq |u(t) - Tu(t)| + |Tu(t) - Tv(t)| + |Tv(t) - v(t)| \\ &\leq |u(t) - Tu(t)| + \frac{5}{6}|u(t) - v(t)| + |Tv(t) - v(t)| \end{aligned}$$

$$\frac{1}{6}|u(t) - v(t)| \leq |u(t) - Tu(t)| + |Tv(t) - v(t)|$$

so for some positive number l *we have*

$$\|u - v\| \leq 4l \min\{\|u - Tu\|, \|v - Tv\|\}.$$

All conditions of Example (3) *with* $F(r, s) = 4ls$ *hold, and integral equation* (12) *has a unique solution* $u = 0$.

Availability of data and material

Not applicable.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have read and approved the final manuscript.

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CONTINUOUS-TIME ZERO-SUM GAMES FOR MARKOV DECISION PROCESSES WITH RISK-SENSITIVE FINITE-HORIZON COST CRITERION ON A GENERAL STATE SPACE

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ABSTRACT

In this manuscript, we study continuous-time risk-sensitive finite-horizon time-homogeneous zero-sum dynamic games for controlled Markov decision processes (MDP) on a Borel space. Here, the transition and payoff functions are extended real-valued functions. We prove the existence of the game's value and the uniqueness of the solution of Shapley equation under some reasonable assumptions. Moreover, all possible saddle-point equilibria are completely characterized in the class of all admissible feedback multi-strategies. We also provide an example to support our assumptions.

KEYWORDS

Zero-sum stochastic game, Borel state space, risk-sensitive utility, finite-horizon cost criterion, optimality equation, saddle-point

1 INTRODUCTION

In the literature of game theory, there are two types of game models: a zero-sum model and a nonzero-sum game model. We know that in the zero-sum two-person game, one player tries to maximize his/her payoff and another tries to minimize his/her payoff, whereas in the nonzero-sum game, both players try to minimize their payoff. We can study game theory either in discrete-time or in continuous-time. In continuous time, the players observe the state space continuously, whereas in discrete-time, they observe the state space in discrete-time. Also, there are two types of game models with respect to the risk measure; one is a risk-neutral game, and another is a risk-sensitive game. Risk-sensitive, or "exponential of integral utility" cost criterion is popular, particularly in finance (see, e.g., Bielecki and Pliska (1999)) since it has the property to capture the effects of more than first order (expectation) moments of the cost.

There are large number of literatures for the risk-neutral utility cost criterion for continuous-time controlled Markov decision processes (CTCMDPs) with different setup, see Guo (2007), Guo and Hernandez-Lerma (2009), Guo et al. (2015), Guo et al. (2012), Guo and Piunovskiy (2011), Huang (2018), Piunovskiy and Zhang (2011), Piunovskiy and Zhang (2014) for single controller model and Guo and Hernandez-Lerma (2003), Guo and Hernandez-Lerma (2005), Guo and Hernandez-Lerma (2007), wei and Chen (2016), Zhang and Guo (2012) for game models. Players want to ignore risk in risk-neutral stochastic games because of the additive feature of this criterion. If the variance is high, the risk-neutral criterion is not useful since there can be issues with optimal control. Regarding risk preferences, different controllers may exhibit various perspectives. So, risk preferences are considered by the decision-makers to be the performance criterion. Bell (1995) gave a model containing the interpretation of risk-sensitive utility. This paper considers finite-horizon risk-sensitive two-person zero-sum dynamic games for controlled CTMDPs with unbounded rates (transition and payoff rates) under admissible feedback strategies. State and action spaces are considered to be Borel spaces. The main target of this manuscript is to find the solution of the optimality equation (6) (Shapley equation), to provide the proof of the existence of game's value, and to give a proof of complete characterization of saddle-point equilibrium.

The finite-horizon optimality criterion generally comes up in real-life scenarios. where the cost criterion may not be risk-neutral. For finite-horizon risk-neutral CTMDPs, see Guo et al. (2015), Huang (2018) while for the corresponding game, see Wei and Chen (2016) and its references. In this context, for risk-sensitive finite-horizon controlled CTMDP, one can see Ghosh and Saha (2014), Guo et al. (2019), Wei (2016), while the research for infinite-horizon risk-sensitive CTMDP are available in, Ghosh and Saha (2014), Golui and Pal (2022), Guo and Zhang (2018), Kumar and Pal (2013), Kumar and Pal (2015), Zhang (2017) and the references therein. At the same time the corresponding finite/infinite-horizon dynamic games are studied in Ghosh et at. (2022), Golui and Pal (2021a), Golui and Pal (2021b), Golui et al. (2022), wei (2019). Study on CTMDPs for risk-sensitive control on a denumerable state space are available greatly, see Ghosh and saha (2014), Guo and Liao (2019), Guo et al. (2019) but some times we see the countable state space dose not help to study some models specially in chemical reactions problem, water reservoir management problem, inventory problem, cash-flow problem, insurance problem etc. We see that the literature in controlled CTMDPs considering on general state space is very narrow. Some exceptions for single controller are Golui and Pal (2022), Guo et al. (2012), Guo and Zhang (2019), Pal and Pradhan (2019), Piunovskiy and Zhang (2014), Piunovskiy and Zhang (2020) and for corresponding stochastic games are Bauerle and Rieder (2017), Golui and Pal (2021b), Guo and Hernandez-Lerma (2007), Wei (2017). So, it is very interesting and very important to consider the game problem in some general state space. In Guo and Zhang (2019), the authors studied the same as in Guo et al. (2019) but on general state space, whereas in Wei (2017), the finite-horizon risk-sensitive zero-sum game for a controlled Markov jump process with bounded costs and unbounded transition rates was studied. Where in Ghosh et al. (2016), the authors studied dynamic games on the infinite-horizon for controlled CTMDP by considering bounded transition and payoff rates. However this boundedness condition is a restrictive conditon for many real life scenarios. Someone may note queuing and population processes for the requirement of unboundedness in transition and payoff functions. In Golui and Pal (2021a), finite-horizon continuous-time risk-sensitive zero-sum games for

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unbounded transition and payoff function on countable state space is considered. But the extension of the same results to a general Borel state space were unknown to us. We solve this problem in this paper. Here we are dealing with finite-horizon risk-sensitive dynamic games employing the unbounded payoff and transition rates in the class of all admissible feedback strategies on some general Borel state space, whose results were unknown until now. In this paper, we try to find the solution to the risk-sensitive finite-horizon optimality equation and, at the same time, try to obtain the existence of an optimal equilibrium point for this jump process. We take homogeneous game model. In Theorem 4, we prove our final results, i.e., we show that if the cost rates are real-valued functions, then the Shapley equation (6), has a solution. The existence of optimal-point equilibria is proved by using the measurable selection theorem in Nowak (1985). The claim of uniqueness of the solution is due to the well known Feynman-Kac formula. The value of the game has also been established.

The remaining portions of this work are presented. Section 2 describes the model of our stochastic game, some definitions, and the finite-horizon cost criterion. In Section 3, preliminary results, conditions, and the extension of the Feynman-Kac formula are provided. Also, we establish the probabilistic representation of the solution of the finite horizon optimality equation (6) there. The uniqueness of this optimal solution as well as the game's value are proved in section 4. We also completely characterize the Nash equilibrium among the class of admissible Markov strategies for this game model here. In Section 5, we verify our results with an example.

2 THE ZERO-SUM DYNAMIC GAME MODEL

First, we introduce a time-homogeneous continuous-time zero-sum dynamic game model in this section, which contains the following:

$$\mathcal{G} := \{\boldsymbol{X}, U, V, (U(x) \subset U, x \in \boldsymbol{X}), (V(x) \subset V, x \in \boldsymbol{X}), q(\cdot|x,u,v), c(x,u,v), g(x)\}. \quad (1)$$

Here $\boldsymbol{X}$ is our state space which is a Borel space and the corresponding Borel σ-algebra is $\mathcal{B}(\boldsymbol{X})$. The action spaces are U and V for first and second players, respectively, and are considered to be Borel spaces. Their corresponding Borel σ-algebras are, respectively, $\mathcal{B}(U)$ and $\mathcal{B}(V)$. For each $x \in \boldsymbol{X}$, the admissible action spaces are denoted by $U(x) \in \mathcal{B}(U)$ and $V(x) \in \mathcal{B}(V)$, respectively and these spaces are assumed to be compact. Now let us define a Borel subset of $\boldsymbol{X} \times U \times V$ denoted by $\mathcal{K} := \{(x,u,v)|x \in \boldsymbol{X}, u \in U(x), v \in V(x)\}$.

Next, for any $(x,u,v) \in \mathcal{K}$, we know that the transition rate of the CTMDPs denoted by $q(\cdot|x,u,v)$ is a signed kernel on $\boldsymbol{X}$ such that $q(D|x,u,v) \geq 0$ where $(x,u,v) \in \mathcal{K}$ and $x \notin D$. Also, $q(\cdot|x,u,v)$ is assumed to be conservative i.e., $q(\boldsymbol{X}|x,u,v) \equiv 0$, as well as stable i.e.,

$$q^*(x) := \sup_{u \in U(x), v \in V(x)} [q_x(u,v)] < \infty \quad \forall x \in \boldsymbol{X}, \quad (2)$$

$q_x(u,v) := -q(\{x\}|x,u,v) \geq 0$ for all $(x,u,v) \in \mathcal{K}$. Our running cost is c, assumed to be measurable on $\mathcal{K}$ and the terminal cost is g, assumed to be measurable on $\boldsymbol{X}$. These costs are taken to be real-valued.

The dynamic game is played as following. The players take actions continuously. At time moment $t \geq 0$, if the system's state is $x \in S$, the players take their own actions $u_t \in U(x)$ and $v_t \in V(x)$ independently as their corresponding strategies. As a results the following events occurs:

- the first player gets an reward at rate $c(x,u_t,v_t)$ immediately and second player gives a cost at a rate $c(x,u_t,v_t)$; and
- staying for a random time in state x, the system leaves the state x at a rate given by the quantity $q_x(u_t,v_t)$, and it jumps to a set D, $(x \notin D)$ with some probability determined by $\dfrac{q(D|x,u_t,v_t)}{q_x(u_t,v_t)}$ (for details, see Proposition B.8 in Guo and Hernandez-Lerma (2009), p. 205 for details).

Now suppose the system is at a new state y. Then the above operation is replicated till the fixed time $\hat{T} > 0$. Moreover, at time $\hat{T}$ if the system occupies a state $y_{\hat{T}}$, second player pays a terminal cost $g(y_{\hat{T}})$ to first player.

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Consequently, first player always tries to maximize his/her payoff, whereas second player wants to minimize his/her payoff according to some cost measurement criterion $\mathcal{H}^{\cdot,\cdot}(\cdot,\cdot)$, that is presented below by equation (4). Next the construction of the CTMDPs will be presented under possibly pair of admissible feedback strategies. For construction of the corresponding CTMDPs (as in Kitaev (1986), Piunovskiy and Zhang (2011)), we imopse some usefull notations: define $\boldsymbol{X}(\Delta) := \boldsymbol{X} \cup \{\Delta\}$ (for some $\Delta \notin \boldsymbol{X}$),
$\Omega^0 := (\boldsymbol{X} \times (0,\infty))^\infty$, $\Omega := \Omega^0 \cup \{(x_0,\theta_1,x_1,\cdots,\theta_{\hat{k}},x_{\hat{k}},\infty,\Delta,\infty,\Delta,\cdots)|x_0 \in \boldsymbol{X}, x_l \in \boldsymbol{X}, \theta_l \in (0,\infty)$, for each $1 \le l \le \hat{k}, \hat{k} \ge 1\}$, and suppose $\mathcal{F}$ be the corresponding Borel σ-algebra on Ω. Then we get a Borel measurable space $(\Omega,\mathcal{F})$. For each $\hat{k} \ge 0$, $\omega := (x_0,\theta_1,x_1,\cdots,\theta_{\hat{k}},x_{\hat{k}},\cdots) \in \Omega$, let us define $T_0(\omega) := 0$, $T_{\hat{k}}(\omega) - T_{\hat{k}-1}(\omega) := \theta_{\hat{k}}$, $T_\infty(\omega) := \lim_{\hat{k}\to\infty} T_{\hat{k}}(\omega)$. Now in view of the definition of $\{T_{\hat{k}}\}$, we define the state process $\{\xi_t\}_{t\ge0}$ defined by

$$\xi_t(\omega) := \sum_{\hat{k}\ge0} I_{\{T_{\hat{k}} \le t < T_{\hat{k}+1}\}} x_{\hat{k}} + I_{\{t \ge T_\infty\}}\Delta,\ t \ge 0. \tag{3}$$

Here I_E is the standard notation for indicator function corresponding to a set E, and we use $0+z =: z$ and $0z =: 0$ for any $z \in \boldsymbol{X}(\Delta)$ as convention. The process after the time T_∞ is treated for absorbtion in the state Δ. Hence, let us define $q(\cdot|\Delta,u_\Delta,v_\Delta) :\equiv 0$, $U(\Delta) := U \cup U_\Delta$, $V(\Delta) := V \cup V_\Delta$, $U_\Delta := \{u_\Delta\}$, $V_\Delta := \{v_\Delta\}$, $c(\Delta,u,v) :\equiv 0$ for all $(u,v) \in U(\Delta) \times V(\Delta)$, u_Δ, v_Δ are treated as isolated points. Furthermore, define $\mathcal{F}_t := \sigma(\{T_{\hat{k}} \le s, \xi_{T_{\hat{k}}} \in D\} : D \in \mathcal{B}(\boldsymbol{X}), 0 \le s \le t, \hat{k} \ge 0)$ $\forall t \in \mathbb{R}_+$, and $\mathcal{F}_{s-} =: \bigvee_{0\le t<s} \mathcal{F}_t$. Lastly the σ-algebra of predictable sets on $\Omega \times [0,\infty)$ corresponding to $\{\mathcal{F}_t\}_{t\ge0}$ is denoted by $\mathcal{P} := \sigma(\{U \times \{0\}, U \in \mathcal{F}_0\} \cup \{V \times (s,\infty), V \in \mathcal{F}_{s-}\})$. Now we intoduce strategies of players to define the risk sensitive cost criterion:

Definition 1. *An admissible feedback strategy for player 1, denoted by $\zeta^1 = \{\zeta^1_t\}_{t\ge0}$, is defined to be a transition probability $\zeta^1(du|\omega,t)$ from $(\Omega \times [0,\infty),\mathcal{P})$ onto $(U_\Delta,\mathcal{B}(U_\Delta))$, for which $\zeta^1(U(\xi_{t-}(\omega))|\omega,t) = 1$.*

For more informations, one can see [Guo and Song (2011), Definition 2.1, Remark 2.2], Piunovskiy and Zhang (2011), Zhang (2017).

Let Π^1_{Ad} denote the set of all admissible feedback strategies for player 1. A strategy $\zeta^1 \in \Pi^1_{Ad}$ for player 1, is said to be Markov if for every $\omega \in \Omega$ and $t \ge 0$ the relation $\zeta^1(du|\omega,t) = \zeta^1(du|\xi_{t-}(\omega),t)$ holds, $\lim_{s\uparrow t}\xi_s(\omega) := \xi_{t-}(w)$. We call a Markov strategy $\{\zeta^1_t\}$ as a stationary Markov for player 1, if it not explicitly dependent on time t. The family of all Markov strategies and all stationary strategies are denoted by Π^1_M and Π^1_{SM}, respectively, for first player. The sets Π^2_{Ad}, Π^2_M, Π^2_{SM} stand for all admissible feedback strategies, all Markov strategies, and all stationary strategies, respectively, for second player are defined similarly. In view of Assumption 1, below, for any initial distribution γ on $\boldsymbol{X}$ and any multi-strategy $(\zeta^1,\zeta^2) \in \Pi^1_{Ad} \times \Pi^2_{Ad}$, in view of Theorem 4.27 in Kitaev and Rykov (1985) a unique probability measure exists and denoted by $P^{\zeta^1,\zeta^2}_\gamma$ (depending on γ and (ζ^1,ζ^2)) on $(\Omega,\mathcal{F})$ for which $P^{\zeta^1,\zeta^2}_\gamma(\xi_0 = x) = 1$. Let us define the corresponding expectation operator as $E^{\zeta^1,\zeta^2}_\gamma$. Particularly, when γ represents the Dirac measure at a state $x \in \boldsymbol{X}$, $P^{\zeta^1,\zeta^2}_\gamma$ and $E^{\zeta^1,\zeta^2}_\gamma$ will be written as P^{ζ^1,ζ^2}_x and E^{ζ^1,ζ^2}_x, respectively. For any compact metric space Y, the space of probability measures on Y is denoted by $\mathcal{P}(Y)$ with Prohorov topology. As $U(x)$ and $V(x)$ are compact sets for each $x \in \boldsymbol{X}$, $\mathcal{P}(U(x))$ and $\mathcal{P}(V(x))$ are also compact and convex metric spaces. Now for each fixed $x \in \boldsymbol{X}$, $\vartheta \in \mathcal{P}(U(x))$ and $\eta \in \mathcal{P}(V(x))$, the corresponding transition and payoff rates are defined, as below:

$$q(D|x,\vartheta,\eta) := \int_{V(x)}\int_{U(x)} q(D|x,u,v)\vartheta(du)\eta(dv),\ D \subseteq \boldsymbol{X}.$$

$$c(x,\vartheta,\eta) := \int_{V(x)}\int_{U(x)} c(x,u,v)\vartheta(du)\eta(dv),$$

Note that $\zeta^1 \in \Pi^1_{SM}$ can be identified by a mapping $\zeta^1 : \boldsymbol{X} \to \mathcal{P}(U)$ for which $\zeta^1(\cdot|x) \in \mathcal{P}(U(x))$ for each $x \in \boldsymbol{X}$. So, we can write $\Pi^1_{SM} = \Pi_{x\in S}\mathcal{P}(U(x))$ and $\Pi^2_{SM} = \Pi_{x\in\boldsymbol{X}}\mathcal{P}(V(x))$. So, the sets Π^1_{SM} and Π^2_{SM} are compact metric spaces by using Tychonoff theorem.

 https://doi.org/10.17993/3cemp.2022.110250.76-92

Next take $\lambda \in (0,1]$ as a fixed risk-sensitivity coefficient and fix a finite time horizon $\hat{T} > 0$. Then for each $x \in \boldsymbol{X}$, $t \in [0,\hat{T}]$ and $(\zeta^1,\zeta^2) \in \Pi^1_{Ad} \times \Pi^2_{Ad}$, define the risk-sensitive finite-horizon ($\hat{T}$-horizon) cost criterion as

$$\mathcal{H}^{\zeta^1,\zeta^2}(0,x) := E_x^{\zeta^1,\zeta^2}\left[e^{\lambda \int_0^{\hat{T}} \int_V \int_U c(\xi_t,u,v)\zeta^1(da|\omega,t)\zeta^2(dv|\omega,t)dt+\lambda g(\xi_{\hat{T}})}\right], \tag{4}$$

whence it is given that the integral is well defined. For each $(\zeta^1,\zeta^2) \in \Pi^1_M \times \Pi^2_M$, we know that $\{\xi_t, \geq 0\}$ is a controlled Markov Process on $(\Omega, \mathcal{F}, P_\gamma^{\zeta^1,\zeta^2})$, and hence for any γ (initial distribution on $\boldsymbol{X}$), for each $x \in \boldsymbol{X}$, $t \in [0,\hat{T}]$,

$$\mathcal{H}^{\zeta^1,\zeta^2}(t,x) := E_\gamma^{\zeta^1,\zeta^2}\left[e^{\lambda \int_t^{\hat{T}} \int_V \int_U c(\xi_t,u,v)\zeta^1(du|\xi_t,t)\zeta^2(dv|\xi_t,t)dt+\lambda g(\xi_{\hat{T}})}|\xi_t = x\right], \tag{5}$$

is well defined.
We define the lower value of the game on $\boldsymbol{X}$ as $\mathcal{L}(x) := \sup_{\zeta^2 \in \Pi^2_{Ad}} \inf_{\zeta^1 \in \Pi^1_{Ad}} \mathcal{H}^{\zeta^1,\zeta^2}(0,x)$.
Similarly, define the upper value of the game on $\boldsymbol{X}$ as $\mathcal{U}(x) := \inf_{\zeta^1 \in \Pi^1_{Ad}} \sup_{\zeta^2 \in \Pi^2_{Ad}} \mathcal{H}^{\zeta^1,\zeta^2}(0,x)$.
It is easy to see that

$$\mathcal{L}(x) \leq \mathcal{U}(x) \text{ for each } x \in \boldsymbol{X}.$$

If $\mathcal{L}(x) = \mathcal{U}(x)$, $\forall x \in \boldsymbol{X}$, define $\mathcal{L}(\cdot) \equiv \mathcal{U}(\cdot) :\equiv \mathcal{H}^*(\cdot)$, and then the function $\mathcal{H}^*(x)$ is called the value of the game. Also, if $\sup_{\zeta^2 \in \Pi^2_M} \inf_{\zeta^1 \in \Pi^1_M} \mathcal{H}^{\zeta^1,\zeta^2}(t,x) = \inf_{\zeta^1 \in \Pi^1_M} \sup_{\zeta^2 \in \Pi^2_M} \mathcal{H}^{\zeta^1,\zeta^2}(t,x)$, $\forall (t,x) \in [0,\hat{T}] \times \boldsymbol{X}$, the common function is denoted by $\mathcal{H}^*(\cdot,\cdot)$.
A strategy $\zeta^{*1} \in \Pi^1_{Ad}$ is called optimal for first player if

$$\mathcal{H}^{\zeta^{*1},\zeta^2}(x,c) \leq \sup_{\zeta^2 \in \Pi^2_{Ad}} \inf_{\zeta^1 \in \Pi^1_{Ad}} \mathcal{H}^{\zeta^1,\zeta^2}(x) = \mathcal{L}(x) \;\forall\; x \in \boldsymbol{X},\; \forall \zeta^2 \in \Pi^2_{Ad}.$$

Similarly, for second player, the strategy $\zeta^{*2} \in \Pi^2_{Ad}$ is optimal if

$$\mathcal{H}^{\zeta^1,\zeta^{*2}}(x,c) \geq \inf_{\zeta^1 \in \Pi^1_{Ad}} \sup_{\pi^2 \in \Pi^2_{Ad}} \mathcal{H}^{\zeta^1,\zeta^2}(x) = \mathcal{U}(x) \;\forall\; x \in \boldsymbol{X},\; \forall \zeta^1 \in \Pi^1_{Ad}.$$

If for k^{th} player, (k=1,2), $\zeta^{*k} \in \Pi^k_{Ad}$ is optimal, then (ζ^{*1},ζ^{*2}) is said to be a pair of optimal strategies.
Now for the pair of strategies (ζ^{*1},ζ^{*2}) if

$$\mathcal{H}^{\zeta^{*1},\zeta^2}(x,c) \leq \mathcal{H}^{\zeta^{*1},\zeta^{*2}}(x,c) \leq \mathcal{H}^{\zeta^1,\zeta^{*2}}(x,c),\;\; \forall \zeta^1 \in \Pi^1_{Ad},\; \forall \zeta^2 \in \Pi^2_{Ad},$$

then (ζ^{*1},ζ^{*2}) is said to a saddle-point equilibrium, and then the strategies ζ^{*1} and ζ^{*2} are optimal strategies corresponding to first player and second player, respectively.

3 PRELIMS

For proving the existence of an optimal pair of strategies, we recall some statndard results for the risk-sensitive finite time horizon CTMDPs. Due to the unboundedness of the rates $q(dy|x,u,v)$ and $c(x,u,v)$, we impose some conditions to make the processes $\{\xi_t, t \geq 0\}$ nonexplosive, and to make $\mathcal{H}^{\pi^1,\pi^2}(0,x)$ finite, which were used greatly in CTMDPs; see, Golui and Pal (2021a), Guo and Liao (2019), Guo et al. (2019), Guo and Zhang (2019) and references therein. For bounded rates, following Assumption 1 (ii)-(iii) are not required, see Ghosh and Saha (2014), Kumar and Pal (2015).

Assumption 1. *There exists a function $\mathcal{W} : \boldsymbol{X} \to [1,\infty)$ for which the followings hold:*

(i) The relation $\int_S \mathcal{W}(y)q(dy|x,u,v) \leq \rho_1 \mathcal{W}(x) + b_1$ holds, for each $(x,u,v) \in \mathcal{K}$, for some constants $\rho_1 > 0$, $b_1 \geq 0$;

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(ii) $q^*(x) \leq M_1\mathcal{W}(x)$, $\forall x \in \boldsymbol{X}$, *for some nonnegative constant* $M_1 \geq 1$, *where* $q^*(x)$ *is as in (2.2);*

(iii) $e^{2(\hat{T}+1)\lambda|c(x,u,v)|} \leq M_2\mathcal{W}(x)$ *for any* $(x,u,v) \in \mathcal{K}$, *and* $e^{2(\hat{T}+1)\lambda|g(x)|} \leq M_2\mathcal{W}(x)$ *for each* $x \in \boldsymbol{X}$, *for some constant* $M_2 \geq 1$.

The non-exlposion of the state process $\{\xi_t, t \geq 0\}$ and the finiteness of $\mathcal{H}^{\zeta^1,\zeta^2}(0,x)$ is shown in following Lemma. Here we see that the function $\mathcal{H}^{\zeta^1,\zeta^2}(0,x)$ has upper and lower bound in terms of the function $\mathcal{W}$.

Lemma 1. *We grant Assumption 1. Then for each* $(\zeta^1,\zeta^2) \in \Pi^1_{Ad} \times \Pi^2_{Ad}$, *we obtain the following results.*

(a) $P^{\zeta^1,\zeta^2}_x(T_\infty = \infty) = 1$, $P^{\zeta^1,\zeta^2}_x(\xi_t \in \boldsymbol{X}) = 1$, *and* $P^{\zeta^1,\zeta^2}_x(\xi_0 = x) = 1$ *for each* $t \geq 0$ *and* $x \in \boldsymbol{X}$.

(b) (b_1) $e^{-L_1\mathcal{W}(x)} \leq \mathcal{H}^{\zeta^1,\zeta^2}(0,x) \leq L_1\mathcal{W}(x)$ *for* $x \in \boldsymbol{X}$ *and* $(\zeta^1,\zeta^2) \in \Pi^1_{Ad} \times \Pi^2_{Ad}$, *where* $L_1 := M_2 e^{\rho_1\hat{T}}\left[1 + \frac{b_1}{\rho_1}\right]$.

(b_2) $e^{-L_1\mathcal{W}(x)} \leq \mathcal{H}^{\zeta^1,\zeta^2}(t,x) \leq L_1\mathcal{W}(x)$ *for* $(t,x) \in [0,\hat{T}] \times \boldsymbol{X}$ *and* $(\zeta^1,\zeta^2) \in \Pi^1_M \times \Pi^2_M$.

Proof. These results can be proved by using Guo et al. (2019), Lemma 3.1 and Guo and Zhang (2019), Lemma 3.1.

In order to apply the extended Feynman-Kac formula, we impose the following assumption for unbounded functions. If the rates are bounded, the following Assumption is not required, see Ghosh and Saha (2014). Since we are dealing with unbounded rates, we require the following condition.

Assumption 2. *There exists* $[1,\infty)$*-valued function* $\mathcal{W}_1$ *on* $\boldsymbol{X}$ *such that*

(i) $\int_{\boldsymbol{X}} \mathcal{W}_1^2(y)q(dy|x,u,v) \leq \rho_2\mathcal{W}_1^2(x) + b_2$, *for each* $(x,u,v) \in \mathcal{K}$ *for some constants* $\rho_2 > 0$ *and* $b_2 > 0$*;*

(ii) $\mathcal{W}^2(x) \leq M_3\mathcal{W}_1(x)$, $\forall x \in \boldsymbol{X}$, *for some constant* $M_3 \geq 1$, *where the function* $\mathcal{W}$ *is as in Assumption 1.*

In addition of Assumptions 1, 2, we impose the following conditions to guarantee the existence of a pair of optimal strategies.

Assumption 3. *(i) The cost and transition rate functions,* $c(x,u,v)$ *and* $q(\cdot|x,u,v)$ *are continuous on* $U(x) \times V(x)$, *for each* $x \in \boldsymbol{X}$.

(ii) The integral functions $\displaystyle\int_{\boldsymbol{X}} f(y)q(dy|x,u,v)$ *and* $\displaystyle\int_{\boldsymbol{X}} \mathcal{W}(y)q(dy|x,u,v)$ *are continuous on* $U(x) \times V(x)$, *for each* $x \in \boldsymbol{X}$, *for all bounded measurable functions* f *on* $\boldsymbol{X}$ *and* $\mathcal{W}$ *as previous in Assumption 1.*

We next introduce some useful notations. Let $\mathcal{A}_c(\Omega \times [0,\hat{T}] \times \boldsymbol{X})$ denote the space of all real-valued, $\mathcal{P} \times \mathcal{B}(\boldsymbol{X})$-measurable functions $\varphi(\omega,t,x)$ which are differentiable in $t \in [0,\hat{T}]$ a.e. i.e., $\mathcal{A}_c(\Omega \times [0,\hat{T}] \times \boldsymbol{X})$ contains the said measurable functions φ with the following facets: Given any $x \in \boldsymbol{X}$, $(\zeta^1,\zeta^2) \in \Pi^1_{Ad} \times \Pi^2_{Ad}$, and a.s. $\omega \in \Omega$, there exists a $\mathcal{E}_{(\varphi,\omega,x,\zeta^1,\zeta^2)} \subseteq [0,\hat{T}]$ (a Borel subset of $[0,\hat{T}]$ that depends on $\varphi,\omega,x,\zeta^1,\zeta^2$) such that $\frac{\partial\varphi}{\partial t}$ (the partial derivative with respect to time $t \in [0,\hat{T}]$) exists for every $t \in \mathcal{E}_{(\varphi,\omega,x,\zeta^1,\zeta^2)}$ and $m_L(\mathcal{E}^c_{(\varphi,\omega,x,\zeta^1,\zeta^2)}) = 0$, where m_L is the Lebesgue measure on $\mathbb{R}$. Now if for some $(\omega,t,x) \in \Omega \times [0,\hat{T}] \times \boldsymbol{X}$, $\frac{\partial\varphi}{\partial t}(\omega,t,x)$ does not exists, we take this as any real number, and so $\frac{\partial\varphi}{\partial t}(\cdot,\cdot,\cdot)$ can be made definable on $\Omega \times [0,\hat{T}] \times \boldsymbol{X}$. For any given function $W \geq 1$ on $\boldsymbol{X}$, a function f (real-valued) on $\Omega \times [0,\hat{T}] \times \boldsymbol{X}$ is said to be a W-bounded if $\|f\|^\infty_W := \sup_{(\omega,t,x)\in\Omega\times[0,\hat{T}]\times\boldsymbol{X}} \frac{|f(\omega,t,x)|}{W(x)} < \infty$. The W-bounded Banach space is denoted by $\mathcal{B}_W(\Omega \times [0,\hat{T}] \times \boldsymbol{X})$. Note that if $W \equiv 1$, $\mathcal{B}_1(\Omega \times [0,\hat{T}] \times \boldsymbol{X})$ is the space of all bounded functions on $\Omega \times [0,\hat{T}] \times \boldsymbol{X}$.

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Now define $\mathcal{C}^1_{W_0,W_1}(\Omega \times [0,\hat{T}] \times \boldsymbol{X}) := \{\psi \in \mathcal{B}_{W_0}(\Omega \times [0,\hat{T}] \times \boldsymbol{X}) \cap \mathcal{A}_c(\Omega \times [0,\hat{T}] \times \boldsymbol{X}) : \frac{\partial \psi}{\partial t} \in \mathcal{B}_{W_1}(\Omega \times [0,\hat{T}] \times \boldsymbol{X})\}$. If any function $\psi(\omega,t,x) \in \mathcal{C}^1_{W_0,W_1}(\Omega \times [0,\hat{T}] \times \boldsymbol{X})$ does not depend on ω, we write it as $\psi(t,x)$ and the corresponding space is $\mathcal{C}^1_{W_0,W_1}([0,\hat{T}] \times \boldsymbol{X})$.
In the the next theorem, we state the extended Feynman-Kac formula, which is very useful for us.

Theorem 1. *We grant Assumptions 1 and 2.*

(a) Then, for each $x \in \boldsymbol{X}$, $(\zeta^1,\zeta^2) \in \Pi^1_{Ad} \times \Pi^2_{Ad}$ *and* $\psi \in \mathcal{C}^1_{\mathcal{W},\mathcal{W}_1}(\Omega \times [0,\hat{T}] \times \boldsymbol{X})$,

$$E_x^{\zeta^1,\zeta^2}\left[\int_0^{\hat{T}}\left(\frac{\partial \psi}{\partial t}(\omega,t,\xi_t) + \int_{\boldsymbol{X}} \psi(\omega,t,y)\int_V\int_U q(dy|\xi_t,u,v)\zeta^1(du|\omega,t)\zeta^2(dv|\omega,t)\right)dt\right]$$
$$= E_x^{\zeta^1,\zeta^2}[\psi(\omega,\hat{T},\xi_{\hat{T}})] - E_x^{\zeta^1,\zeta^2}\psi(\omega,0,x).$$

Note that since $(\zeta^1,\zeta^2) \in \Pi^1_{Ad} \times \Pi^2_{Ad}$ *may be dependent on histories,* $\{\xi_t, t \geq 0\}$ *may be not Markovian.*

(b) For each $x \in \boldsymbol{X}$, $(\zeta^1,\zeta^2) \in \Pi^1_M \times \Pi^2_M$ *and* $\psi \in \mathcal{C}^1_{\mathcal{W},\mathcal{W}_1}([0,\hat{T}] \times \boldsymbol{X})$,

$$E_\gamma^{\zeta^1,\zeta^2}\left[\int_s^{\hat{T}}\left(\left(\frac{\partial \psi}{\partial t}(t,\xi_t) + \lambda c(\xi_t,\zeta^1_t,\zeta^2_t)\right)e^{\int_s^t \lambda c(\xi_\beta,\zeta^1_\beta,\zeta^2_\beta)d\beta}\psi(t,\xi_t)\right.\right.$$
$$\left.\left. + \int_{\boldsymbol{X}} e^{\int_s^t \lambda c(\xi_\beta,\zeta^1_\beta,\zeta^2_\beta)d\beta}\psi(t,y)q(dy|\xi_t,\zeta^1_t,\zeta^2_t)\right)dt|\xi_s = x\right]$$
$$= E_\gamma^{\zeta^1,\zeta^2}\left[e^{\int_s^{\hat{T}} \lambda c(\xi_\beta,\zeta^1_\beta,\zeta^2_\beta)d\beta}\psi(\hat{T},\xi_{\hat{T}})|\xi_s = x\right] - \psi(s,x).$$

Proof. See Guo and Zhang (2019), Theorem 3.1.

Next, we present a theorem which shows that the solutions of the optimality equations (Shapley equations) have unique probabilistic representations. In section 4, we also illustrate how this verification theorem can be used to determine the game's value.

Theorem 2. *Assume that Assumptions 1 and 2 are true. If there exist a function* $\psi \in \mathcal{C}^1_{\mathcal{W},\mathcal{W}_1}([0,\hat{T}] \times \boldsymbol{X})$ *and a pair of stationary strategies* $(\zeta^{*1},\zeta^{*2}) \in \Pi^1_{SM} \times \Pi^2_{SM}$ *for which*

$$\psi(s,x) - e^{\lambda g(x)}$$
$$= E_1 = \int_s^{\hat{T}} \sup_{\vartheta \in \mathcal{P}(U(x))} \inf_{\eta \in \mathcal{P}(V(x))}\left[\lambda c(x,\vartheta,\eta)\psi(t,x) + \int_{\boldsymbol{X}} \psi(t,y)q(dy|x,\vartheta,\eta)\right]dt$$
$$= E_2 = \int_s^{\hat{T}} \inf_{\eta \in \mathcal{P}(V(x))} \sup_{\vartheta \in \mathcal{P}(U(x))}\left[\lambda c(x,\vartheta,\eta)\psi(t,x) + \int_{\boldsymbol{X}} \psi(t,y)q(dy|x,\vartheta,\eta)\right]dt$$
$$= \int_s^{\hat{T}} \inf_{\eta \in \mathcal{P}(V(x))}\left[\lambda c(x,\zeta^{*1}(\cdot|x,t),\eta)\psi(t,x) + \int_{\boldsymbol{X}} \psi(t,y)q(dy|x,\zeta^{*1}(\cdot|x,t),\eta)\right]dt$$
$$= \int_s^{\hat{T}} \sup_{\vartheta \in \mathcal{P}(U(x))}\left[\lambda c(x,\vartheta,\zeta^{*2}(\cdot|x,t))\psi(t,x) + \int_{\boldsymbol{X}} \psi(t,y)q(dy|x,\vartheta,\zeta^{*2}(\cdot|x,t))\right]dt$$
$$s \in [0,\hat{T}], \ x \in \boldsymbol{X}, \tag{6}$$

then

(a)

$$\psi(0,x) = \sup_{\zeta^1 \in \Pi^1_{Ad}} \inf_{\zeta^2 \in \Pi^2_{Ad}} \mathcal{H}^{\zeta^1,\zeta^2}(0,x) = \inf_{\zeta^2 \in \Pi^2_{Ad}} \sup_{\zeta^1 \in \Pi^1_{Ad}} \mathcal{H}^{\zeta^1,\zeta^2}(0,x)$$
$$= \inf_{\zeta^2 \in \Pi^2_{Ad}} \mathcal{H}^{\zeta^{*1},\zeta^2}(0,x) = \sup_{\zeta^1 \in \Pi^1_{Ad}} \mathcal{H}^{\zeta^1,\zeta^{*2}}(0,x), \ x \in \boldsymbol{X} \tag{7}$$

and

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(b)

$$\psi(t,x) = \sup_{\zeta^1\in\Pi^1_M}\inf_{\zeta^2\in\Pi^2_M}\mathcal{H}^{\zeta^1,\zeta^2}(t,x) = \inf_{\zeta^2\in\Pi^2_M}\sup_{\zeta^1\in\Pi^1_M}\mathcal{H}^{\zeta^1,\pi^2}(t,x)$$
$$= \inf_{\zeta^2\in\Pi^2_M}\mathcal{H}^{\zeta^{*1},\zeta^2}(t,x) = \sup_{\zeta^1\in\Pi^1_M}\mathcal{H}^{\zeta^1,\zeta^{*2}}(t,x) = \mathcal{H}^*(t,x),\ t\in[0,\hat{T}],\ x\in\boldsymbol{X}. \tag{8}$$

Proof.

(a) See Golui and Pal (2021a), Corollary 3.1.

(b) This proof follows from part (a).

4 THE EXISTENCE OF OPTIMAL SOLUTION AND SADDLE POINT EQUILIBRIUM

This section provides the proof that optimality equation (6) has a solution in the space $\mathcal{C}^1_{\mathcal{W},\mathcal{W}_1}([0,\hat{T}]\times\boldsymbol{X})$. Furthermore, we use the optimality equation (6) to prove the existence of saddle point equilibrium. The next Proposition proves the optimality equation (6) has a solution when the rates are bounded.

Proposition 1. *Suppose Assumption 3 holds. Also, assume that $\|q\| < \infty$, $\|c\| < \infty$, $\|g\| < \infty$, $c(x,u,v)\geq 0$ and $g(x)\geq 0$, for all $(x,u,v)\in\mathcal{K}$. Then the following results are true.*

(a) There exists a bounded function $\psi\in\mathcal{B}_1([0,\hat{T}]\times\boldsymbol{X})$ satisfying first two equations (E_1 and E_2) of (6).

(b) There exists a pair of strategies $(\zeta^{*1},\zeta^{*2})\in\Pi^1_{SM}\times\Pi^2_{SM}$ satisfying the equations (6), (7) and (8) and hence this forms a saddle-point equilibrium.

(c) $\mathcal{H}^*(t,x)$ (and so $\psi(t,x)$) is non-increasing in t for fixed $x\in\boldsymbol{X}$, where $t\in[0,\hat{T}]$.

Proof. (a) From Wei (2017), Theorem 4.1, there exists $\psi\in\mathcal{B}_1([0,\hat{T}]\times\boldsymbol{X})$ satisfying first two equations (E_1 and E_2) of (6).

(b) In view of measurable selection theorem as in Nowak (1985), we get the existence of $(\zeta^{*1},\zeta^{*2})\in\Pi^1_{SM}\times\Pi^2_{SM}$ for which (6) holds. So, by Theorem 2, we get

$$\sup_{\zeta^1\in\Pi^1_{Ad}}\inf_{\zeta^2\in\Pi^2_{Ad}}\mathcal{H}^{\zeta^1,\zeta^2}(0,x) = \inf_{\zeta^2\in\Pi^2_{Ad}}\sup_{\zeta^1\in\Pi^1_{Ad}}\mathcal{H}^{\zeta^1,\zeta^2}(0,x) = \sup_{\zeta^1\in\Pi^1_{Ad}}\mathcal{H}^{\zeta^1,\zeta^{*2}}(0,x)$$
$$= \inf_{\zeta^2\in\Pi^2_{Ad}}\mathcal{H}^{\zeta^{*1},\zeta^2}(0,x) = \psi(0,x) \tag{9}$$

and

$$\sup_{\zeta^1\in\Pi^1_M}\inf_{\zeta^2\in\Pi^2_M}\mathcal{H}^{\zeta^1,\zeta^2}(t,x) = \inf_{\zeta^2\in\Pi^2_M}\sup_{\zeta^1\in\Pi^1_M}\mathcal{H}^{\zeta^1,\zeta^2}(t,x) = \sup_{\zeta^1\in\Pi^1_M}\mathcal{H}^{\zeta^1,\zeta^{*2}}(t,x)$$
$$= \inf_{\zeta^2\in\Pi^2_M}\mathcal{H}^{\zeta^{*1},\zeta^2}(t,x) = \mathcal{H}^*(t,x) = \psi(t,x). \tag{10}$$

Thus the game's value exists and $(\zeta^{*1},\zeta^{*2})\in\Pi^1_{SM}\times\Pi^2_{SM}$ forms a saddle-point equilibrium.

(c) First we fix any $s,t\in[0,\hat{T}]$ where $s<t$. Also fix any $(\zeta^1,\zeta^2)\in\Pi^1_M\times\Pi^2_M$. Now for each $x\in\boldsymbol{X}$, define a Markov strategy corresponding to $\zeta^1\in\Pi^1_M$ as

$$\zeta^1_{s,t}(du|x,\beta) = \begin{cases}\zeta^1(du|x,\beta+t-s) \text{ if } \beta\geq s\\ \zeta^1(du|x,\beta) \text{ otherwise.}\end{cases} \tag{11}$$

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Similarly, for each $\zeta^2 \in \Pi_M^2$, we define $\zeta_{s,t}^2$.
Then, for each $\beta \in [s, s+\hat{T}-t]$ and $x \in \boldsymbol{X}$, $q(dy|x, \zeta_{s,t}^1(du|x,\beta), \zeta_{s,t}^2(dv|x,\beta)) = q(dy|x, \zeta^1(du|x, \beta + t - s), \zeta^2(dv|x, \beta + t - s))$,
$c(x, \zeta_{s,t}^1(du|x,\beta), \zeta_{s,t}^2(dv|x,\beta)) = c(x, \zeta^1(du|x, \beta+t-s), \zeta^2(dv|x, \beta+t-s))$. Next define

$$\mathcal{H}^{\zeta^1,\zeta^2}(s \rightsquigarrow t, x) := E_\gamma^{\zeta^1,\zeta^2}\left[e^{\lambda\int_s^t c(\xi_\beta,\zeta^1(du|\xi_\beta,\beta),\zeta^2(dv|\xi_\beta,\beta))d\beta+\lambda g(\xi_t)}|\xi_s = x\right], \tag{12}$$

$$\mathcal{H}^*(s \rightsquigarrow t, x) := \inf_{\zeta^2\in\Pi_M^2}\sup_{\zeta^1\in\Pi_M^1} \mathcal{H}^{\zeta^1,\zeta^2}(s \rightsquigarrow t, x). \tag{13}$$

Now in view of the Markov property of $\{\xi_t, t \geq 0\}$ under any $(\zeta^1, \zeta^2) \in \Pi_M^1 \times \Pi_M^2$ and (11)-(13), we have $\mathcal{H}^{\zeta^1,\zeta^2}(t \rightsquigarrow \hat{T}, x) = \mathcal{H}^{\zeta_{s,t}^1,\zeta_{s,t}^2}(s \rightsquigarrow \hat{T}+s-t, x)$.
It can be easily shown that $\sup_{\zeta_{s,t}^1\in\Pi_M^1} \mathcal{H}^{\zeta_{s,t}^1,\zeta_{s,t}^2}(s \rightsquigarrow \hat{T}+s-t, x) \leq \sup_{\zeta^1\in\Pi_M^1} \mathcal{H}^{\zeta^1,\zeta^2}(t \rightsquigarrow T, x)$ and $\sup_{\zeta^1\in\Pi_M^1} \mathcal{H}^{\zeta^1,\zeta^2}(t \rightsquigarrow \hat{T}, x) \leq \sup_{\zeta_{s,t}^1\in\Pi_M^1} \mathcal{H}^{\zeta_{s,t}^1,\zeta_{s,t}^2}(s \rightsquigarrow \hat{T}+s-t, x)$ for all $\zeta^2 \in \Pi_2^M$. Hence, $\sup_{\zeta^1\in\Pi_M^1} \mathcal{H}^{\zeta^1,\zeta^2}(t \rightsquigarrow \hat{T}, x) = \sup_{\zeta_{s,t}^1\in\Pi_M^1} \mathcal{H}^{\zeta_{s,t}^1,\zeta_{s,t}^2}(s \rightsquigarrow \hat{T}+s-t, x)$ for all $\zeta^2 \in \Pi_2^M$. Similarly, we can show that $\mathcal{H}^*(t \rightsquigarrow \hat{T}, x) = \mathcal{H}^*(s \rightsquigarrow \hat{T}+s-t, x)$. Now since $c(x,u,v) \geq 0$ on $\mathcal{K}$, by (13) and $t > s$, we have $\mathcal{H}^*(t \rightsquigarrow \hat{T}, x) = \mathcal{H}^*(s \rightsquigarrow \hat{T}+s-t, x) \leq \mathcal{H}^*(s \rightsquigarrow \hat{T}, x)$. But by (10), (12) and (13), we have $\mathcal{H}^*(t \rightsquigarrow \hat{T}, x) = \mathcal{H}^*(t, x)$. Hence, we obtain $\mathcal{H}^*(s, x) \geq \mathcal{H}^*(t, x)$ i.e. $\mathcal{H}^*(t, x)$ is decreasing in t. Now from part (b), we have $\mathcal{H}^*(t, x) = \psi(t, x)$. Hence, $\psi(t, x)$ is also decreasing in t.

Theorem 3. *Suppose Assumptions 1, 2 and 3 hold. Also, in addition suppose* $c(x,u,v) \geq 0$ *and* $g(x) \geq 0$ *for all* $(x,u,v) \in \mathcal{K}$. *Then there exist a unique* $\psi \in \mathcal{C}_{\mathcal{W},\mathcal{W}_1}^1([0,\hat{T}] \times \boldsymbol{X})$ *and some pair of strategies* $(\zeta^{*1}, \zeta^{*2}) \in \Pi_{SM}^1 \times \Pi_{SM}^2$ *satisfying the equations (6), (7) and (8) and hence this is a saddle-point equilibrium.*
Proof. *First observe that* $1 \leq e^{2(\hat{T}+1)\lambda c(x,u,v)} \leq M_2\mathcal{W}(x)$ *and* $1 \leq e^{2(\hat{T}+1)\lambda g(x)} \leq M_2\mathcal{W}(x)$. *For each integer* $n \geq 1$, $x \in \boldsymbol{X}$, *define* $\boldsymbol{X}_n := \{x \in \boldsymbol{X} | \mathcal{W}(x) \leq n\}$, $U_n(x) := U(x)$ *and* $V_n(x) := V(x)$. *Also for each* $(x,u,v) \in \mathcal{K}_n := \{(x,u,v) : x \in \boldsymbol{X}, u \in U_n(x), v \in V_n(x)\}$, *define*

$$q_n(dy|x,u,v) := \begin{cases} q(dy|x,u,v) & \text{if } x \in \boldsymbol{X}_n, \\ 0 \quad \text{if} \quad x \notin \boldsymbol{X}_n, \end{cases} \tag{14}$$

$$c_n^+(x,u,v) := \begin{cases} c(x,u,v) \wedge \ min\left\{n, \frac{1}{\lambda(\hat{T}+1)}\ln\sqrt{M_2\mathcal{W}(x)}\right\} & \text{if } x \in \boldsymbol{X}_n, \\ 0 \quad \text{if} \quad x \notin \boldsymbol{X}_n. \end{cases} \tag{15}$$

and

$$g_n^+(x) := \begin{cases} g(x) \wedge \ min\left\{n, \frac{1}{\lambda(\hat{T}+1)}\ln\sqrt{M_2\mathcal{W}(x)}\right\} & \text{if } x \in \boldsymbol{X}_n, \\ 0 \quad \text{if} \quad x \notin \boldsymbol{X}_n. \end{cases} \tag{16}$$

By (14), obviously $q_n(dy|x,u,v)$ *is transition rates on* $\boldsymbol{X}$ *satisfying conservative and stable conditions. Now consider the sequence of CTMDPs models with bounded rates* $\mathcal{G}_n^+ := \{\boldsymbol{X}, U, V, (U_n(x), V_n(x), x \in \boldsymbol{X}), c_n^+, g_n^+, q_n\}$. *Fix a n. Corresponding to a pair of Markov strategies* $(\zeta^1,\zeta^2) \in \Pi_M^1 \times \Pi_M^2$, *suppose for this model the risk-sensitive cost criterion is* $\mathcal{H}_n^{\zeta^1,\zeta^2}(t,x)$ *and the value function is*

$$\mathcal{H}_n(t,x) := \sup_{\zeta^1\in\Pi_M^1}\inf_{\zeta^2\in\Pi_M^2} \mathcal{H}_n^{\zeta^1,\zeta^2}(t,x).$$

Then by Proposition 1, for each $n \geq 1$, *we get a unique* ψ_n *in* $\mathcal{C}_{1,1}^1([0,\hat{T}]) \times S$ *and* $(\zeta_n^{*1}, \zeta_n^{*2}) \in \Pi_{SM}^1 \times \Pi_{SM}^2$ *satisfying*

$$\psi_n(s,x) - e^{\lambda g_n^+(x)}$$

 https://doi.org/10.17993/3cemp.2022.110250.76-92

$$
\begin{aligned}
&= \int_s^{\hat{T}} \inf_{\eta\in\mathcal{P}(V(x))}\left[\lambda c_n^+(x,\zeta_n^{*1}(\cdot|x,t),\eta)\psi_n(t,x)+\int_{\boldsymbol{X}}\psi_n(t,y)q_n(dy|x,\zeta_n^{*1}(\cdot|x,t),\eta)\right]dt\\
&= \int_s^{\hat{T}} \sup_{\vartheta\in\mathcal{P}(U(x))}\left[\lambda c_n^+(x,\vartheta,\zeta_n^{*2}(\cdot|x,t))\psi_n(t,x)+\int_{\boldsymbol{X}}\psi_n(t,y)q_n(dy|x,\vartheta,\zeta_n^{*2}(\cdot|x,t))\right]dt\\
&s\in[0,\hat{T}],\ x\in\boldsymbol{X}.
\end{aligned}
\tag{17}
$$

Now, $e^{2\lambda(\hat{T}+1)c_n^+(x,u,v)}\le M_2\mathcal{W}(x)$, $e^{2\lambda(\hat{T}+1)g_n^+(x)}\le M_2\mathcal{W}(x)$ *and* $\psi_n(\hat{T},x)=e^{\lambda g_n^+(x)}$. *Hence by Lemma 1, Theorem 2 and (17), we have*

$$
e^{-L_1\mathcal{W}(x)}\le\psi_n(t,x)=\sup_{\zeta^1\in\Pi_1^M}\mathcal{H}_n^{\zeta^1,\zeta_n^{*2}}(t,x)\le L_1\mathcal{W}(x)\ \forall\ n\ge1. \tag{18}
$$

Moreover, since $\psi_n(t,x)\ge0$, $c_{n-1}^+(x,u,v)\le c_n^+(x,u,v)$, *and* $g_{n-1}^+(t,x)\le g_n^+(x)\ \forall(x,u,v)\in\mathcal{K}$, *using (14), (15), (17) and Proposition 1,* $\forall x\in\boldsymbol{X}$ *and a.e.* t, *we obtain,*

$$
\begin{cases}
\frac{\partial\psi_n}{\partial t}(t,x)+\left[\lambda c_{n-1}^+(x,\vartheta,\zeta_n^{*2}(\cdot|x,t))\psi_n(t,x)+\int_{\boldsymbol{X}}\psi_n(t,y)q_{n-1}(dy|x,\vartheta,\zeta_n^{*2}(\cdot|x,t))\right]\\
\le 0 \qquad \text{if } x\in\boldsymbol{X}_{n-1}
\end{cases}
\tag{19}
$$

and

$$
\begin{cases}
\frac{\partial\psi_n}{\partial t}(t,x)+\left[\lambda c_{n-1}^+(x,\vartheta,\zeta_n^{*2}(\cdot|x,t))\psi_n(t,x)+\int_{\boldsymbol{X}}\psi_n(t,y)q_{n-1}(dy|x,\vartheta,\zeta_n^{*2}(\cdot|x,t))\right]\\
=\frac{\partial\psi_n}{\partial t}(t,x)\le 0 \qquad \text{if } x\notin\boldsymbol{X}_{n-1},
\end{cases}
\tag{20}
$$

(for details see, Golui and Pal (2021b), Theorem 4.1, p. 24). So, for any $\zeta^1\in\Pi_M^1$, *by Feynman-Kac formula (similar proof as in Theorem 2), we get*

$$
\mathcal{H}_{n-1}^{\zeta^1,\zeta_n^{*2}}(t,x)\le\psi_n(t,x).
$$

Since $\zeta^1\in\Pi_M^1$ *is arbitrary*

$$
\inf_{\zeta^2\in\Pi_M^2}\sup_{\zeta^1\in\Pi_M^1}\mathcal{H}_{n-1}^{\zeta^1,\zeta^2}(t,x)\le\sup_{\zeta^1\in\Pi_M^1}\mathcal{H}_{n-1}^{\zeta^1,\zeta_n^{*2}}(t,x)\le\psi_n(t,x). \tag{21}
$$

Also using (17) and Feynman-Kac formula (similar proof as in Theorem 2), we have

$$
\sup_{\zeta^1\in\Pi_M^1}\inf_{\zeta^2\in\Pi_M^2}\mathcal{H}_{n-1}^{\zeta^1,\zeta^2}(t,x)=\inf_{\zeta^2\in\Pi_M^2}\sup_{\zeta^1\in\Pi_M^1}\mathcal{H}_{n-1}^{\zeta^1,\zeta^2}(t,x)=\psi_{n-1}(t,x). \tag{22}
$$

From (21) and (22), we obtain $\psi_{n-1}(t,x)\le\psi_n(t,x)$. *Also, since* ψ_n *has an upper bound,* $\lim_{n\to\infty}\psi_n$ *exists. Let*

$$
\lim_{n\to\infty}\psi_n(t,x):=\psi(t,x)\ \forall t\in[0,\hat{T}],\ \forall x\in\boldsymbol{X}. \tag{23}
$$

Next by Lemma 1, we get

$$
|\psi(t,x)|\le L_1\mathcal{W}(x)\quad\forall t\in[0,\hat{T}]. \tag{24}
$$

Let

$$
\begin{aligned}
I_n(t,x):=&\sup_{\vartheta\in\mathcal{P}(U_n(x))}\inf_{\eta\in\mathcal{P}(V_n(x))}\left[\lambda c_n^+(x,\vartheta,\eta)\psi_n(t,x)+\int_{\boldsymbol{X}}\psi_n(t,y)q_n(dy|x,\vartheta,\eta)\right],\\
&\forall t\in[0,\hat{T}],\ \forall x\in\boldsymbol{X}.
\end{aligned}
$$

Then, applying Fan's minimax theorem, Fan, (1953), we obtain

$$
I_n(t,x):=\inf_{\eta\in\mathcal{P}(V_n(x))}\sup_{\vartheta\in\mathcal{P}(U_n(x))}\left[\lambda c_n^+(x,\vartheta,\eta)\psi_n(t,x)+\int_{\boldsymbol{X}}\psi_n(t,y)q_n(dy|x,\vartheta,\eta)\right],
$$

$\forall t \in [0,\hat{T}],\ \forall x \in \boldsymbol{X}.$

Then, by Assumptions 1 and 2 and the fact that $\lambda \leq 1$, *we get the following result*

$$\begin{aligned}|I_n(t,x)| &\leq L_1\Big(M_2\mathcal{W}^2(x)+(b_1+\rho_1)\mathcal{W}^2(x)+2M_1\mathcal{W}^2(x)\Big)\\ &\leq L_1M_3\mathcal{W}_1(x)(M_2+b_1+\rho_1+2M_1)=:\mathfrak{R}(x),\ (t,x)\in[0,\hat{T}]\times\boldsymbol{X}.\end{aligned}\tag{25}$$

Let

$$I(t,x):=\sup_{\vartheta\in\mathcal{P}(U(x))}\inf_{\eta\in\mathcal{P}(V(x))}\left[\lambda c(x,\vartheta,\eta)\psi(t,x)+\int_{\boldsymbol{X}}\psi(t,y)q(dy|x,\vartheta,\eta)\right],$$
$$\forall t\in[0,\hat{T}],\ \forall x\in\boldsymbol{X}.$$

Hence in view of Fan's minimax theorem, Fan, (1953), we obtain

$$I(t,x):=\inf_{\eta\in\mathcal{P}(V(x))}\sup_{\vartheta\in\mathcal{P}(U(x))}\left[\lambda c(x,\vartheta,\eta)\psi(t,x)+\int_{\boldsymbol{X}}\psi(t,y)q(dy|x,\vartheta,\eta)\right],$$
$$\forall t\in[0,\hat{T}],\ \forall x\in\boldsymbol{X}.$$

We next prove that for each fixed $x\in\boldsymbol{X}$ *and* $t\in[0,\hat{T}]$, *along some suitable subsequence of* $\{n\}$ *(if necessary),* $\lim_{n\to\infty}I_n(t,x)=I(t,x)$. *Now, using Assumption 3, the functions* $c(x,\vartheta,\eta)$ *and* $\int_{\boldsymbol{X}}q(dy|x,\vartheta,\eta)\psi_n(t,y)$ *are continuous on* $\mathcal{P}(U(x))\times\mathcal{P}(V(x))$ *for each* $x\in\boldsymbol{X}$. *So, we find a sequence of pair of measurable functions* $(\vartheta_n^*,\eta_n^*)\in\mathcal{P}(U(x))\times\mathcal{P}(V(x))$ *such that*

$$\begin{aligned}I_n(t,x):&=\inf_{\eta\in\mathcal{P}(V(x))}\left[\lambda c_n^+(x,\vartheta_n^*,\eta)\psi_n(t,x)+\int_{\boldsymbol{X}}\psi_n(t,y)q_n(dy|x,\vartheta_n^*,\eta)\right]\\ &=\sup_{\vartheta\in\mathcal{P}(U(x))}\left[\lambda c_n^+(x,\vartheta,\eta_n^*)\psi_n(t,x)+\int_{\boldsymbol{X}}\psi_n(t,y)q_n(dy|x,\vartheta,\eta_n^*)\right].\end{aligned}\tag{26}$$

Now, $\mathcal{P}(U(x))$ *and* $\mathcal{P}(V(x))$ *are compact. So, there exists a subsequences (here, we take the same sequence for simplicity) that* $\vartheta_n^*\to\vartheta^*$ *and* $\eta_n^*\to\eta^*$ *as* $n\to\infty$ *for some* $(\vartheta^*,\eta^*)\in\mathcal{P}(U(x))\times\mathcal{P}(V(x))$.

Taking $n\to\infty$ *in* (26), *by the generalized version of Fatou's lemma Feinberge et al. (2014), Hernandez-Lerma and Lasserre (1999), Lemma 8.3.7, for arbitrarily fixed* $\vartheta\in\mathcal{P}(U(x))$, *we have*

$$\liminf_{n\to\infty}I_n(t,x)\geq\left[\lambda c(x,\vartheta,\eta^*)\psi(t,x)+\int_{\boldsymbol{X}}\psi(t,y)q(dy|x,\vartheta,\eta^*)\right].$$

Since $\vartheta\in\mathcal{P}(U(x))$ *is arbitrary,*

$$\begin{aligned}\liminf_{n\to\infty}I_n(t,x)&\geq\sup_{\vartheta\in\mathcal{P}(U(x))}\left[\lambda c(x,\vartheta,\eta^*)\psi(t,x)+\int_{\boldsymbol{X}}\psi(t,y)q(dy|x,\vartheta,\eta^*)\right]\\ &\geq\inf_{\eta\in\mathcal{P}(V(x))}\sup_{\vartheta\in\mathcal{P}(U(x))}\left[\lambda c(x,\vartheta,\eta)\psi(t,x)+\int_{\boldsymbol{X}}\psi(t,y)q(dy|x,\vartheta,\eta)\right].\end{aligned}\tag{27}$$

Using analogous arguments from (26), *by the generalized version of Fatou's Lemma, Feinberge et al. (2014), Hernandez-Lerma and lasserre (1999), Lemma 8.3.7, we have*

$$\limsup_{n\to\infty}I_n(t,x)\leq\sup_{\vartheta\in\mathcal{P}(U(x))}\inf_{\eta\in\mathcal{P}(V(x))}\left[\lambda c(x,\vartheta,\eta)\psi(t,x)+\int_{\boldsymbol{X}}\psi(t,y)q(dy|x,\vartheta,\eta)\right].\tag{28}$$

So, by (27) and (28), we get

$$\lim_{n\to\infty}I_n(t,x)=I(t,x)\ \forall t\in[0,\hat{T}],\ \forall x\in\boldsymbol{X}.\tag{29}$$

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Since $\lim_{n\to\infty}\psi_n(t,x)=\psi(t,x)$ *and* $\forall t\in[0,\hat{T}]$, $\forall x\in\boldsymbol{X}$, *in view of (29) and the dominated convergent theorem (since* $|I_n(t,x)|\le\mathcal{R}(x)$*), taking limit* $n\to\infty$ *in (17), we say that* ψ *satisfies first two equations (*E_1 *and* E_2*) of (6) and hence* $\psi(\cdot,x)$ *is differentiable almost everywhere on* $[0,\hat{T}]$*, see Athreya (2006), Theorem 4.4.1. Again, by the analogous arguments as in (25), we obtain*

$$\left|\frac{\partial\psi(t,x)}{\partial t}\right|=|I(t,x)|\le R(x),\quad \forall t\in[0,\hat{T}],\ \forall x\in\boldsymbol{X}.$$

Therefore, we see that $\psi\in\mathcal{C}^1_{\mathcal{W},\mathcal{W}_1}([0,\hat{T}]\times\boldsymbol{X})$*. Furthermore, using analogous arguments as in Proposition 1 (b),* ψ *is the unique solution of (6) satisfying (7) and (8) and hence saddle-point equilibrium exists.*

Next we state the main optimal results that provide the proof of the existence of saddle point equilibrium and game's value when payoff rates are extended real valued functions.

Theorem 4. *We grant Assumptions 1, 2 and 3. Then, the following claims are true.*

(a) There exists a unique function $\psi\in\mathcal{C}^1_{\mathcal{W},\mathcal{W}_1}([0,\hat{T}]\times\boldsymbol{X})$ *that satisfies first two equations (*E_1 *and* E_2*) of (6).*

(b) There exists a pair of strategies $(\zeta^{*1},\zeta^{*2})\in\Pi^1_{SM}\times\Pi^2_{SM}$ *that satisfies the equations (6), (7) and (8) and hence this pair of strategies becomes a saddle-point equilibrium.*

Proof. *We only need prove part (a) since part (b) follows from Proposition 1 (b). Now, for each* $n\ge 1$*, define* c_n *and* g_n *on* $\mathcal{K}$ *as:*

$$c_n(x,u,v):=\max\{-n,c(x,u,v)\},\ \ g_n(x):=\max\{-n,g(x)\}$$

for each $(x,u,v)\in\mathcal{K}$*. Then* $\lim_{n\to\infty}c_n(x,u,v)=c(x,u,v)$ *and* $\lim_{n\to\infty}g_n(x)=g(x)$*. Define* $r_n(x,u,v):=c_n(x,u,v)+n$ *and* $\tilde{g}_n(x):=g_n(x)+n$*. So,* $r_n(x,u,v)\ge 0$ *and* $\tilde{g}_n(x)\ge 0$ *for each* $n\ge 1$ *and* $(x,u,v)\in\mathcal{K}$*. Now by Assumption 1, we have*

$$-\frac{\ln\sqrt{M_2\mathcal{W}(x)}}{\lambda(\hat{T}+1)}\le\max\left\{-n,-\frac{\ln\sqrt{M_2\mathcal{W}(x)}}{\lambda(\hat{T}+1)}\right\}\le c_n(x,u,v)\le\frac{\ln\sqrt{M_2\mathcal{W}(x)}}{\lambda(\hat{T}+1)}\tag{30}$$

and

$$-\frac{\ln\sqrt{M_2\mathcal{W}(x)}}{\lambda(\hat{T}+1)}\le\max\left\{-n,-\frac{\ln\sqrt{M_2\mathcal{W}(x)}}{\lambda(\hat{T}+1)}\right\}\le g_n(x)\le\frac{\ln\sqrt{M_2\mathcal{W}(x)}}{\lambda(\hat{T}+1)}.\tag{31}$$

So, we have $e^{2\lambda(\hat{T}+1)r_n(x,u,v)}\le e^{2\lambda(\hat{T}+1)n}M_2\mathcal{W}(x)$ *and* $e^{2\lambda(\hat{T}+1)\tilde{g}_n(x)}\le e^{2\lambda(\hat{T}+1)n}M_2\mathcal{W}(x)$, $\forall n\ge 1$ *and* $(x,u,v)\in\mathcal{K}$*. Define a new model* $\mathcal{R}_n:=\{\boldsymbol{X},U,V,(U_n(x),V_n(x),x\in\boldsymbol{X}),r_n,\tilde{g}_n,q\}$*. Now for any real-valued measurable functions* $\tilde{\psi}$ *and* ϕ *defined on* $\mathcal{K}$ *and* $[0,\hat{T}]\times\boldsymbol{X}$*, respectively, define*

$$\mathcal{H}(s,x,\tilde{\psi},\phi):=\sup_{\zeta^1\in\Pi^1_M}\inf_{\zeta^2\in\Pi^2_M}E^{\zeta^1,\zeta^2}_{\gamma}\left[\exp\left(\lambda\int_s^{\hat{T}}\tilde{\psi}(\xi_t,\pi^1_t,\pi^2_t)dt+\lambda\phi(\xi_{\hat{T}})\right)\bigg|\xi_s=x\right]\tag{32}$$

assuming that the integral exists. Now since $r_n\ge 0$, $\tilde{g}_n\ge 0$ *and all Assumptions hold for the model* $\mathcal{R}_n$*, by Theorem 3, we have*

$$\begin{aligned}
&-\frac{\partial\mathcal{H}(s,x,r_n,\tilde{g}_n)}{\partial s}\\
&=\sup_{\vartheta\in\mathcal{P}(U(x))}\inf_{\eta\in\mathcal{P}(V(x))}\left[\lambda r_n(x,\vartheta,\nu)\mathcal{H}(s,x,r_n,\tilde{g}_n)+\int_{\boldsymbol{X}}\mathcal{H}(s,y,r_n,\tilde{g}_n)q(dy|x,\vartheta,\eta)\right]\\
&=\inf_{\eta\in\mathcal{P}(V(x))}\sup_{\vartheta\in\mathcal{P}(U(x))}\left[\lambda r_n(x,\vartheta,\nu)\mathcal{H}(s,x,r_n,\tilde{g}_n)+\int_{\boldsymbol{X}}\mathcal{H}(s,y,r_n,\tilde{g}_n)q(dy|x,\vartheta,\eta)\right]
\end{aligned}\tag{33}$$

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for almost all $s \in [0, \hat{T}]$. *Now*

$$\mathcal{H}(s, x, r_n, \tilde{g}_n) = \mathcal{H}(s, x, c_n + n, g_n + n) = \mathcal{H}(s, x, c_n, g_n)e^{\lambda(\hat{T}-s+1)n}.$$

So, by (33), we can write for a.e. $s \in [0, \hat{T}]$,

$$\begin{aligned}-\frac{\partial \mathcal{H}(s, x, c_n, g_n)}{\partial s} &= \sup_{\vartheta \in \mathcal{P}(U(x))} \inf_{\eta \in \mathcal{P}(V(x))} \left[\lambda c_n(x, \vartheta, \eta)\mathcal{H}(s, x, c_n, g_n) + \int_{\boldsymbol{X}} \mathcal{H}(s, y, c_n, g_n)q(dy|x, \vartheta, \eta)\right] \\ &= \inf_{\eta \in \mathcal{P}(V(x))} \sup_{\vartheta \in \mathcal{P}(U(x))} \left[\lambda c_n(x, \vartheta, \eta)\mathcal{H}(s, x, c_n, g_n) + \int_{\boldsymbol{X}} \mathcal{H}(s, y, c_n, g_n)q(dy|x, \vartheta, \eta)\right].\end{aligned}$$

Hence

$$\begin{aligned}&\mathcal{H}(s, x, c_n, g_n) - e^{\lambda g_n(x)} \\ &= \int_s^{\hat{T}} \sup_{\vartheta \in \mathcal{P}(U(x))} \inf_{\eta \in \mathcal{P}(V(x))} \left[\lambda c_n(x, \vartheta, \eta)\mathcal{H}(t, x, c_n, g_n) + \int_{\boldsymbol{X}} \mathcal{H}(t, y, c_n, g_n)q(dy|x, \vartheta, \eta)\right] dt \\ &= \int_s^{\hat{T}} \inf_{\eta \in \mathcal{P}(V(x))} \sup_{\vartheta \in \mathcal{P}(U(x))} \left[\lambda c_n(x, \vartheta, \eta)\mathcal{H}(t, x, c_n, g_n) + \int_{\boldsymbol{X}} \mathcal{H}(t, y, c_n, g_n)q(dy|x, \vartheta, \eta)\right] dt. \qquad (34)\end{aligned}$$

Now by (34) and Lemma 1, we obtain

$$|\mathcal{H}(t, x, c_n, g_n)| \leq L_1 \mathcal{W}(x) \quad n \geq 1. \qquad (35)$$

Now since $c_n(x, u, v)$ *and* $g_n(x)$ *are non-increasing in* $n \geq 1$, *hence its corresponding value function* $\mathcal{H}(t, x, c_n, g_n)$ *is also non-increasing in* n. *Also by Lemma 1, we know that* $\mathcal{H}(\cdot, \cdot, c_n, g_n)$ *has a lower bound. So,* $\lim_{n\to\infty} \mathcal{H}(t, x, c_n, g_n)$ *exists. Let* $\lim_{n\to\infty} \mathcal{H}(t, x, c_n, g_n) =: \psi(t, x)$, $(t, x) \in [0, \hat{T}] \times \boldsymbol{X}$. *Then using analogous arguments as Theorem 4.1, and using the function* $\mathcal{H}(t, x, c_n, g_n)$ *in the place of the function* $\psi_n(t, x)$ *here, by (34), (35), Assumptions 1, and 2, we see that (a) is true.*

The converse of Theorem 4 is given below.

Theorem 5. *Under Assumptions 1, 2 and 3, suppose* $(\hat{\zeta}^{*1}, \hat{\zeta}^{*2}) \in \Pi^1_{SM} \times \Pi^2_{SM}$ *is a saddle-point equilibria. Then* $(\hat{\zeta}^{*1}, \hat{\zeta}^{*2})$ *is a mini-max selector of eq. (6).*

Proof. Using the definition of saddle-point equilibrium, we have

$$\begin{aligned}\mathcal{H}^{\hat{\zeta}^{*1}, \hat{\zeta}^{*2}}(0, x) &= \sup_{\zeta^2 \in \Pi^2_{Ad}} \inf_{\zeta^1 \in \Pi^1_{Ad}} \mathcal{H}^{\zeta^1, \zeta^2}(0, x) \\ &= \inf_{\zeta^1 \in \Pi^1_{Ad}} \sup_{\zeta^2 \in \Pi^2_{Ad}} \mathcal{H}^{\zeta^1, \zeta^2}(0, x) = \sup_{\zeta^2 \in \Pi^2_{Ad}} \mathcal{H}^{\hat{\zeta}^{*1}, \zeta^2}(0, x) = \inf_{\zeta^1 \in \Pi^1_{Ad}} \mathcal{H}^{\zeta^1, \hat{\zeta}^{*2}}(0, x). \qquad (36)\end{aligned}$$

Now arguing as in Theorem 4, it follows that for $\hat{\zeta}^{*1} \in \Pi^1_{SM}$ there exists a function $\tilde{\psi} \in C^1_{\mathcal{W}, \mathcal{W}_1}([0, \hat{T}] \times \boldsymbol{X})$ such that

$$\begin{aligned}&\tilde{\psi}(s, x) - e^{\lambda g(x)} \\ &= \int_s^{\hat{T}} \inf_{\eta \in \mathcal{P}(V(x))} \left[\lambda c(x, \hat{\zeta}^{*1}(\cdot|x, t), \eta)\tilde{\psi}(t, x) + \int_{\boldsymbol{X}} \tilde{\psi}(t, y)q(dy|x, \hat{\zeta}^{*1}(\cdot|x, t), \eta)\right] dt \\ &\quad s \in [0, \hat{T}], \ x \in \boldsymbol{X}, \qquad (37)\end{aligned}$$

satisfying

$$\tilde{\psi}(0, x) = \inf_{\zeta^2 \in \Pi^2_{Ad}} \mathcal{H}^{\hat{\zeta}^{*1}, \zeta^2}(0, x) \qquad (38)$$

and

$$\tilde{\psi}(t, x) = \inf_{\zeta^2 \in \Pi^2_m} \mathcal{H}^{\hat{\zeta}^{*1}, \zeta^2}(t, x). \qquad (39)$$

Then by (6), (36), (37), (38), (39), Theorem 2, Theorem 4, we say that $\hat{\zeta}^{*1}$ is outer maximizing selector of (6). By analogous arguments, $\hat{\zeta}^{*2}$ is outer minimizing selector of (6).

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5 EXAMPLE

This section is dedicated for an example to validate assumptions in this paper, where transition and cost functions are not bounded.

Example 1. *Consider a model of a zero-sum game as*

$$\mathcal{G} := \{\boldsymbol{X}, (U, U(x), x \in \boldsymbol{X}), (V, V(x), x \in \boldsymbol{X}), c(x,u,v), q(dy|x,u,v)\}.$$

Suppose our state space is $\boldsymbol{X} = (-\infty, \infty)$ *and transition rate is given by*

$$q(\hat{D}|x,u,v) = \hat{\lambda}(x,a,b)\left[\int_{y\in\hat{D}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}} dy - \delta_x(\hat{D})\right], \; x \in \boldsymbol{X}, \hat{D} \in \mathcal{B}(\boldsymbol{X}), (u,v) \in U(x) \times V(x). \tag{40}$$

We take the following requirements to see if our model has a saddle-point equilibrium.

(I) $U(x)$ *and* $V(x)$ *are compact subsets of the Borel spaces* U *and* V*, respectively, for each fixed* $x \in \boldsymbol{X}$*,.*

(II) The payoff function $c(x,u,v)$ *and the rate function* $\hat{\lambda}(x,u,v)$ *are continuous on* $U(x) \times V(x)$*, for each* $x \in S$*. Also, assume that* $e^{2\lambda(\hat{T}+1)|c(x,u,v)|} \leq M_2\mathcal{W}(x)$*,* $e^{2\lambda(\hat{T}+1)|g(x)|} \leq M_2\mathcal{W}(x)$ *and* $0 < \sup_{(u,v)\in U(x)\times V(x)} \hat{\lambda}(x,u,v) \leq M_0(x^2+1)$ *for each* $(x,u,v) \in \mathcal{K}$*.*

Proposition 2. *In view of conditions (I)-(II), Assumptions 1, 2, and 3 are satisfied by above controlled system. Therefore, the existence of a saddle point equilibrium is proved by Theorem 4.*

Proof. See Guo and Zhang (2019), Proposition 5.1.

6 CONCLUSIONS

A finite-time horizon dynamic zero-sum game with risk-sensitive cost criteria on a Borel state space is studied. Here for each state x, the admissible action spaces ($U(x)$ and $V(x)$) are compact metric spaces and costs and transition rate functions are unbounded. Under certain assumptions, we have solved the Shapley equation and have established a saddle point equilibrium.

Risk-sensitive non-zero-sum game with unbounded rates (costs and transition rates) over countable state space was investigated in Wei (2019). It would be a challenging problem to study the same problem on the Borel state space.

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CONCAVITY METHOD: A CONCISE SURVEY

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ABSTRACT

This short review article discusses the concavity method, one of the most effective ways to deal with parabolic equations with unbounded solutions in finite time. If the solution ceases to exist for some time, we say it blows up. The solution or some of its derivatives become singular depending on the equation. We focus on situations where the solution becomes unbounded in finite time, and our objective is to review some of the key blowup theory papers utilising the concavity method.

KEYWORDS

Parabolic equations, Concavity method, weak solutions, blowup, variable exponent spaces, $p(x)$-Laplacian operator.

1 INTRODUCTION

The idea of unbounded solutions, known as blowup theory, holds a specific place in the study of nonlinear equations. Blow-up is a phenomenon where solutions of differential equations cease to exist because of the infinite growth of the variables describing the evolution processes. Before the successful calculation of mathematical methods to deal with the unboundedness of solutions, the physical significance of blowup was understood because it occurs in processes such as heat conduction, combustion, volcanic eruption, gas dynamics, etc. In addition, some of the current notable works in this field include the blowing up of cancer cells, nuclear blowup, electrical blow up, laser fusion, blow up in pandemic simulations, etc.

The solutions to a problem can be unbounded at a finite or infinite time. In this work, we only deal with papers studying finite time blowup of solutions using the concavity method. Finite time blowup is a sufficient condition for the nonexistence of global solutions, since the solutions grow without bound in finite time intervals. There are nonlinear PDEs, with local solutions for time $t < T^\star$, which blowup at a finite time $T^\star$. Thus, we can give a formal definition to finite time blowup as, if $T^\star < \infty$ and

$$\limsup_{t \to T^\star} \|z(t)\| = \infty, \tag{1}$$

then we say the solution z of a given problem blows up at a finite time $T^\star$.

From 1960s onward, the following equations,

$$z_t = \Delta z + |z|^{p-1} z, \tag{2}$$

and

$$z_t = \Delta z + \lambda e^z, \lambda > 0 \tag{3}$$

have become fundamental models for blow-up study [4,20,38,52]. Fujita, Hayakawa [16,17,22], Kaplan [24] and Friedman [18] studied these problems and obtained several critical results on blowup of solutions. These fundamental results from the 1960s initiated deep research of blowup solutions for various nonlinear evolution PDEs in the next decade [2, 3, 25, 28–30, 48, 49]. Some of the fascinating problems in this area are finding whether the solutions blow up at a finite time, obtaining lower and upper bounds for blowup time and getting the blowup rate. Challenges in this theoretical study have attracted many scientists, and useful techniques have been developed to deal with certain nonlinear parabolic problems. They include eigenfunction method, explicit inequality method, logarithmic convexity method, Fourier coefficient method, comparison method, concavity method and differential inequality techniques. We solely concentrate on the concavity approach out of all these methods.

Concavity Method

The concavity method was introduced by H. A. Levine [28] and proved very successful with a wide range of applications. The method uses the concavity of an auxiliary functional, say $M^{-\eta}$, with $\eta > 0$. Here $M(t)$ is a positive energy functional of the solution to a PDE. Since $M^{-\eta}$ is concave, we get

$$\frac{d^2 M(t)}{dt^2} \leq 0. \tag{4}$$

Hence by integrating the above inequality, one can arrive at

$$M^\eta(t) \geq \frac{M^{\eta+1(t)}}{M(0) - t\eta M'(0)}, \tag{5}$$

this implies if $M'(0) > 0$, then $M^\eta(t)$ is bounded below by a function which becomes unbounded at a finite time. To apply the concavity method, we need to show that $M(t)$ obeys the inequality (4). Hence we have the following inequality,

$$\frac{d^2 M^{-\eta}(t)}{dt^2} = -\eta M^{-\eta-2}(t)[M(t)M''(t) - (1+\eta)(M'(t))^2]. \tag{6}$$

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Now, for $M(t) > 0$ we get

$$M(t)M''(t) - (1+\eta)(M'(t))^2 > 0. \tag{7}$$

Hence this inequality (7) is a sufficient condition for the existence of blowup. Moreover, the inequality (5) helps to derive an upper bound for blowup time.

In [28], Levine studied an abstract parabolic equation

$$\begin{cases} P\dfrac{dz}{dt} = -A(t)z + f(z(t)), t \in [0,\infty) \\ z(0) = z_0, \end{cases} \tag{8}$$

where P and A are positive linear operators defined on a dense subdomain D of a real or complex Hilbert space. In which he obtained blowup results under the conditions

$$2(\alpha+1)F(x) \leq (x, f(x)), F(z_0(x)) > \frac{1}{2}(z_0(x), Az_0(x)), \tag{9}$$

for every $x \in D$, where $F(x) = \int_0^1 (f(\rho x), x)d\rho$. This study has been acknowledged as an innovative and elegant way, known as "the concavity method,"for providing criteria for the blowup.

Later, Philippin and Proytcheva [39] transformed the procedure from its abstract form into a concrete form and used it to solve the same equation with $P = 2$. Since then, the concavity method has been used for several variations of the equations (8) or other equations to get the blowup solutions. Levine, together with Payne [29, 30], established nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backwards in time using the concavity argument. In [37] Payne, Philippin and Piro dealt with the blowup of the solutions to a semilinear second-order parabolic equation with nonlinear boundary conditions. They demonstrated that blowup would occur at some finite time under specific conditions on the nonlinearities and data. Junning [23] investigated the following initial boundary value problem

$$\begin{cases} z_t = \operatorname{div}(|\nabla z|^{(p-2)}\nabla z) + f(\nabla z, z, x, t), & (x,t) \in \Omega \times (0,t) \\ z(x,0) = z_0(x), & x \in \Omega \\ z(x,t) = 0, & x \in \partial\Omega \end{cases} \tag{10}$$

and obtained results on the existence and nonexistence of solutions under specific conditions. Erdem [15] estabilished sufficient conditions for the global nonexistence of solutions of a second order quasilinear parabolic equations.

In 2017, the finiteness of the time for blow up of a semilinear parabolic equation with Dirichlet boundary conditions discussed by Chung and Choi [9]. They continued the study and obtained a new condition for the concavity method of blowup solutions to p-Laplacian parabolic equations [10]. The authors then developed a condition for blowup solutions to discrete p-Laplacian parabolic equations under the mixed boundary conditions on networks with Hwang [11]. For a reaction-diffusion equation with a generalised Lewis function and nonlinear exponential growth, Dai and Zhang established results on nonexistence of solutions using concavity method [12]. In [47], the authors took into account the results of finite time blowup for a parabolic equation coupled with a superlinear source term and a local linear boundary dissipation. The adequate conditions for the solutions to blow up in a finite time were deduced using the concavity argument. The existence of finite time blowup solutions with arbitrarily high initial energy and the upper and lower bound of the blowup time were specifically obtained. Galaktionov [19] provide the sufficient conditions for the unboundedness of the solutions of boundary value problems for a class of quasilinear equations and systems of parabolic type, describing the propagation of heat in media with nonlinear heat conduction and volume liberation of energy.

Blow up of solutions for a semilinear heat equation with a viscoelastic term was studied in [21] for nonlinear flux on the boundary. The authors obtained blowup results for initial negative energy by employing the concavity method. Li and Han [32] then improved these results for positive initial energy with the support of the potential well method. Sun et. al. [46] investigated an initial boundary value problem for a pseudo-parabolic equation under the influence of a linear memory term and a nonlinear source term and obtained results on finite time blowup of solutions under suitable assumptions on the initial data and the relaxation function.

The existence of solutions for a pseudo-parabolic equation with memory was deduced by Di and Shang [13] using Galerkin method and potential well theory. Then using the concavity method, derived finite time blowup results for both negative and non-negative initial energy. Sun et. al. [45] studied the problem and came up with existence and finite time blow-up results using Galerkin method, concavity argument and potential well theory by making a slight change in the source term. They derived an upper bound for the blowup time and obtained the existence of solutions which blow up in finite time with arbitrary initial energy conditions. Di and Shang [14] worked on a class of nonlinear pseudo-parabolic equations with a memory term under Dirichlet boundary condition. They proved a finite time blowup result for specific initial energy and relaxation function. In 2019, Messaoudi and Talahmeh [33] studied a semilinear viscoelastic pseudo-parabolic problem with variable exponent and demonstrated any weak solution with initial data at arbitrary energy level blows up in finite time. Furthermore, they obtained an upper bound for the blowup time using the concavity method. Chen and Xu [5] studied a finitely degenerate semilinear pseudo-parabolic problem and showed the global existence and blowup in finite time of solutions with sub-critical and critical initial energy. The asymptotic behaviour of the global solutions and a lower bound for blowup time of the local solution are also obtained. A pseudo-parabolic equation with variable exponents under initial and Dirichlet boundary value conditions is the subject of study in [53]. In [53], Zhou et. al. established the global existence and blowup results of weak solutions with arbitrarily high initial energy.

Existence and blow-up studies of the following $p(x)$-Laplacian parabolic equation with memory was studied by Lakshmipriya and Gnanavel [34],

$$\begin{cases} z_t - \Delta z - \mu \nabla(|\nabla z|^{p(x)-2}\nabla z) + \int_0^t h(t-\tau)\Delta z(x,\tau)d\tau = \beta |z|^{b(x)-2}z, & x \in \Omega, t \geq 0 \\ z(x,t) = 0, & x \in \partial\Omega, t \geq 0 \\ z(x,0) = z_0(x), & x \in \Omega \end{cases} \tag{11}$$

where $\Omega \subset \mathbb{R}^N, (N \geq 1, N \neq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. $\beta > 0, \mu \geq 0$ are constants. The authors established the existence and finite time blow up of weak solutions of the problem. Further, obtained upper and lower bounds for the blowup time of solutions, by employing the concavity method and differential inequality technique, respectively. In [35], the authors analysed and interpreted unbounded solutions of a viscoelastic $p(x)-$ Laplacian parabolic equation with logarithmic nonlinearity. Here the problem was considered for initial data corresponding to the sub-critical initial energy. In this attempt, Lakshmipriya and Gnanavel obtained the local existence of solutions on an interval $[0, T)$. Moreover, it extracted an upper bound for the blowup time by applying the concavity method.

The paper [27] deals with the existence and blowup of weak solutions of the following pseudo-parabolic equation with logarithmnonlinearityity

$$\begin{cases} w_t - \Delta w_t - div(|\nabla w|^{p(x)-2}\nabla w) = |w|^{s(x)-2}w + |w|^{h-2}w \log|w|, & (x,t) \in \Omega \times (0,\infty) \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty) \\ w(x,0) = w_0(x), & x \in \overline{\Omega} \end{cases} \tag{12}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The model consider is used to describe the non-stationary process in semiconductors in the presence of sources; the first two terms represent the free electron density rate and logarithmic and polynominonlinearityity stands for the source of free electron current [26]. Lakshmipriya and Gnanavel [36] analysed the blowup of solutions

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to the following problem

$$\begin{cases} z_t(x,t) = \Delta_{p(x)} z(x,t) + g(z(x,t)), & (x,t) \in \Omega \times (0,\infty) \\ z(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty) \\ z(x,0) = z_0(x) \geq 0, & x \in \overline{\Omega} \end{cases} \tag{13}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The model is involved in image processing, elastic mechanics and electro-rheological fluids [1, 40, 42]. The authors considered a condition on the nonlinear function $g(z)$ given by,

$$\varsigma \int_0^z g(s)ds \leq zg(z) + \eta z^{b(x)} + \mu, z > 0.$$

Obtained results on blowup and established an upper bound for the blowup time with the help of the concavity method.

Now, we consider the most recent works involving the Concavity method. Ruzhansky et.al. [41] proved a global existence and blowup of the positive solutions to the initial-boundary value problem of the nonlinear porous medium equation and the nonlinear pseudo-parabolic equation on the stratified Lie groups based on the concavity argument and the Poincare inequality. A nonlinear porous medium equation under a new nonlinearity condition is considered in a bounded domain by Sabitbek and Torebek [43]. They presented the blowup of the positive solution to the considered problem for the negative initial energy. For the subcritical and critical initial energy cases, obtained a global existence, asymptotic behaviour and blowup phenomena in a finite time of the positive solution to the nonlinear porous medium equation. In [31], Li and Fang are concerned with the blowup phenomena for a semilinear pseudo-parabolic equation with general nonlinearity under the null Dirichlet boundary condition. When the nonlinearity satisfies a new structural condition, they obtain some new blowup criteria with different initial energy levels. They derived the growth estimations and life span of blowup solutions.

The concavity method is also used to understand the blowup behaviour of system of nonlinear parabolic equations [8]. Apart from parabolic and pseudo-parabolic equations, the method plays a significant role in studying unbounded solutions of hyperbolic equations and systems. Some of the latest works are as follows [6, 7, 44, 50, 51]. Hence the Concavity method is a simple and powerful tool to use in the blow up studies of solutions to differential equation problems.

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RETRIEVING THE MISSING DATA FROM DIFFERENT INCOMPLETE SOFT SETS

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ABSTRACT

Dealing an incomplete information has been a major issue in the theory of soft sets. In this paper, we have presented an approach to deal with incomplete soft set, incomplete fuzzy soft set and incomplete intuitionistic fuzzy soft set. For this purpose we have discussed about the notion of distance between two objects (parameters) which will be used to compute the degree of interdependence between them. This approach will use the full information of known data and the relationships between them. Data filling converts an incomplete soft set into complete one which makes the soft sets applicable not only to decision making but also to other fields.

KEYWORDS

Soft set, incomplete information system, fuzzy soft set, object parameter approach.

1 INTRODUCTION

There are various real life problems involving uncertainties and classical mathematical tools are not sufficient for handling them. There are many theories developed recently for dealing with them. Some of them are probability theory, theory of fuzzy sets [16], theory of intuitionistic fuzzy sets [1, 2], theory of vague sets [4], theory of interval mathematics [11] and theory of rough sets [13]. All of these theories have their own advantages and some limitations as well. For example, in the theory of fuzzy sets and intuitionistic fuzzy sets, it is very difficult to choose the membership and non membership functions that give us the desired result; in the theory of probability the outcomes of an event must be unbiased; in the theory of rough sets, the indiscernibility relation may create a situation where two completely different objects are same. One major common drawback of these theories is probably the inadequacy of parameterization tools which was observed by Molodtsov in 1999. Consequently he introduced the concept of soft set theory [10] that is free from the difficulties that have troubled the usual theoretical approaches. The absence of any restriction on the approximate description in soft set theory makes it easily applicable in practice. A soft set model requires no prior knowledge of data sets. Molodtsov provided several applications of soft set theory in his work. Maji et al. [9] introduced fuzzy soft set by allowing the parameters to be mapped to the fuzzy sets. Further allowing the parameters to be mapped to the intuitionistic fuzzy sets, Maji et al. [8] introduced the concept of intuitionistic fuzzy soft set which is a generalization of standard soft set and fuzzy soft set in the sense that it is a soft set whose approximate values are the intuitionistic fuzzy sets.

Lots of research work are currently active in the field of theoretical and practical soft sets. The major portion of these works is based on complete information. However, incomplete information widely exists in real life due to mishandling data, mistakes in processing or transferring data, mistakes in measuring and collecting data or any other factor. Soft set under incomplete information is referred to as an incomplete soft set. Similarly fuzzy soft set and intuitionistic fuzzy soft set under incomplete information are referred to as incomplete fuzzy soft set and incomplete intuitionistic fuzzy soft set respectively.

The simplest approach to transform an incomplete data set to a complete one is to delete all objects related to missing information. But in this process we may deduce wrong information from it. On the other hand, predicting the unknown information gives more fruitful results. Zou et al. [17] initiated the study of incomplete soft sets. For incomplete soft set, they computed decision values rather than filling the empty cells in the corresponding incomplete information system. The decision values are calculated by the weighted average of all the choice values and the weight of each choice value is decided by the distribution of other available objects. Incomplete fuzzy soft set is completed by the method of average probability. Zou's method is too complicated and it does not fill the empty cells of the corresponding information system. So the soft set obtained by this method is only useful in decision making. Using average probability method we can predict individual unknown value of fuzzy soft set but all the predicted values of a parameter for different objects are equal, so this method is also of low accuracy. Kong et al. [6] proposed a simple method equivalent to that of Zou which fills the empty cells. To fill the empty cells in the incomplete information system, Kong's method uses the values of target parameter (the parameter for which the cell is empty) on the objects other than target object (the object for which the cell is empty). Qin et al. [14] presented a method called DFIS (data filling approach for incomplete soft set). In that paper, empty cells are filled in terms of the association degree between the parameters, when a strong association exists between the parameters, otherwise they are filled in terms of probability of other available objects. Khan et al. [5] proposed an alternative data filling approach for incomplete soft set (ADFIS) to predict the missing data in soft sets. In ADFIS, the value of the empty cell whose corresponding parameter has strongest association is computed first. Unlike the DFIS, before filling second empty cell, the value of the first is inserted in the information table. But the drawback of DFIS and ADFIS is that a parameter can have strongest association or maximal association with more than one parameters having opposite type of association. In that case empty cell can't be filled. This method considers only the relation between parameters and does not take the effect of objects into account. However there may be some relationship between objects too. For example the houses in same locality have nearly same price. Deng et al. [3] introduced an object-parameter

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approach which uses the full information between object and between parameters. Deng's approach has some drawbacks as: (i) the estimated value may not be in the interval [0,1]; (ii) the information between objects and between parameters is not comprehensive. To overcome these drawbacks Liu et al. [7] improved Deng's approach by redefining the notion of distance and dominant degree.

It is a review paper. In paper [15] we have put forward an algorithm to predict missing data in an incomplete soft set and incomplete fuzzy soft set. For this we have defined the notion of distance (emerged from the concept of Euclidean distance in $\mathbb{R}^n$) between two objects (parameters) and defined the degree of interdependence between two objects (parameters). And thus we have taken account of the effect of other objects (parameters) on the target object (parameter). This algorithm uses the full available data to reveal the hidden relationship between objects (parameters). Moreover we have introduced an approach to predict missing data in an incomplete intuitionistic fuzzy soft set with the help of algorithm for incomplete soft set and incomplete fuzzy soft set.

Rest of the paper has been organized as follows. Section 2 recalls the basic definitions and concepts of soft set theory and information system. In section 3 we have introduced an algorithm to predict missing data in an incomplete soft set and incomplete fuzzy soft set and given an application through an example. In section 4, we have given an algorithm to predict the missing data in an incomplete intuitionistic fuzzy soft set and given an application through an example. Finally we have concluded this paper in section 5.

2 PRELIMINARIES

Let $U = \{u_1, u_2, \ldots, u_m\}$ be a universe set of objects and $E = \{e_1, e_2, \ldots, e_n\}$ be a set of parameters.

Definition 1 (Fuzzy set). *[16] A fuzzy set A over U is given by*

$$A = \{\langle u, \mu_A(u)\rangle | u \in U\}$$

where $\mu_A : U \to [0,1]$ is called the membership function of the fuzzy set A. $\mu_A(u)$ is said to be the degree of membership of u in A.

Definition 2 (Intuitionistic fuzzy set). *[1] An intuitionistic fuzzy set (IFS) A over U is given by*

$$A = \{\langle u, \mu_A(u), \nu_A(u)\rangle | u \in U; \mu_A(u), \nu_A(u) \in [0,1] \text{ and } \mu_A(u) + \nu_A(u) \leq 1\}\}$$

where $\mu_A : U \to [0,1]$ and $\nu_A : U \to [0,1]$ are said to be the membership and non membership functions of the intuitionistic fuzzy set A respectively.

Definition 3 (Soft set). *[10] A pair $A = (F, E)$ is said to be a soft set over U, where F is a mapping from E to $\mathcal{P}(U)$ (set of all crisp subsets of U). Sometimes it is also called a crisp soft set to emphasize the fact that $F(e)$ is a crisp set for every $e \in E$.*

Alternatively, a soft set A is given by

$$A = \{F(e) | e \in E\}$$

where F is a mapping from E to $\mathcal{P}(U)$.

Definition 4 (Fuzzy soft set). *[9] A pair $A = (F, E)$ is said to be a fuzzy soft set over U, where F is a mapping from E to $\mathcal{F}(U)$ (set of all fuzzy sets over U).*

Alternatively, a fuzzy soft set A is given by

$$A = \{F(e) | e \in E\}$$

where F is a mapping from E to $\mathcal{F}(U)$.

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Definition 5 (Intuitionistic fuzzy soft set). *[8] A pair $A = (F, E)$ is said to be an intuitionistic fuzzy soft set (IFSS) over U, where F is a mapping from E to $\mathcal{IF}(U)$ (set of all intuitionistic fuzzy sets over U).*

Alternatively, an intuitionistic fuzzy soft set A is given by

$$A = \{F(e)|e \in E\}$$

where F is mapping from E to $\mathcal{IF}(U)$.

Definition 6 (Information system). *[12] A quadruple $S = (U, A, F, V)$ is called an information system, where $U = \{u_1, u_2, \ldots, u_m\}$ is a universe of discourse, $A = \{a_1, \ldots, a_n\}$ is a set of attributes and $V = \bigcup_{j=1}^{n} V_j$, where each V_j is the value set of the attribute a_j and $F = \{f_1, \ldots, f_n\}$ where $f_j : U \to V_j$ for every j.*

If $V_j = \{0, 1\}$ for every $1 \leq j \leq n$ then the corresponding information system is called Boolean valued information system and if $V_j = [0, 1]$ for every $1 \leq j \leq n$ then the corresponding information system is called fuzzy information system. In an information system $u_{ik} = f_k(u_i)$ denotes the value of the attribute a_k on the object u_i. An information system is often represented by an information table.

Remark: (i) Every soft set can be considered as a Boolean valued information system with each entry filled by 1 or 0 depending on whether an object belongs to range of the parameter or not.
(ii) Every fuzzy soft set can be considered as a fuzzy information system with each entry filled by a quantity in $[0, 1]$ which represents the membership degree of object in the range of the related parameter.
(iii) Every intuitionistic fuzzy soft set can be considered as an information system with each entry filled by an element of $[0, 1] \times [0, 1]$ where the first and second coordinates represent the membership degree and non membership degree of the object in the range of the related parameter respectively.

Example 2.1. Every incomplete soft set can be considered as an incomplete information system. Examples of incomplete soft set, incomplete fuzzy soft set and incomplete intuitionistic fuzzy soft set are given in table 1, 2 and 3 respectively. The unknown value in incomplete information system is denoted by '$*$'.

Table 1 – Incomplete soft set

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	1	0	1	0	1	0
u_2	1	0	0	1	0	0
u_3	0	1	0	0	1	0
u_4	0	1	$*$	1	0	$*$
u_5	1	0	1	1	0	0
u_6	0	1	0	0	$*$	0
u_7	1	$*$	1	0	1	0
u_8	0	0	1	1	0	0

Table 2 – Incomplete fuzzy soft set

U	e_1	e_2	e_3	e_4	e_5	e_6	e_7
u_1	0.9	0.4	0.1	0.9	0.6	0.3	0.4
u_2	0.8	0.6	0.5	$*$	0.5	0.3	0.3
u_3	$*$	0.8	0.9	$*$	0.9	0.9	0.9
u_4	0.9	0.8	0.9	0.8	$*$	0.8	0.9
u_5	0.9	0.2	0.2	0.6	0.3	0.4	$*$
u_6	0.9	0.2	0.4	0.4	0.4	0.3	0.3

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Table 3 – Incomplete Intuitionistic fuzzy soft set

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	(0.8,0.1)	(0.2,0.1)	(0.8,0.1)	(0.4,0.5)	(0.4,0.5)	(0.6,0.2)
u_2	(0.8,0.2)	(0.8,0.1)	(0.7,0.2)	(0.6,0.4)	(0.5,0.5)	(0.6,0.2)
u_3	(0.7,0.2)	(0.3,0.1)	(0.8,0.2)	$*$	(0.6,0.1)	(0.4,0.2)
u_4	(0.6,0.2)	(0.7,0.2)	(0.7,0.3)	(0.4,0.3)	(0.7,0.1)	(0.6,0.1)
u_5	(0.5,0.3)	(0.6,0.3)	(0.4,0.5)	(0.7,0.3)	(0.8,0.1)	$*$
u_6	(0.2,0.4)	(0.4,0.4)	(0.5,0.5)	(0.4,0.3)	(0.4,0.3)	(0.5,0.1)
u_7	(0.7,0.2)	(0.8,0.1)	(0.5,0.4)	(0.9,0.1)	(0.5,0.3)	(0.4,0.1)

3 ALGORITHM TO PREDICT MISSING DATA IN AN INCOMPLETE SOFT SET AND INCOMPLETE FUZZY SOFT SET AND ITS APPLICATION

3.1 A PREPARATORY STEP

There is always a direct or indirect relationship between objects (parameters). To measure this relationship, we will define the 'degree of interdependence' between objects (parameters). To determine the unknown value in the incomplete soft set we will examine the remaining known values and interdependence between target object (parameter) and other objects (parameters).

Let $U = \{u_1, u_2, \ldots, u_m\}$ be universe set of objects and $E = \{e_1, e_2, \ldots, e_n\}$ be set of parameters. Suppose that $\mu_{F(e_k)}(u_i) = u_{ik}$. For every $1 \leq i \leq m$; denote $E^{(i)} = \{k|u_{ik} \neq *\}$ and for every $1 \leq k \leq n$; $U^{(k)} = \{i|u_{ik} \neq *\}$.

Now we will define distance and degree of interdependence between two objects and between two parameters.

Definition 7 (Distance). *For u_i and u_j in U, the distance between u_i and u_j is defined by*

$$d(u_i, u_j) = \left(\sum_{k \in E^{(i)} \cap E^{(j)}} (u_{ik} - u_{jk})^2 \right)^{1/2} \tag{1}$$

where $E^{(i)} \cap E^{(j)} = \{k|u_{ik} \neq * \text{ and } u_{jk} \neq *\}$.

Similarly, for e_k and e_l in E, distance between e_k and e_l is defined by

$$d(e_k, e_l) = \left(\sum_{i \in U^{(k)} \cap U^{(l)}} (u_{ik} - u_{il})^2 \right)^{1/2} \tag{2}$$

where $U^{(k)} \cap U^{(l)} = \{i|u_{ik} \neq * \text{ and } u_{il} \neq *\}$.

Definition 8 (Degree of Interdependence). *For u_i and u_j in U, the degree of interdependence between u_i and u_j is denoted by α_{ij} and is defined as* $\alpha_{ij} = \frac{1}{1+d(u_i,u_j)}$.

Similarly, for e_k and e_l in E, the degree of interdependence between e_k and e_l is denoted by β_{kl} and is defined as $\beta_{kl} = \frac{1}{1+d(e_k,e_l)}$.

Suppose that the value u_{ik} is missing, then we will call u_i as target object and e_k as target parameter. The prediction of u_{ik} will contain two parts: (i) object part u_{ik}^{obj} and (ii) parameter part u_{ik}^{par}. As the distance between two objects (parameters) increases, the interdependence between them decreases. So the objects (parameters) which are nearer to target object (parameter) will be more reliable to determine the object (parameter) part of the unknown value. Object part of an unknown value is determined using the values of the target parameter on the objects other than target object and the parameter part is determined using the values of all parameters other than target parameter on target object.

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3.2 ALGORITHM

Suppose we have to predict the value of u_{ik}. Before giving Algorithm we define some notations here:

$U_i^* = \{p|u_p \in U - \{u_i\}$ and $u_{pk} \neq *\}$ and

$E_k^* = \{q|e_q \in E - \{e_k\}$ and $u_{iq} \neq *\}$

And we define U_r and E_r recursively as

$U_r = \{j_r|d(u_i, u_{j_r}) = \min_{j \in U_i^* - (U_0 \cup U_1 \cup \cdots \cup U_{r-1})} d(u_i, u_j)\}$; where $U_0 = \emptyset$.

$E_r = \{l_r|d(e_k, e_{l_r}) = \min_{l \in E_k^* - (E_0 \cup E_1 \cup \cdots \cup E_{r-1})} d(e_k, e_l)\}$; where $E_0 = \emptyset$.

First we compute the object part:

1. Input the incomplete soft set (F, E).
2. Find u_i such that u_{ik} is unknown.
3. Compute $d(u_i, u_j)$ for all $j \in U_i^*$.
4. Let $\bar{u}_{ik}^{1,obj} = \frac{\sum_{j_1 \in U_1} u_{j_1 k}}{|U_1|}$.
5. Compute degree of interdependence between u_i and u_{j_1}, which is given by $\alpha_{ij_1} = \frac{1}{1+d(u_i,u_{j_1})}$ where $j_1 \in U_1$.
6. Define $u_{ik}^{1,obj} = \bar{u}_{ik}^{1,obj} \times \alpha_{ij_1}$.
7. Let $\bar{u}_{ik}^{2,obj} = \frac{\sum_{j_2 \in U_2} u_{j_2 k}}{|U_2|}$.
8. Compute degree of interdependence between u_i and u_{j_2}, which is given by $\alpha_{ij_2} = \frac{1}{1+d(u_i,u_{j_2})}$ where $j_2 \in U_2$.
9. Define $u_{ik}^{2,obj} = \bar{u}_{ik}^{2,obj} \times \alpha_{ij_2}$.
10. Continue in this way until $U_1 \cup U_2 \cup \cdots \cup U_t = U_i^*$.
11. Hence object part of unknown value $u_{ik}^{obj} = \frac{\sum_{r=1}^{t} u_{ik}^{r,obj}}{\sum_{r=1}^{t} \alpha_{ij_r}}$.

Now we compute the parameter part:

1. Input the incomplete soft set (F, E).
2. Find e_k such that u_{ik} is unknown.
3. Compute $d(e_k, e_l)$ for all $l \in E_k^*$.
4. Let $\bar{u}_{ik}^{1,par} = \frac{\sum_{l_1 \in E_1} u_{il_1}}{|E_1|}$.
5. Compute degree of interdependence between e_k and e_{l_1}: $\beta_{kl_1} = \frac{1}{1+d(e_k,e_{l_1})}$ for $l_1 \in E_1$.
6. Define $u_{ik}^{1,par} = \bar{u}_{ik}^{1,par} \times \beta_{kl_1}$.
7. Let $\bar{u}_{ik}^{2,par} = \frac{\sum_{l_2 \in E_2} u_{il_2}}{|E_2|}$.
8. Compute degree of interdependence between u_k and u_{l_2}: $\beta_{kl_2} = \frac{1}{1+d(e_k,e_{l_2})}$ for $l_2 \in E_2$.

https://doi.org/10.17993/3cemp.2022.110250.104-114

9. Define $u_{ik}^{2,par} = \bar{u}_{ik}^{2,par} \times \beta_{kl_2}$.
10. Continue in this way until $E_1 \cup E_2 \cup \cdots \cup E_t = E_k^*$.
11. Hence parameter part of unknown value is $u_{ik}^{par} = \frac{\sum_{r=1}^{t} u_{ik}^{r,par}}{\sum_{r=1}^{t} \beta_{kl_r}}$.

Now the unknown value u_{ik} of a fuzzy soft set can be predicted by the equation

$$u_{ik} = w_1.u_{ik}^{obj} + w_2.u_{ik}^{par} \tag{3}$$

where w_1 and w_2 are weights of the objects and parameters measuring the impact on unknown data, respectively. The weights can be assigned according to the given problem. If the objects and parameters are treated equally, the weights can be set as $w_1 = w_2 = \frac{1}{2}$.

In case u_{ik} is an unknown value of a soft set then we compute $h_{ik} = w_1.u_{ik}^{obj} + w_2.u_{ik}^{par}$ as above. If $h_{ik} < \frac{1}{2}$, put $u_{ik} = 0$ and if $h_{ik} \geq \frac{1}{2}$, put $u_{ik} = 1$.

3.3 APPLICATION OF ALGORITHM FOR INCOMPLETE SOFT SET

Consider the incomplete soft set represented in tabel 1. In this table there are eight objects, six parameters and four unknown values to be predicted. Suppose that the weights of objects and parameters be equal, i.e., $w_1 = w_2 = \frac{1}{2}$. By using our algorithm we compute $h_{43} = 0.5628$, $h_{46} = 0.2682$, $h_{65} = 0.4903$, $h_{72} = 0.4334$. Since $h_{43} > \frac{1}{2}$, $h_{46} < \frac{1}{2}$, $h_{65} < \frac{1}{2}$ and $h_{72} < \frac{1}{2}$, therefore we obtain $h_{43} = 1$, $h_{46} = 0$, $h_{65} = 0$, $h_{72} = 0$.

3.4 APPLICATION OF ALGORITHM FOR INCOMPLETE FUZZY SOFT SET

Consider the incomplete fuzzy soft set represented in table 2. In this table there are six objects, seven parameters and five unknown values to be predicted. Here also we suppose that the weights of objects and parameters are equal, i.e., $w_1 = w_2 = \frac{1}{2}$. By using our algorithm we obtain the unknown values as $u_{24} = 0.5825$, $u_{31} = 0.8815$, $u_{34} = 0.7910$, $u_{45} = 0.7202$, $u_{57} = 0.4575$.

4 ALGORITHM TO PREDICT MISSING DATA IN AN INCOMPLETE INTUITIONISTIC FUZZY SOFT SET AND ITS APPLICATION

4.1 A PREPARATORY STEP

To predict the unknown values of incomplete intuitionistic fuzzy sets we will construct four fuzzy soft sets from given intuitionistic fuzzy soft set as follows. We take an example given in table 3 to make it more clear. For this incomplete intuitionistic fuzzy soft set we construct four tables; first by using membership degrees (table 4), second by using non membership degrees (table 5), third by using the sum of membership and non membership degrees (table 6) and fourth by their differences (table 7).

Table 4 – Membership degrees

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	0.8	0.2	0.8	0.4	0.4	0.6
u_2	0.8	0.8	0.5	0.6	0.5	0.6
u_3	0.7	0.3	0.8	$*$	0.6	0.4
u_4	0.6	0.7	0.7	0.4	0.7	0.6
u_5	0.5	0.6	0.4	0.7	0.8	$*$
u_6	0.2	0.4	0.5	0.4	0.4	0.5
u_7	0.7	0.8	0.5	0.9	0.5	0.4

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Table 5 – Non membership degrees

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	0.1	0.1	0.1	0.5	0.5	0.2
u_2	0.2	0.1	0.2	0.4	0.5	0.2
u_3	0.2	0.1	0.2	$*$	0.1	0.2
u_4	0.2	0.2	0.3	0.3	0.1	0.1
u_5	0.3	0.3	0.5	0.3	0.1	$*$
u_6	0.4	0.4	0.5	0.3	0.3	0.1
u_7	0.2	0.1	0.4	0.1	0.3	0.1

Table 6 – Sum of membership and non membership degrees

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	0.9	0.3	0.9	0.9	0.9	0.8
u_2	1	0.9	0.7	1	1	0.8
u_3	0.9	0.4	1	$*$	0.7	0.6
u_4	0.8	0.9	1	0.7	0.8	0.7
u_5	0.8	0.9	0.9	1	0.9	$*$
u_6	0.6	0.8	1	0.7	0.7	0.6
u_7	0.9	0.9	0.9	1	0.8	0.5

Table 7 – Difference of membership and non membership degrees

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	0.7	0.1	0.7	0.1	0.1	0.4
u_2	0.6	0.7	0.3	0.2	0	0.4
u_3	0.5	0.2	0.6	$*$	0.5	0.2
u_4	0.4	0.5	0.4	0.1	0.6	0.5
u_5	0.2	0.3	0.1	0.4	0.7	$*$
u_6	0.2	0	0	0.1	0.1	0.4
u_7	0.5	0.7	0.1	0.8	0.2	0.3

4.2 ALGORITHM

Suppose we have to predict the unknown value of $(\mu_{F(e_k)}(u_i), \nu_{F(e_k)}(u_i)) = (u_{ik}, v_{ik})$. Let m_{ik}, n_{ik}, s_{ik} and t_{ik} denote the corresponding unknown values of fuzzy soft set of membership degrees, fuzzy soft set of non membership degrees, fuzzy soft set of sum of membership and non membership degrees and fuzzy soft set of difference of membership and non membership degrees respectively.

1. Input the incomplete intuitionistic fuzzy soft set.
2. Compute $s_{ik} = u_{ik} + v_{ik}$ using algorithm 3.2.
3. Compute $t_{ik} = |u_{ik} - v_{ik}|$ using algorithm 3.2.
4. Compute m_{ik} and n_{ik} using algorithm 3.2.
5. If $m_{ik} > n_{ik}$, put $|u_{ik} - v_{ik}| = u_{ik} - v_{ik}$ otherwise put $|u_{ik} - v_{ik}| = v_{ik} - u_{ik}$. Accordingly we get $t_{ik} = u_{ik} - v_{ik}$ or $t_{ik} = v_{ik} - u_{ik}$.
6. Solve equations obtained from step (ii) and step (v) to get the values of u_{ik} and v_{ik}.

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4.3 APPLICATION

Consider the incomplete intuitionistic fuzzy soft set given in table 3. In this table there are seven objects, six parameters and two unknown values to be predicted. Here also we suppose that the weights of objects and parameters are equal, i.e., $w_1 = w_2 = \frac{1}{2}$. Now we predict the two unknown values (u_{34}, v_{34}) and (u_{56}, v_{56}) as follows:

For (u_{34}, v_{34}) we obtain $m_{34} = 0.5609$, $n_{34} = 0.2360$, $s_{34} = 0.8086$ and $t_{34} = 0.3331$. Using algorithm 4.2 we get $u_{34} = 0.5709$ and $v_{34} = 0.2377$.

For (u_{56}, v_{56}) we obtain $m_{56} = 0.5598$, $n_{56} = 0.2273$, $s_{56} = 0.7804$ and $t_{56} = 0.3523$. Using algorithm 4.2 we get $u_{56} = 0.5664$ and $v_{56} = 0.2140$.

5 CONCLUSION

This paper analyzes the effect of known data on unknown ones in an incomplete data set and proposes algorithms to predict unknown values. The concept of Euclidean distance on $\mathbb{R}^n$ is used to measure the distance between objects (parameters). This distance is further used in measuring the degree of interdependence between objects (parameters). An approach to predict the missing data in an intuitionistic fuzzy set is also given.

Our proposed methodology has the following advantages:

1. There is only one basic algorithm (algorithm 3.2) given in this paper which is used to predict the missing data in each of incomplete soft set, incomplete fuzzy soft set and incomplete intuitionistic fuzzy soft set.

2. Algorithms given in this paper makes full use of known data so that the predicted values have higher accuracy.

3. The basic algorithm 3.2 produces a finite sequence of predictions based on the distance and degree of interdependence between objects (parameters).

4. In this paper the relation between objects (parameters) is determined using degree of interdependence. If the degree of interdependence between an object (parameter) and the target object (parameter) is less, then the missing values corresponding to the target object (parameter) is less expected to be same as the corresponding values of former object (parameter).

5. The algorithm 3.2 predicts the unknown values of incomplete soft set to be in $\{0, 1\}$ precisely.

6. Algorithm 4.2 predicts the unknown values of incomplete intuitionistic fuzzy soft set.

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$MAP/PH/1$ QUEUE WITH DISCARDING CUSTOMERS HAVING IMPERFECT SERVICE

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ABSTRACT

In this paper, we consider two queueing models. Model I is on a single-server queueing system in which the arrival process follows MAP with representation $D = (D_0, D_1)$ *of order* m *and service time follows phase-type distribution* $(\boldsymbol{\beta}, S)$ *of order* n*. When a customer enters into service, a generalized Erlang clock is started simultaneously. The clock has* k *stages. The* p^{th} *stage parameter is* θ_p *for* $1 \leq p \leq k$*. If a customer completes the service in between the realizations of stages* k_1 *and* k_2 $(1 < k_1 < k_2 < k)$ *of the clock, it is a perfect one. On the other hand, if the service gets completed either before the* k_1^{th} *stage realization or after the* k_2^{th} *stage realization, it is discarded because of imperfection. We analyse this model using the matrix-geometric method. We obtain the expected service time and expected waiting time of a tagged customer. Additional performance measures are also computed. We construct a revenue function and numerically analyse it. In Model II, a single server queueing system in which all assumptions are the same as in Model I except the assumption on service time, is considered. Up to stage* k_1 *service time follows phase-type distribution* $(\boldsymbol{\alpha}', T')$ *of order* n_1 *and beyond stage* k_1*, the service time follows phase type distribution* $(\boldsymbol{\beta}', S')$ *of order* n_2*. We compare the values of the revenue function of the two models*

KEYWORDS

Markovian Arrival Process, Phase-type distribution, Erlang Clock, Imperfect Service.

1 INTRODUCTION

Queueing models play an important role in our everyday life. Important application areas of queueing models are production systems, transportation and stocking systems, communication systems, information processing systems, etc. In a manufacturing system, a product goes through several stages to getting processed; the processing time of a product is very important.

Phase type distribution was introduced by Neuts (1975) as a generalization of the exponential distribution . Phase type distribution is defined as the distribution of time to absorption of a Markov chain with finite transient states and one absorbing state. Let $\bar{X} = \{X(t) : t \geq 0\}$ denote a continuous time Markov chain with state space $S = \{1, 2, 3, \ldots, m, m+1\}$ where the first m states are transient and the last state is absorbing and with infinitesimal generator matrix

$\tilde{Q} = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$, where T is a square matrix of order m and T^0 is a column vector and $T^0 = -T\boldsymbol{e}$. The initial probability distribution of $\bar{X}$ is $\bar{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, \alpha_{m+1})$ where $\boldsymbol{\alpha}$ is a row vector of dimension m and $\alpha_{m+1} = 1 - \boldsymbol{\alpha e}$. Let $Z = inf\{t \geq 0 : X(t) = m+1\}$ be a random variable of time until absorption in state $m+1$. The distribution of Z is called a continuous phase-type distribution (PH distribution) with parameter $(\boldsymbol{\alpha}, T)$. The distribution function of a continuous phase type distribution $PH(\boldsymbol{\alpha}, T)$ is given by $F(t) = 1 - \boldsymbol{\alpha} e^{Tt}\boldsymbol{e}$ for $t \geq 0$ and probability density function is $f(t) = \boldsymbol{\alpha} e^{Tt} T^0$ for $t \geq 0$. The Laplace Stieltjes transform of $PH(\boldsymbol{\alpha}, T)$ is given by $\phi(s) = \alpha_{m+1} + \boldsymbol{\alpha}(sI - T)^{-1} T^0$ for all $s \in C$ with $Re(s) \geq 0$.

The Markovian Arrival Process(MAP) was introduced by David M. Lucantoni (1990) as a simpler version of an earlier model proposed by Neuts (1979). It is a generalization of the Markov process where arrivals are governed by an underlying m-states Markov chain. A continuous time Markov chain $\{(N(t), J(t)) : t \geq 0\}$ with state space $\{(i, j) : i = 0, 1, 2, \ldots; 1 \leq j \leq m\}$ and infinitesimal generator matrix

$$\bar{Q} = \begin{bmatrix} D_0 & D_1 & & & \\ & D_0 & D_1 & & \\ & & D_0 & D_1 & \\ & & & \ddots & \ddots \end{bmatrix}$$ is called a MAP with matrix representation (D_0, D_1).

D_0 and D_1 are square matrices of order m. $N(t)$ counts the number of arrivals during $(0, t)$ and $J(t)$ represents the phase of the arrival process. D_0 has negative diagonal elements and non-negative off-diagonal elements, and its elements correspond to state transition without an arrival. D_1 is a non-negative matrix whose elements represent state transition with one arrival. Let the matrix D be defined as $D = D_0 + D_1$. Then D is an irreducible infinitesimal generator of the underlying Markov chain $\{J(t)\}$. Let $\boldsymbol{\pi}$ be the invariant probability vector of D, then
$\boldsymbol{\pi} D = 0, \boldsymbol{\pi e} = 1$. The average rate of events in a MAP, which is called the fundamental rate of the MAP, is given by $\lambda = \boldsymbol{\pi} D_1 \boldsymbol{e}$.

The arrival of a negative customer to a queueing system causes the removal of one ordinary customer (called a positive customer) who is present in the queue. But the Negative arrivals have no effect if the system is empty. We can therefore represent a Negative customer as a type of work canceling signal. Queues with negative arrivals were first introduced by Gelenbe (1991a). So queues with negative arrivals are called G-queues. Those who are interested in a comprehensive analysis of G-queues may refer to Gelenbe et al. (1991b), Artalejo (2000), and Bocharov and Vishnevskii (2003).

Valentina Klimenok and Alexander Dudin (2012) consider a multi-server queueing system with finite and infinite buffers. The input flow is described by Batch Markovian Arrival Process(BMAP) and the service time has the PH distribution. Besides positive customers, the negative customers arrive according to the Markovian Arrival Process. A negative customer can remove an ordinary customer in service if the state service process does not belong to protected phases.

S R Chakravarthy (2009) has considered a single server queueing system in which arrivals occur according to a Markovian arrival process. All the customers in the system are lost when the system undergoes disastrous failures. In G-queues a regular customer is pushed out of the system by a negative customer. But here we consider a queueing system in which a customer is discarded if his service completion is not within a stipulated time interval.

The queueing models considered so far in the literature did not look at the possibility of service completion of customers before a threshold or beyond a second threshold. Several real-life situations warrant the completion of services between the lower and upper thresholds. This is necessitated by the fact that the raw material used for the production of a specified item may not get completely processed if completed before time. Similarly, it could get over-processed if the processing completion time gets beyond a threshold. The subject matter of this paper addresses this important aspect in production and manufacturing.

This Queueing model can be applied in various fields in our day-to-day life. For example, in a food manufacturing unit, the correct baking time of a product is a crucial factor. If the baking time exceeds a threshold, the product gets burnt. On the other hand, if the baking time is not sufficient, the product will only be half cooked and will not be acceptable.

Another example is the manufacturing of Nylon wires and films. In the manufacture of nylon, caprolactam (a chemical used as raw material), is melted and the molten caprolactam is catalytically polymerized at previously optimized conditions of temperature, pressure, the concentration of the catalyst, etc. Further, the output of the above process is subjected to another process like extrusion or calendering. Extrusion is used to produce nylon wires, whereas calendering is used to produce nylon films. The condition of this is also an optimized one, in which any variation will cause defective wires and films which will not be suitable for end-use. The condition is optimized based on laboratory and pilot plant situations.

In this paper, we first consider a single-server queueing system in which the arrival process follows MAP and service time follows the continuous phase-type distribution. When a customer enters into service, a generalized Erlang clock is started simultaneously. The clock has k stages. The p^{th} stage parameter is θ_p for $1 \leq p \leq k$. If a customer completes the service in between the realizations of stages k_1 and k_2 $(1 < k_1 < k_2 < k)$ of the clock, the final product is perfect. If it gets completed either before the k_1^{th} stage realization or after the k_2^{th} stage realization, it has to be discarded.

Salient features of this paper are

- it deviates from the classical assumption of merely specifying a service time distribution.
- the lower and upper thresholds for service are the most important additions.
- When a customer enters into service, a generalized Erlang clock is started simultaneously.
- If a customer completes service in between the realizations of stages k_1 and k_2 $(1 < k_1 < k_2 < k)$ of the Erlang clock, it is perfect.
- If a customer completes the service either before the k_1^{th} stage realization or after the k_2^{th} stage realization, it is discarded.
- To maximise revenue, in Model II we consider the service time as phase-type distributed with representation $(\boldsymbol{\gamma}', L)$ of order $n = n_1 + n_2$, which is the convolution of the two phase type distributions $(\boldsymbol{\alpha}', T')$ of order n_1 and $(\boldsymbol{\beta}', S')$ of order n_2.

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Notations and abbreviations used

- $LIQBD$: Level independent Quasi-Birth and Death.
- MAP: Markovian Arrival Process.
- $CTMC$: Continuous time Markov chain.
- I_P: Identity matrix of order P.
- $\boldsymbol{e_a}$: Column vector of $1'$s of order a.
- $\boldsymbol{e}$: Column vector of $1'$s of appropriate order.
- $\boldsymbol{x}'$: Transpose of a vector $\boldsymbol{x}$.

The remaining part of this paper is organized as follows. In section 2 the model under study is mathematically formulated. In section 3 we perform the steady-state analysis of the queueing model. Service time analysis and waiting time analysis of a customer are discussed in sections 4 and 5 respectively. Some additional performance measures are provided in section 6. A revenue function is discussed in section 7. Model description and mathematical formulation of model 2 are given in section 8. In section 9 we perform the steady state analysis of model 2. Numerical results are discussed in section 10.

2 Mathematical formulation of Model I

We consider a single-server queueing system in which the arrival process follows MAP with representation $D = (D_0, D_1)$ of order m and service time follows continuous phase-type distribution $(\boldsymbol{\beta}, S)$ of order n. When a customer enters into service, a generalized Erlang clock is started simultaneously. The clock has k stages. The p^{th} stage parameter is θ_p for $1 \leq p \leq k$. If a customer completes the service in between the realizations of stages k_1 and k_2 $(1 < k_1 < k_2 < k)$ of the clock, it is perfect. If a customer completes the service either before the k_1^{th} stage realization or after the k_2^{th} stage realization, it is discarded. The expected service rate is $\mu = [\boldsymbol{\beta}(-S)^{-1}\mathbf{e}]^{-1}$. Let $D = D_0 + D_1$ be the infinitesimal generator matrix of the arrival process and $\boldsymbol{\delta}$ be its stationary probability vector, then $\boldsymbol{\delta} D = 0, \boldsymbol{\delta}\mathbf{e} = 1$. The constant $\lambda = \boldsymbol{\delta} D_1 \mathbf{e}$ referred to as the fundamental rate, gives the expected number of arrivals per unit of time.

2.1 The QBD process

The model described in section 1 can be studied as a LIQBD process. First, we define the following notations:

$N(t)$: number of customers in the system at time t,

$J(t) = j$, if the Erlang clock is in the jth stage at time t, $j = 1, 2, ..., k_2$,

$I_s(t)$: the phase of service process at time t,

$I_a(t)$: the phase of arrival process at time t,

$(N(t), J(t), I_s(t), I_a(t) : t \geq 0\}$ is a LIQBD with state space

$\Omega = \{\{(0, j)/1 \leq j \leq m\} \bigcup \{(q, p, i, j)/q \geq 1, 1 \leq p \leq k_2, 1 \leq i \leq n, 1 \leq j \leq m\}\}$

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The infinitesimal generator of this CTMC is

$$\mathcal{Q}^* = \begin{bmatrix} B_1 & B_0 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Here B_1 is an $m \times m$ matrix which contains the transition within the level 0; B_0 is an $m \times k_2mn$ matrix which contains transitions from level 0 to level 1; B_2 is a $k_2mn \times m$ matrix that contains transitions from level 1 to level 0; A_0 represents transitions from level q to level $q+1$ for $q \geq 1$, A_1 represents transitions within the level q for $q \geq 1$ and A_2 represents transitions from level q to $q-1$ for $q \geq 2$. All these are square matrices of order $k_2mn \times k_2mn$.

$$B_1 = D_0$$

$$B_0 = \begin{bmatrix} \boldsymbol{\beta} \otimes D_1 & \mathbf{0} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} S^0 \otimes I_m \\ S^0 \otimes I_m \\ S^0 \otimes I_m \\ \vdots \\ (S^0 + e_n\theta_{k_2}) \otimes I_m \end{bmatrix}$$

$$A_1 = \begin{bmatrix} C_1 & I_{mn}\theta_1 & & & & & & & & \\ & C_2 & I_{mn}\theta_1 & & & & & & & \\ & & C_3 & I_{mn}\theta_3 & & & & & & \\ & & & \ddots & \ddots & & & & & \\ & & & & C_{k_1} & I_{mn}\theta_{k_1} & & & & \\ & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & C_{k_2-1} & I_{mn}\theta_{k_2-1} \\ & & & & & & & & C_{k_2} \end{bmatrix}$$

where $C_h = S \otimes I_m + I_n \otimes D_0 - Imn\theta_h, 1 \leq h \leq k_2$

$$A_2 = \begin{bmatrix} S^0\boldsymbol{\beta} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ S^0\boldsymbol{\beta} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ S^0\boldsymbol{\beta} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \vdots & \vdots & \vdots & & \\ (S^0 + e_n\theta_{k_2})\boldsymbol{\beta} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \end{bmatrix}$$

$$A_0 = \begin{bmatrix} I_n \otimes D_1 & & & & & & \\ & I_n \otimes D_1 & & & & & \\ & & I_n \otimes D_1 & & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & & & I_n \otimes D_1 \end{bmatrix}$$

3 Steady State Analysis

In this section, we perform the steady state analysis of the queueing model under study by first establishing the stability condition of the queueing system.

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3.1 Stability Condition

The generator matrix $A = A_0 + A_1 + A_2$

$$A=\begin{bmatrix} I_n\otimes D_1+C_1+S^0\boldsymbol{\beta}\otimes I_m & I_{mn}\theta_1 & & & & & \\ S^0\boldsymbol{\beta}\otimes I_m & I_n\otimes D_1+C_2 & I_{mn}\theta_2 & & & & \\ S^0\boldsymbol{\beta}\otimes I_m & \mathbf{0} & I_n\otimes D_1+C_3 & I_{mn}\theta_3 & & & \\ & & & \ddots & \ddots & & \\ S^0\boldsymbol{\beta}\otimes I_m & & & & I_n\otimes D_1+C_{k_1} & I_{mn}\theta_{k_1} & \\ & & & & \ddots & \ddots & \\ (S^0\boldsymbol{\beta}+\mathbf{e}\theta_{k_2}\boldsymbol{\beta})\otimes I_m & & & & & & I_n\otimes D_1+C_{k_2} \end{bmatrix}.$$

Let $\boldsymbol{\pi} = (\boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \boldsymbol{\pi_3}, .., \boldsymbol{\pi_{k_1}}, ..., \boldsymbol{\pi_{k_2}})$ denote the steady state probability vector of the generator matrix A.

Here $O(\boldsymbol{\pi}) = 1 \times k_2mn$ and the $O(\boldsymbol{\pi_r}) = 1 \times nm$ for $1 \le r \le k_2$.
Steady state probability vector π satisfying the equations

.

$$\boldsymbol{\pi A} = 0, \boldsymbol{\pi}\mathbf{e} = 1. \tag{1}$$

Using equation (1), we get

$$\boldsymbol{\pi_1}[I_n\otimes D_1+C_1+S^0\boldsymbol{\beta}\otimes I_m]+(\boldsymbol{\pi_2}+\boldsymbol{\pi_3}+\boldsymbol{\pi_4}+.....+\boldsymbol{\pi_{k_1}}+.....+\boldsymbol{\pi_{k_2-1}})[S^0\boldsymbol{\beta}\otimes I_m]+\boldsymbol{\pi_{k_2}}[(S^0\boldsymbol{\beta}+\mathbf{e}\theta_{k_2}\boldsymbol{\beta})\otimes I_m] = \mathbf{0} \tag{2}$$

$$\boldsymbol{\pi_1} I_{mn}\theta_1 + \boldsymbol{\pi_2}[I_n \otimes D_1 + C_2] = \mathbf{0} \tag{3}$$

$$\boldsymbol{\pi_2} I_{mn}\theta_2 + \boldsymbol{\pi_3}[I_n \otimes D_1 + C_3] = \mathbf{0} \tag{4}$$

$$\boldsymbol{\pi_3} I_{mn}\theta_3 + \boldsymbol{\pi_4}[I_n \otimes D_1 + C_4] = \mathbf{0} \tag{5}$$

$$\boldsymbol{\pi_{k_1}} I_{mn}\theta_{k_1} + \boldsymbol{\pi_{k_1+1}}[I_n \otimes D_1 + C_{k_1+1}] = \mathbf{0} \tag{6}$$

$$\boldsymbol{\pi_{k_2-1}} I_{mn}\theta_{k_2-1} + \boldsymbol{\pi_{k_2}}[I_n \otimes D_1 + C_{k_2}] = \mathbf{0} \tag{7}$$

$$\boldsymbol{\pi_1} \times \mathbf{e} + \boldsymbol{\pi_2} \times \mathbf{e} + ... + \boldsymbol{\pi_{k_1}} \times \mathbf{e} + ... + +\boldsymbol{\pi_{k_2}} \times \mathbf{e} = 1 \tag{8}$$

From equation (7);

$$\boldsymbol{\pi_{k_2-1}} = -\boldsymbol{\pi_{k_2}}[I_n \otimes D_1 + C_{k_2}]\frac{1}{\theta_{k_2-1}} I_{mn} \tag{9}$$

By back substitution and using equation (8) we get all the values of $\boldsymbol{\pi_r}'s$. Thus we get the steady-state probability vector of A.

The $LIQBD$ description of the model indicates that the queueing system is stable if and only if the left drift exceeds that of the right drift. That is,

$$\boldsymbol{\pi} A_0\mathbf{e} < \boldsymbol{\pi} A_2\mathbf{e}. \tag{10}$$

$$\boldsymbol{\pi} A_0\mathbf{e} = \boldsymbol{\pi_1}(I_n\otimes D_1)\mathbf{e}+\boldsymbol{\pi_2}(I_n\otimes D_1)\mathbf{e}+........+\boldsymbol{\pi_{k_1}}(I_n\otimes D_1)\mathbf{e}+.......+\boldsymbol{\pi_{k_2}}(I_n\otimes D_1)\mathbf{e} = \sum_{r=1}^{k_2}\pi_r(I_n\otimes D_1)\mathbf{e} \tag{11}$$

$$\begin{aligned}\boldsymbol{\pi} A_2\mathbf{e} &= \boldsymbol{\pi_1}[S^0\boldsymbol{\beta}\otimes I_m]\mathbf{e}+\boldsymbol{\pi_2}[S^0\boldsymbol{\beta}\otimes I_m]\mathbf{e}+.......+\boldsymbol{\pi_{k_1}}[S^0\boldsymbol{\beta}\otimes I_m]\mathbf{e}+........+\boldsymbol{\pi_{k_2}}[(S^0+e_n\theta_{k_2})\boldsymbol{\beta}\otimes I_m]\mathbf{e} \\ &= \sum_{r=1}^{k_2-1}\boldsymbol{\pi_r}[S^0\boldsymbol{\beta}\otimes I_m]+\boldsymbol{\pi_{k_2}}[(S^0+e_n\theta_{k_2})\boldsymbol{\beta}\otimes I_m]\mathbf{e}\end{aligned} \tag{12}$$

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Therefore the stability condition is

$$\sum_{r=1}^{k_2} \boldsymbol{\pi_r}(I_n \otimes D_1)\mathbf{e} < \sum_{r=1}^{k_2-1} \boldsymbol{\pi_r}[S^0\boldsymbol{\beta} \otimes I_m] + \boldsymbol{\pi_{k_2}}[(S^0 + e_n\theta_{k_2})\boldsymbol{\beta} \otimes I_m]\mathbf{e} \tag{13}$$

3.2 The Steady State Probability Vector of Q

Let $\boldsymbol{x}$ be the steady state probability vector of Q.

$\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2}\ldots)$, where $\boldsymbol{x_0}$ is of dimension $1 \times m$ and $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots$ are each of dimension $1 \times k_2mn$.

Under the stability condition, we have $\boldsymbol{x_i} = \boldsymbol{x}_1 R^{i-1}, i \geq 2$, where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2A_2 + RA_1 + A_0 = 0$$

and the vectors $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$are obtained by solving the equations

$$\boldsymbol{x_0}B_1 + \boldsymbol{x_1}B_2 = 0 \tag{14}$$

$$\boldsymbol{x_0}B_0 + \boldsymbol{x_1}(A_1 + RA_2) = 0 \tag{15}$$

subject to the normalizing condition

$$\boldsymbol{x_0 e} + \boldsymbol{x_1}(I - R)^{-1}\boldsymbol{e} = 1 \tag{16}$$

Solving equations (15),(16) and(17), we get $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$. Hence we can find all $\boldsymbol{x_i}$'s.

4 Analysis of Service Time of a Customer

We consider a Markov Process $Y(t) = \{(J(t), I_s(t)) : t \geq 0\}$ where
$J(t)= j$, if the Erlang clock is in the jth stage at time t, $j = 1, 2, ..., k_2$.
$I_s(t)$: the phase of service process at time t

The state space of this process is
$\Omega_1 = \{1, 2, ..k_1, ...k_2\} \times \{1, 2, 3, ..., n\} \bigcup\{\Delta_1\} \bigcup\{\Delta_2\}$,where Δ_1 and Δ_2 denote the absorbing states. Δ_1 denotes the absorption occur due to service completion and Δ_2 denotes absorption occur due to realization of k_2^{th} stage of the Erlang clock.

The infinitesimal generator matrix is

$$Q_1 = \begin{bmatrix} S-\theta_1 I & \theta_1 I & & & & & S^0 & \mathbf{0} \\ & S-\theta_2 I & \theta_2 I & & & & S^0 & \mathbf{0} \\ & & \ddots & \ddots & & & & \\ & & & S-\theta_{k_1} I & \theta_{k_1} I & & & \\ & & & & \ddots & \ddots & & \\ & & & & & S-\theta_{k_2} I & S^0 & \mathbf{e}\theta_{k_2} \end{bmatrix}.$$

$$\text{where } S_1 = \begin{bmatrix} S-\theta_1 I & \theta_1 I & & & & \\ & S-\theta_2 I & \theta_2 I & & & \\ & & \ddots & \ddots & & \\ & & & S-\theta_{k_1} I & \theta_{k_1} I & \\ & & & & \ddots & \ddots \\ & & & & & S-\theta_{k_2} I \end{bmatrix}.$$

The initial probability vector is $\boldsymbol{\alpha} = (\boldsymbol{\beta}, \mathbf{0}, \mathbf{0}, ..., \mathbf{0})$

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The expected service time of a customer is the time until absorption of the above process which is given by $ES = \boldsymbol{\alpha}(-S_1^{-1})\mathbf{e}$

5 Waiting Time Analysis

To find the expected waiting time of a tagged customer who joins as the rth customer in system, we consider the Markov Processes
$W = \{W(t) : t \geq 0\} = \{(N(t), J(t), I_s(t)) : t \geq 0\}$ where

$N(t)$-Rank of the customer in the system at time t

$J(t)= j$, if the Erlang clockis in the jth stage at time t, $j = 1, 2, ..., k_2$.

$I_s(t)$ - Phase of the service at time t

The rank of the customer decrease by one when a customer ahead of him completes the service. The rank of the customer is assumed to be r if he joins as the rth customer in the system. State-space of $W(t)$ is $\Omega_2 = \{\{r, r-1, r-2, \cdots, 2\} \times \{1, 2, 3, ...k_2\} \times \{1, 2, 3, ..., n\}\} \cup \{\Delta^*\}$
where Δ^* denotes the absorbing state. That is Δ^* denotes the state that the tagged customer selected for service.

The infinitesimal generator is

$$\mathbf{W} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & & & & \\ T^0 & T & & & & \\ & T^0\boldsymbol{\beta} & T & & & \\ & & T^0\boldsymbol{\beta} & T & & \\ & & & \ddots & \ddots & \end{bmatrix}.$$

$$\text{where } T = \begin{bmatrix} S-\theta_1 I & \theta_1 I & & & & & \\ & S-\theta_2 I & \theta_2 I & & & & \\ & & \ddots & \ddots & & & \\ & & & S-\theta_{k_1} I & \theta_{k_1} I & & \\ & & & & \ddots & \ddots & \\ & & & & & & S-\theta_{k_2} I \end{bmatrix}.$$

$$T^0 = \begin{bmatrix} S^0 & \mathbf{0} & \mathbf{0} \\ S^0 & \mathbf{0} & \mathbf{0} \\ S^0 & \mathbf{0} & \mathbf{0} \\ \vdots & & \\ S^0 + \mathbf{e}\theta_{k_2} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Let y_{rpi} be the steady-state probability that an arriving customer finds the server in busy with current service phase i, Erlang clock is in p^{th} level and the number of customers in the system including the current arrival tobe r for $1 \leq p \leq k_2$ and $1 \leq i \leq n$

Let $y_r = (y_{r11}, y_{r12}, , ...y_{r1n}, y_{r21}, y_{r22}, ...y_{r2n}, ..., y_{rk_21}, y_{rk_22}, ...y_{rk_2n})$
and $y = (0, y_2, y_3.....y_r)$

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Then $y_r = x_{r-1}(I \otimes \frac{D_1}{\lambda}), r \geq 2$

Waiting time is the time until absorption of the Markov chain is given by Ω_2. Let $W(s)$ denote the Laplace Stieltjes Transform (LST) of waiting time in the queue of an arrival.

Theorem 1. *The LST of the waiting time distribution of an arriving customer is* $W(s) = c\sum_{r=2}^{\infty} y_r(sI-T)^{-1}T^0[\boldsymbol{\beta}(sI-T)^{-1}T^0]^{r-2}, Re(s) \geq 0$, *where the normalising constant* c *is given by* $c = [\sum_{r=2}^{\infty} y_r e]^{-1}$

6 Additional Performance Measures

- probability that the system is empty:
$$P_0 = \boldsymbol{x_0}\mathbf{e}.$$
- Probability that q customers in the system:
$$P_q = \boldsymbol{x_q}\mathbf{e}.$$
- Probability that the server is busy:
$$P_{busy} = \sum_{q=1}^{\infty}\sum_{p=1}^{k_2}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{qpij}}.$$
- Expected number of customers in the queue:
$$ECQ = \sum_{q=1}^{\infty}(q-1)\boldsymbol{x_q}\boldsymbol{e}.$$
- Expected number of customers in the system:
$$ECS = \sum_{q=0}^{\infty} q\boldsymbol{x_q}\boldsymbol{e}.$$
.
- Rate at which customers discarded before k_1^{th} stage realization of Erlang clock
$$RK_1 = \sum_{q=1}^{\infty}\sum_{p=1}^{k_1}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{qpij}} S^0\mathbf{e}.$$
- Rate at which customers discard after k_2^{th} stage realization of Erlang clock
$$RK_2 = \sum_{q=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{qk_2ij}}\theta_{k_2}.$$
- Rate at which customers depart with successful completion of service
$$RP = \sum_{q=1}^{\infty}\sum_{p=k_1+1}^{k_2}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{qpij}} S^0 e.$$

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7 Revenue Function

Based on the above performance measures, we construct a revenue function as follows.
CK_1 - Unit time cost of service when customer discarded before the k_1^{th} stage realization of Erlang Clock.

CK_2 - Unit time cost of service when customer discarded after k_2^{th} stage realization of Erlang clock.

RS - Revenue per unit time for successful service.

Then the expected revenue per unit time, $ER = RP \times RS - RK_1 \times CK_1 - RK_2 \times CK_2$.

In this model, customers are discarded when either their service completes before reachig the stage k_1 or goes beyond the stage k_2. To minimise the rate of discarding customers before reaching stage k_1, we have to slow down the service rate up to k_1^{th} stage realization of Erlang clock so as to get the service cross the stage k_1. Similarly to minimse the rate of discarding customers after k_2^{th} stage, we have to increase the service rate beyond k_1^{th} stage realization of the Erlang clock to get the service completed before crossing the boundary k_2. Accordingly, we can reduce the loss to the system due to imperfect service. The extra cost involved while increasing the service rate beyond k_1 gets compensated through slow down of service rate up to the stage k_1, and also through reduced imperfect service.

Next, we proceed to the analysis of Model II.

8 Model II

8.1 Model description and Mathematical Formulation

We consider a single server queueing system in which all assumptions are exactly same as in Model I except the assumption on service time. Upto the stage k_1 service time follows phase-type distribution $(\boldsymbol{\alpha}', T')$ of order n_1 and beyond the stage k_1, the service time follows phase-type distribution $(\boldsymbol{\beta}', S')$ of order n_2. Therefore the entire service time follows phase-type distribution $(\boldsymbol{\gamma}', L)$ of order $n = n_1 + n_2$, which is the convolution of the two phase-type distributions $(\boldsymbol{\alpha}', T')$ of order n_1 and $(\boldsymbol{\beta}', S')$ of order n_2.

Then $\gamma' = (\boldsymbol{\alpha}', \alpha'_{n_1+1}\boldsymbol{\beta}') = (\boldsymbol{\alpha}', \mathbf{0})$, $L = \begin{bmatrix} T' & T'^0\boldsymbol{\beta}' \\ \mathbf{0} & S' \end{bmatrix}$.

Here we take $\alpha'_{n_1+1} = 0$ and $\beta'_{n_2+1} = 0$

The above described model can be studied as a LIQBD process.
Let

$N(t)$: Number of customers in the system at time t,

$J(t)= p$, if the Erlang clock is in the pth stage at time t, $p = 1, 2, ..., k_2$,

$I_s(t)$: the phase of service process at time t,

$I_a(t)$: the phase of arrival process at time t.

$(N(t), J(t), I_s(t), I_a(t) : t \geq 0\}$ is a LIQBD with state space

$\Omega_3 = \{\{(0, j)/1 \leq j \leq m\} \bigcup \{(q, p, i, j)/q \geq 1, 1 \leq p \leq k_1, 1 \leq i \leq n_1, 1 \leq j \leq m\} \bigcup \{(q, p, i, j)/q \geq 1, (k_1 + 1) \leq p \leq k_2, (n_1 + 1) \leq i \leq (n_1 + n_2), 1 \leq j \leq m\}\}$

The infinitesimal generator of this CTMC is

$$\mathcal{Q}^* = \begin{bmatrix} B_1^{'} & B_0^{'} & & & \\ B_2^{'} & A_1^{'} & A_0^{'} & & \\ & A_2^{'} & A_1^{'} & A_0^{'} & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Here $B_1^{'}$ is an $m \times m$ matrix that contains the transition within the level 0; $B_0^{'}$ is an $m \times [k_1 n_1 m + (k_2 - k_1)mn_2]$ matrix which contains transitions from level 0 to level 1; $B_2^{'}$ is a $[k_1 n_1 m + (k_2 - k_1)mn_2] \times m$ matrix which contains transitions from level 1 to level 0; $A_0^{'}$ represents transitions from level q to $q+1$ for $q \geq 1$, $A_1^{'}$ represents transitions within the level q for $q \geq 1$ and $A_2^{'}$ represents transitions from level q to level $q-1$ for $q \geq 2$. All these are square matrices of order $[k_1 n_1 m + (k_2 - k_1)mn_2]$.

$B_1 = B_1^{'} = D_0$

$$B_0^{'} = \begin{bmatrix} \boldsymbol{\alpha}^{'} \otimes D_1 & \mathbf{0} \end{bmatrix}$$

$$B_2^{'} = \begin{bmatrix} T^{'0} \otimes I_m \\ T^{'0} \otimes I_m \\ \vdots \\ T^{'0} \otimes I_m \\ S^{'0} \otimes I_m \\ S^{'0} \otimes I_m \\ \vdots \\ S^{'0} \otimes I_m \\ (S^{'0} + e_{n_2}\theta_{k_2}) \otimes I_m \end{bmatrix}$$

$$A_1^{'} = \begin{bmatrix} F_1 & I_{mn_1}\theta_1 & & & & & & & \\ & F_2 & I_{mn_1}\theta_1 & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & F_{k_1} & \theta_{k_1}\boldsymbol{\beta}^{'} \otimes I_m & & & & \\ & & & & E_{k_1+1} & I_{mn_2}\theta_{k_1+2} & & & \\ & & & & & E_{k_1+2} & I_{mn_2}\theta_{k_1+1} & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & & & E_{k_2} \end{bmatrix}$$

where $F_t = T^{'} \otimes I_m + I_{n_1} \otimes D_0 - Imn_1\theta_t, 1 \leq t \leq k_1$
$E_r = S^{'} \otimes I_m + I_{n_2} \otimes D_0 - Imn_2\theta_r, k_1 + 1 \leq r \leq k_2$

$$A_2^{'} = \begin{bmatrix} T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \vdots & \vdots & \vdots & & \\ T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ S^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ S^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \vdots & \vdots & \vdots & & \\ S^{'0}\boldsymbol{\alpha}^{'} + e_{n_2}\theta_{k_2}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \end{bmatrix}$$

 https://doi.org/10.17993/3cemp.2022.110250.116-137

$$A_0^{'} = \begin{bmatrix} I_{n_1} \otimes D_1 & & & & & & \\ & I_{n_1} \otimes D_1 & & & & & \\ & \ddots & \ddots & & & & \\ & & & I_{n_1} \otimes D_1 & & & \\ & & & & I_{n_2} \otimes D_1 & & \\ & & & & \ddots & \ddots & \\ & & & & & & I_{n_2} \otimes D_1 \end{bmatrix}$$

$$= \begin{bmatrix} I_{k_1} \otimes (I_{n_1} \otimes D_1) & \mathbf{0} \\ \mathbf{0} & I_{(k_2-k_1)} \otimes (I_{n_2} \otimes D_1) \end{bmatrix}$$

9 Steady State Analysis

In this section, we perform the steady state analysis of the queueing model under study by first establishing the stability condition of the queueing system.

9.1 Stability Condition

The generator matrix $A^{'} = A_0^{'} + A_1^{'} + A_2^{'}$

$$A^{'} = \begin{bmatrix} I_{n_1 \otimes D_1} + F_1 + T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & I_{mn_1}\theta_1 & & & & \\ T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & I_{n_1 \otimes D_1} + F_2 & I_{mn_1}\theta_2 & & & \\ & & \ddots & \ddots & & \\ T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & & I_{n_1 \otimes D_1} + F_{k_1} & \theta_{k_1}\boldsymbol{\beta}^{'} \otimes I_m & & \\ S^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & & & & I_{n_2} \otimes D_1 + E_{k_1+1} & I_{mn_2}\theta_{k_1+1} \\ & & & & \ddots & \ddots \\ [(S^{'0} + e_{n_2}\theta_{k_2})\boldsymbol{\alpha}^{'}] \otimes I_m & & & & & I_{n_2} \otimes D_1 + E_{k_2} \end{bmatrix}.$$

Let $\boldsymbol{\pi} = (\boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \boldsymbol{\pi_3}, .., \boldsymbol{\pi_{k_1}}, ..., \boldsymbol{\pi_{k_2}})$ denote the steady state probability vector of the generator matrix A

Here $O(\boldsymbol{\pi}) = 1 \times [k_1 n_1 m + (k_2 - k_1) n_2 m]$ and the $O(\boldsymbol{\pi_r}) = 1 \times n_1 m$ for $1 \leq r \leq k_1$ and $O(\boldsymbol{\pi_r}) = 1 \times n_2 m$ for $k_{1+1} \leq r \leq k_2$.

Steady state probability vector π satisfying the equations

.

$$\boldsymbol{\pi} \boldsymbol{A}^{'} = 0, \boldsymbol{\pi}\mathbf{e} = 1. \tag{17}$$

Using equation (18), we get

$$\begin{gathered} \boldsymbol{\pi_1}[I_{n1} \otimes D_1 + F_1 + T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m] + (\boldsymbol{\pi_2} + \boldsymbol{\pi_3} + \boldsymbol{\pi_4} + ... + \boldsymbol{\pi_{k_1}})[T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m] \\ +(\boldsymbol{\pi_{k_1+1}} + ... + \boldsymbol{\pi_{k_2-1}})S^{'0}\boldsymbol{\alpha}^{'} \otimes I_m + \pi_{k_2}(S^{'0} + e_{n_2}\theta_{k_2})\boldsymbol{\alpha}^{'} \otimes I_m) = \mathbf{0} \end{gathered} \tag{18}$$

$$\boldsymbol{\pi_1} I_{mn_1}\theta_1 + \boldsymbol{\pi_2}[I_{n_1} \otimes D_1 + F_2] = \mathbf{0} \tag{19}$$

$$\boldsymbol{\pi_2} I_{mn_1}\theta_2 + \boldsymbol{\pi_3}[I_{n_1} \otimes D_1 + F_3] = \mathbf{0} \tag{20}$$

$$\boldsymbol{\pi_3} I_{mn_1}\theta_3 + \boldsymbol{\pi_4}[I_{n1} \otimes D_1 + F_4] = \mathbf{0} \tag{21}$$

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$$\boldsymbol{\pi_{k1-1}} I_{mn_2}\theta_{k_1-1} + \boldsymbol{\pi_{k_1}}[I_{n1} \otimes D_1 + F_{k_1}] = \mathbf{0} \tag{22}$$

$$\boldsymbol{\pi_{k_1}}(\theta_{k_1}\boldsymbol{\beta}' \otimes I_m) + \boldsymbol{\pi_{k_1+1}}[I_{n_2} \otimes D_1 + E_{k_1+1}] = \mathbf{0} \tag{23}$$

$$\boldsymbol{\pi_{k_2-2}} I_{mn_2}\theta_{k_2-2} + \boldsymbol{\pi_{k_2-1}}[I_{n_2} \otimes D_1 + E_{k_{2-1}}] = \mathbf{0} \tag{24}$$

$$\boldsymbol{\pi_{k_2-1}} I_{mn_2}\theta_{k_2-1} + \boldsymbol{\pi_{k_2}}[I_{n_2} \otimes D_1 + E_{k_2}] = \mathbf{0} \tag{25}$$

$$\boldsymbol{\pi_1} \times \mathbf{e} + \boldsymbol{\pi_2} \times \mathbf{e} + ... + \boldsymbol{\pi_{k_1}} \times \mathbf{e} + ... + +\boldsymbol{\pi_{k_2}} \times \mathbf{e} = 1 \tag{26}$$

From equation (25);

$$\boldsymbol{\pi_{k_2-1}} = -\boldsymbol{\pi_{k_2}}[I_{n2} \otimes D_1 + E_{k_2}]\frac{1}{\theta_{k_2-1}} I_{mn_2} \tag{27}$$

By back substitution and using equation (26) we get all the values of $\boldsymbol{\pi_r}'s$.Thus we get the steady-state probability vector of A'.

The $LIQBD$ description of the model indicates that the queueing system is stable if and only if the left drift exceeds that of the right drift. That is,

$$\boldsymbol{\pi} A_0'\mathbf{e} < \boldsymbol{\pi} A_2'\mathbf{e}. \tag{28}$$

Therefore the stability condition is

$$\sum_{r=1}^{k_1} \boldsymbol{\pi_r}(I_{n_1} \otimes D_1)\mathbf{e} + \sum_{r=k_1+1}^{k_2} \boldsymbol{\pi_r}(I_{n2} \otimes D_1)\mathbf{e} < \sum_{r=1}^{k_1} \boldsymbol{\pi_r}[T'^0\boldsymbol{\alpha}' \otimes I_m]\mathbf{e} + \sum_{r=k_1+1}^{k_2} \boldsymbol{\pi_r}(S'^0\boldsymbol{\alpha}' \otimes I_m)\mathbf{e} + \boldsymbol{\pi_{k_2}}(e_{n_2}\theta_{k_2}\boldsymbol{\alpha}' \otimes I_m)\mathbf{e} \tag{29}$$

9.2 The Steady State Probability Vector of Q

Let $\boldsymbol{x}$ be the steady state probability vector of Q.

$\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2} \ldots)$, where $\boldsymbol{x_0}$ is of dimension $1 \times m$ and $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots$ are each of dimension $1 \times [k_1 m n_1 + (k_2 - k_1) n_2 m]$.

Under the stability condition, we have $\boldsymbol{x_i} = \boldsymbol{x}_1 R^{i-1}, i \geq 2$, where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = 0$$

and the vectors $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$are obtained by solving the equations

$$\boldsymbol{x_0} B_1 + \boldsymbol{x_1} B_2 = 0 \tag{30}$$
$$\boldsymbol{x_0} B_0 + \boldsymbol{x_1}(A_1 + R A_2) = 0 \tag{31}$$

subject to the normalizing condition

$$\boldsymbol{x_0}\boldsymbol{e} + \boldsymbol{x_1}(I - R)^{-1}\boldsymbol{e} = 1 \tag{32}$$

Solving equations (31), (32) and (33), we get $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$
Hence we can find all $\boldsymbol{x_i}$'s.

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10 Numerical Results

For the arrival process of customers, we consider the following three sets of matrices for D_0 and D_1

MAP with positive correlation (MPC)

$$D_0 = \begin{bmatrix} -1.7615 & 1.7615 & 0 \\ 0 & -1.7615 & 0 \\ 0 & 0 & -11.7054 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1.6294 & 0 & 0.1321 \\ 0.1233 & 0 & 11.5821 \end{bmatrix}$$

MAP with negative correlation (MNC)

$$D_0 = \begin{bmatrix} -5 & 5 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -40.5 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.15 & 0 & 4.85 \\ 40.3 & 0 & 0.2 \end{bmatrix}$$

MAP with zero correlation (MZC)

$$D_0 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -5.25 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.95 & 0 & 0.05 \\ 0.15 & 0 & 5.1 \end{bmatrix}$$

The arrival process labeled MPC has correlated arrivals with the correlation between two successive interarrival times given by 0.5315, the arrival process corresponding to the one labeled MNC has correlated arrivals with the correlation between two successive interarrival times given by -0.4470 and the arrival process labeled MZC has zero correlation between two successive interarrival times.
Service time follows continuous phase-type distribution $(\boldsymbol{\beta}, S)$ of order 6 in Model I.
Here we take $\boldsymbol{\beta} = (0.2, 0.1, 0.2, 0.1, 0.2, 0.2)$

$$S = \begin{bmatrix} -6.7 & 0.5 & 0 & 1.2 & 0 & 0.5 \\ 0 & -5.5 & 0.5 & 0.2 & 1 & 0.8 \\ 0 & 0.1 & -5.5 & 0.9 & 0.3 & 1.2 \\ 0.1 & 0 & 0.8 & -6.5 & 0.3 & 0 \\ 0.3 & 0 & 0.5 & 0 & -4.5 & 1.3 \\ 0.2 & 0.3 & 0 & 0.4 & 0 & -5.5 \end{bmatrix}.$$

In Model II, $\boldsymbol{\alpha'} = (0.2, 0.5, 0.3), \boldsymbol{\beta'} = (0.1, 0.4, 0.5), \boldsymbol{\gamma'} = (\boldsymbol{\alpha'}, \mathbf{0}) = (0.2, 0.5, 0.3, 0, 0, 0)$.

$$S' = \begin{bmatrix} -28.19 & 0.5 & 0 \\ 0 & -28.21 & 0.1 \\ 0 & 0.2 & -28.46 \end{bmatrix}, \quad T' = \begin{bmatrix} -4.5583 & 0.3 & 0 \\ 0 & -4.582 & 0.3 \\ 0 & 0.1 & -4.6 \end{bmatrix},$$

$$L = \begin{bmatrix} T' & T'^0\boldsymbol{\beta'} \\ \mathbf{0} & S' \end{bmatrix} = \begin{bmatrix} -4.5583 & 0.3 & 0 & 0.4258 & 1.7033 & 2.1292 \\ 0 & -4.582 & 0.3 & 0.4282 & 1.7128 & 2.1410 \\ 0 & 0.1 & -4.6 & 0.45 & 1.8 & 2.25 \\ 0 & 0 & 0 & -28.1900 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -28.2100 & 0.1 \\ 0 & 0 & 0 & 0 & 0.2 & -28..46 \end{bmatrix}.$$

Service rate in Model I = Service rate in Model II= 3.7649. Fix $n = 6, n_1 = 3, n_2 = 3, m = 3, k_1 = 2, k_2 = 6, k = 7, CK1 = 12, CK2 = 20, RS = 40$.
Let $ER_1$and ER_2 denote the expected revenue in Model I and Model II respectively.

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10.1 MAP with positive correlation (MPC) and the clock follows generalized Erlang distribution

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12-12.5	51.2838	50.6032	8.0993	0.6468	7.4011	185.9164
13-13.5	48.9982	48.3254	7.8161	0.7322	7.5104	191.9783
14-14.5	46.7988	46.1337	7.5605	0.8178	7.5929	196.6326
15-15.5	44.6870	44.0296	7.3284	0.9032	7.6535	200.1340
16-16.5	42.6620	42.0121	7.1163	0.9881	7.6960	202.6849
17-17.5	40.7213	40.0788	6.9215	1.0722	7.7237	204.4475
18-18.5	38.8617	38.2264	6.7416	1.1555	7.7390	205.5523
19-19.5	37.0791	36.4510	6.5748	1.2378	7.7439	206.1053
20-20.5	35.3697	34.7486	6.4194	1.3190	7.7401	**206.1931**
21-21.5	33.7293	33.1151	6.2741	1.3990	7.7289	205.8866
22-22.5	32.1541	31.5466	6.1377	1.4778	7.7113	205.2444
23-23.5	30.6401	30.0393	6.0093	1.5554	7.6883	204.3152
24-24.5	29.1838	28.5895	5.8879	1.6316	7.6604	203.1393
25-25.5	27.7819	27.1941	5.7727	1.7065	7.6288	201.7505
26-26.5	26.4132	25.8498	5.6633	1.7800	7.5934	200.1771

Table 1 – Effect of θ_i's on performance measures in Model I when the arrival process is MPC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12-12.5	19.3578	18.8077	6.0942	0.0214	7.0839	209.7992
14-14.5	15.2272	14.7401	5.6427	0.0350	7.7778	242.6997
16-16.5	11.8777	11.3801	5.2328	0.0524	8.3303	269.3707
18-18.5	9.168-	8.6946	4.8598	0.0374	8.7588	290.5665
20-20.5	7.0021	6.5515	4.5207	0.0977	9.0821	307.0833
22-22.5	5.3114	4.8821	4.2149	0.1248	9.3219	319.7984
24-24.5	4.0354	3.6254	3.9425	0.1546	9.5012	329.6455
26-26.5	3.1050	2.7124	3.7025	0.1868	9.6401	337.4388
28-28.5	2.4421	2.0649	3.4912	0.2211	9.7500	343.6853
30-30.5	1.9721	1.6087	3.3036	0.2573	9.8350	348.6118
32-32.5	1.6352	1.2844	3.1352	0.2951	9.8966	352.3382
34-34.5	1.3886	1.0493	2.9829	0.3342	9.9367	354.9907
36-36.5	1.2035	0.8749	2.8443	0.3742	9.9582	356.7130
37-37.5	1.1279	0.8044	2.7796	0.3946	9.9629	357.2698
38-38.5	1.0611	0.7425	2.7177	0.4151	9.9640	357.6454
39-39.5	1.0019	0.6879	2.6585	0.4357	9.9618	357.8543
40-40.5	0.9490	0.6395	2.6018	0.4565	9.9565	**357.9098**
41-41.5	0.9015	0.5964	2.5474	0.4774	9.9485	357.8242
42-42.5	0.8587	0.5578	2.4952	0.4983	9.9379	357.6082
43-43.5	0.8200	0.5231	2.4451	0.5194	9.9250	357.2720
44-44.5	0.7847	0.4918	2.3970	0.5405	9.9100	356.8247
45-45.5	0.7525	0.4634	2.3507	0.5616	9.8929	356.2746

Table 2 – Effect of θ_i's on performance measures in Model II when the arrival process is MPC

Table 1 and Table 2 show the effect of $\theta_i's$ on various performance measures and the revenue function in Model I and II respecively when the arrival process is MPC. In Model I, when $\theta_i's$ values increases, the values of ER_1 increase and reach the maximum at θ_i's=19-19.5 and then decreases. The maximum revenue in this case is 206.1931. In Model II, when $\theta_i's$ values increases, the values of ER_2 increase and reach the maximum at $\theta_i's = 40 - 40.5$, and then decreases. The maximum value of ER_2 is 357.9098. When we compare Model I and II, the values expected revenue in Model II is greater than that of the corresponding values of expected revenue in Model I. Also, the values of the rate of perfect service (RP) in Model II are greater than the corresponding values in Model I. In both models values of RK_1 decreases when $\theta_i's$ values increases. Also in both models, RK_2 increases when $\theta_i's$ values increases. This is because when $\theta_i's$ values increase, the expected service time of the customer in each stage decreases.

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10.2 MAP with positive correlation (MPC) and the clock follows Erlang distribution

Now we consider the case when all the values of $\theta_i's$ are equal.

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12	51.8634	51.1808	8.0856	0.6256	7.4584	188.7979
13	49.5574	48.8826	7.8021	0.7110	7.5679	194.8694
14	47.3366	46.6696	7.5466	0.7960	7.6500	199.5104
15	45.2033	44.5439	7.3147	0.8820	7.7100	202.9824
16	43.1571	42.5053	7.1028	0.9670	7.7517	205.4928
17	41.1959	40.5515	6.9084	1.0513	7.7784	207.2073
18	39.3166	38.6795	6.7289	1.1348	7.7926	208.2591
19	37.5154	36.8855	6.5626	1.2174	7.7964	208.7561
20	35.7882	35.1654	6.4076	1.2988	7.7914	**208.7864**
21	34.1312	33.5152	6.2628	1.3792	7.7790	208.4219
22	32.5401	31.9310	6.1270	1.4583	7.7603	207.7220
23	31.0113	30.4089	5.9990	1.5362	7.7362	206.7358
24	29.5411	28.9452	5.8781	1.6127	7.7074	205.5043
25	28.1260	27.5366	5.7635	1.6879	7.6746	204.0613
26	26.7629	26.1799	5.6545	1.7618	7.6381	202.4354

Table 3 – Effect of θ_i's on performance measures in Model I when the arrival process is MPC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12	19.4795	18.9287	6.1059	0.0197	7.0686	209.0797
14	17.2946	16.7574	5.8739	0.0257	7.4369	226.4730
16	15.3286	14.8048	5.6532	0.0327	7.7676	242.2119
18	9.2393	8.7652	4.8685	0.0700	8.7578	290.4893
20	7.0610	6.6097	4.5287	0.0937	9.0849	307.1775
22	5.3588	4.9288	4.2221	0.1204	9.3277	320.0353
24	4.0723	3.6617	3.9489	0.1498	9.5094	329.9940
26	3.1329	2.7397	3.7081	0.1816	9.6501	337.8737
28	2.4629	2.0852	3.4961	0.2156	9.7614	344.1917
30	1.9877	1.6238	3.3080	0.2515	9.8477	349.1815
32	1.6470	1.2957	3.1392	0.2891	9.9104	352.9645
34	1.3977	1.0580	2.9865	0.3280	9.9516	355.6660
36	1.2108	0.8818	2.8476	0.3678	9.9739	357.4294
37	1.1345	0.8105	2.7827	0.3881	9.9790	358.0040
38	1.0671	0.7479	2.7207	0.4085	9.9804	358.3955
39	1.0073	0.6928	2.6614	0.4291	9.9785	358.6188
40	0.9539	0.6440	2.6045	0.4499	9.9735	**358.6872**
41	0.9060	0.6005	2.5500	0.4707	9.9657	358.6131
42	0.8629	0.5615	2.4978	0.4917	9.9553	358.4074
43	0.8238	0.5265	2.4476	0.5127	9.9426	358.0803
44	0.7882	0.4949	2.3993	0.5338	9.9277	357.6409
45	0.7558	0.4663	2.3530	0.5549	9.9108	357.0978

Table 4 – Effect of θ_i's on performance measures in Model II when the arrival process is MPC

Tables 3 and 4 show the effect of θ_i on performance measures and expected revenue (ER) when the arrival process is MPC. In Model I ER is maximum at $\theta = 20$ and the maximum revenue is 208.7864. In Model II ER is maximum at $\theta = 40$ and the maximum revenue is 358.6872. When θ_i's values increases, the values of RK_1 decrease at the same time the values of RK_2 increase. This is because the expected service time of the customer in each stage decreases. When we compare Models I and II, the values of expected revenue in Model II are greater than that of the corresponding values of expected revenue in Model I. Also the values of the rate of perfect service (RP) in Model II are greater than the corresponding values of RP in Model I.

10.3 MAP with negative correlation (MNC) and the clock follows generalized Erlang distribution

Tables 5 and 6 show the effect of $\theta_i's$ on various performance measures and the revenue function when the arrival process is MNC and the clock is a generalized Erlang clock. ER is maximum when θ_i's $= 15 - 15.5$ in Model I and the maximum revenue is 273.7589. In Model II ER is maximum when θ_i's $= 40 - 40.5$ and the maximum revenue is 357.9432. When we compare Model I and II, the values

$\theta_i' s$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12-12.5	18.1197	17.1597	11.4736	0.9058	10.4146	260.7828
13-13.5	12.6645	11.7209	11.0056	1.0202	10.5100	267.9301
14-14.5	9.2711	8.3467	10.5469	1.1295	10.5303	272.0594
15-15.5	7.0945	6.1913	10.1019	1.2330	10.4910	**273.7589**
16-16.5	5.6478	3.7669	9.6761	1.3309	10.4083	273.6003
17-17.5	4.6497	3.7914	9.2740	1.4239	10.2959	272.0712
18-18.5	3.9353	3.0993	8.8974	1.5120	10.1639	269.5493
19-19.5	3.4061	2.5918	.5466	1.5959	10.0197	266.3116
20-20.5	3.0018	2.2084	8.2205	1.6760	9.8681	262.5575
21-21.5	2.6844	1.9112	7.9172	1.7525	9.7122	258.4313
22-22.5	2.4294	1.6755	7.6350	1.8256	9.5543	254.0392
23-23.5	2.2204	1.4850	7.3718	1.8955	9.3958	249.4616
24-24.5	2.0461	1.3283	7.1260	1.9623	9.2380	244.7604
25-25.5	1.8985	1.1978	6.8959	2.0263	9.0815	239.9836

Table 5 – Effect of θ_i's on performance measures in Model I when the arrival process is MNC

$\theta_i' s$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12-12.5	1.2636	0.6835	6.4269	0.0218	7.4496	220.4267
14-14.5	1.0887	0.5487	5.8233	0.0353	8.0099	249.8111
16-16.5	0.9632	0.3591	5.3212	0.0526	8.4581	273.4177
18-18.5	0.8684	0.2997	4.8976	0.0734	8.8175	292.4617
20-20.5	0.7938	0.2558	4.5356	0.0976	9.1056	307.8438
22-22.5	0.7335	0.2221	4.2230	0.1248	9.3354	320.2465
24-24.5	0.6834	0.1957	3.9503	0.1547	9.5174	330.1971
26-26.5	0.6410	0.1744	3.7105	0.1870	9.6594	338.1110
28-28.5	0.6046	0.1569	3.4980	0.2215	9.7681	344.3199
30-30.5	0.5728	0.1424	3.3083	0.2577	9.8486	349.0926
32-32.5	0.5449	0.1301	3.1381	0.2954	9.9054	352.6495
34-34.5	0.5200	0.1196	2.9845	0.3344	9.9418	355.1727
36-36.5	0.4977	0.1105	2.8451	0.3743	9.9611	356.8142
37-37.5	0.4874	0.1064	2.7802	0.3946	9.9650	357.3450
38-38.5	0.4775	0.1026	2.7182	0.4151	9.9655	357.7017
39-39.5	0.4681	0.0990	2.6588	0.4358	9.9630	357.8971
40-40.5	0.4592	0.0957	2.6020	0.4565	9.9574	**357.9432**
41-41.5	0.4506	0.0925	2.5476	0.4774	9.9492	357.8508
42-42.5	0.4424	0.0895	2.4954	0.4984	9.9386	357.6303
43-43.5	0.4346	0.0867	2.4452	0.5194	9.9256	357.2909
44-44.5	0.4270	0.0840	2.3971	0.5405	8.9104	356.8414
45-45.5	0.4197	0.0815	2.3508	0.5617	8.7875	356.2899

Table 6 – Effect of θ_i's on performance measures in Model II when the arrival process is MNC

of expected revenue in Model II is greater than that of the corresponding values of expected revenue in Model I. Also, the values of the rate of perfect service (RP) in Model II are greater than the corresponding values in Model I. In both models values of RK_1 decreases when $\theta_i' s$ values increases. Also in both models, RK_2 increases when $\theta_i' s$ values increases. This is because when $\theta_i' s$ values increase, the expected service time of the customer in each stage decreases.

10.4 MAP with negative correlation (MNC) and the clock follows Erlang distribution

Tables 7 and 8 show the effect of θ on various performance measures and expected revenue, when the arrival process is MNC and the clock is an Erlang clock. ER is maximum at $\theta = 15$ and the maximum revenue is 278.5231 in Model I and ER is maximum at $\theta = 40$ and the maximum revenue is 358.7241 in Model II.

10.5 MAP with zero correlation (MZC) and the clock follows generalized Erlang distribution

Tables 9 and 10 show the effect of θ_i's on various performance measures and the revenue function when the arrival process is MZC and the clock is generalized Erlang clock. ER is maximum when θ_i's $= 16 - 16.5$ and the maximum revenue is 266.1353 in Model I and ER is maximum when θ_i's $= 40 - 40.5$ and the maximum revenue is 350.9024 in Model II . When we compare Model I and II, the values expected revenue in Model II is greater than the corresponding values of expected revenue in Model I. Also, the values of the rate of perfect service (RP) in Model II are greater than the corresponding values

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$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12	19.9412	18.9776	11.4650	0.8768	10.5037	265.0314
13	13.7816	12.8336	11.0061	0.9923	10.6089	272.4372
14	9.9745	9.0450	10.5547	1.1029	10.6364	276.7431
15	7.5522	6.6436	10.1148	1.2078	10.6014	**278.5231**
16	5.9568	5.0703	9.6919	1.3071	10.5201	278.3574
17	4.8662	4.0022	9.2908	1.4012	10.4067	276.7548
18	4.0925	3.2510	8.9143	1.4905	10.2724	274.1164
19	3.5241	2.7044	8.5629	1.5754	10.1250	270.7381
20	3.0930	2.2945	8.2359	1.6565	9.9698	266.8317
21	2.7567	1.9786	7.9317	1.7338	9.8102	262.5493
22	2.4880	1.7294	7.6485	1.8077	9.6485	258.0019
23	2.2688	1.5288	7.3845	1.8784	9.4864	253.2722
24	2.0867	1.3646	7.1379	1.9460	9.3249	248.4235
25	1.9331	1.2282	6.9071	2.0106	9.1651	243.5047

Table 7 – Effect of θ_i's on performance measures in Model I when the arrival process is MNC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12	1.2692	0.6879	6.4435	0.0200	7.4383	219.8071
14	1.0928	0.5517	5.8370	0.0330	8.0032	249.4221
16	0.9663	0.3609	5.3327	0.0497	8.4553	273.2238
18	0.8709	0.3010	4.9073	0.0700	8.8181	292.4344
20	0.7960	0.2567	4.5440	0.0936	9.1090	307.9595
22	0.7353	0.2229	4.2303	0.1204	9.3414	320.4850
24	0.6850	0.1963	3.9567	0.1499	9.5255	330.5411
26	0.6424	0.1749	3.7161	0.1818	9.6694	338.5454
28	0.6059	0.1574	3.5030	0.2159	9.7796	344.8316
30	0.5740	0.1428	3.3128	0.2519	9.8615	349.6701
32	0.5460	0.1305	3.1421	0.2893	9.9194	353.2827
34	0.5210	0.1199	2.9881	0.3281	9.9568	355.8528
36	0.4986	0.1108	2.8484	0.3679	9.9768	357.5335
37	0.4874	0.1067	2.7834	0.3882	9.9811	358.0814
38	0.4784	0.1029	2.7212	0.4086	9.9820	358.4535
39	0.4690	0.0993	2.6617	0.4292	9.9797	358.6628
40	0.4600	0.0959	2.6048	0.4499	9.9744	**358.7241**
41	0.4514	0.0927	2.5502	0.4707	9.9655	358.6404
42	0.4432	0.0897	2.4979	0.4917	9.9560	358.4299
43	0.4353	0.0869	2.4477	0.5127	9.9431	358.0995
44	0.4278	0.0842	2.3995	0.5338	9.9282	357.6579
45	0.4205	0.0817	2.3531	0.5549	9.9112	357.1132

Table 8 – Effect of θ_i's on performance measures in Model II when the arrival process is MNC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12-12.5	16.8978	15.9692	11.963	0.8768	10.0764	252.3641
13-13.5	12.9656	12.0530	10.6422	0.9873	10.1672	259.2338
14-14.5	10.1494	9.2542	10.2102	1.0943	10.1985	263.5313
15-15.5	8.1129	7.2366	9.7986	1.1970	10.1805	265.6982
16-16.5	6.6205	5.7639	9.4070	1.2949	10.1229	**266.1353**
17-17.5	5.5094	4.6729	9.0355	1.3881	10.0346	265.1977
18-18.5	4.6677	3.8516	8.6844	1.4765	9.9233	263.1907
19-19.5	4.0190	3.2230	8.3536	1.5605	9.7956	260.3689
20-20.5	3.5102	2.7339	8.0430	1.6403	9.6565	256.9832
21-21.5	3.1044	2.3473	7.7517	1.7162	9.5102	253.0616
22-22.5	2.7757	2.0372	7.4789	1.7885	9.3595	248.8655
23-23.5	2.5057	1.7850	7.2233	1.8574	9.2069	244.4474
24-24.5	2.2807	1.5773	6.9837	1.9232	9.0538	239.8815
25-25.5	2.0912	1.4043	6.7591	1.9861	8.9014	235.2248

Table 9 – Effect of θ_i's on performance measures in Model I when the arrival process is MZC

in Model I. In both models values of RK_1 decreases when $\theta_i's$ values increases. Also in both models, RK_2 increases when $\theta_i's$ values increases. This is because when $\theta_i's$ values increase, the expected service time of the customer in each stage decreases.

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$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12-12.5	1.2979	0.7292	6,3005	0.0213	7.3031	216.0909
14-14.5	1.0927	0.5633	5.7087	0.0346	7.8524	244.8974
16-16.5	0.9490	0.3537	5.2165	0.0516	8.2917	268.0396
18-18.5	0.8426	0.2848	4.8012	0.0720	8.6441	286.7090
20-20.5	0.7607	0.2350	4.4464	0.0956	8.9265	301.7886
22-22.5	0.6861	0.1942	4.0935	0.1115	9.0690	311.4098
24-24.5	0.6421	0.1691	3.8726	0.1517	9.3302	323.7022
26-26.5	0.5977	0.1466	3.6375	0.1834	9.4694	331.4603
28-28.5	0.5601	0.1285	3.4292	0.2171	9.5760	337.5471
30-30.5	0.5277	0.1138	3.2432	0.2526	9.6549	342.2260
32-32.5	0.4995	0.1016	3.0764	0.2896	9.7105	345.7129
34-34.5	0.4746	0.0913	2.9258	0.3278	9.7463	348.1865
36-36.5	0.4526	0.0827	2.7892	0.3670	9.7651	349.7957
37-37.5	0.4425	0.0788	2.7255	0.3869	9.7690	350.3161
38-38.5	0.4328	0.0752	2.6647	0.4070	9.7695	350.6657
39-39.5	0.4237	0.0719	2.6065	0.4272	9.7670	350.8573
40-40.5	0.4150	0.0688	2.5508	0.4475	9.7616	**350.9024**
41-41.5	0.4068	0.0659	2.4975	0.4680	9.7535	350.8119
42-42.5	0.3989	0.0632	2.4463	0.4886	9.7431	350.5957
43-43.5	0.3914	0.0607	2.3971	0.5092	9.7303	350.2630
44-44.5	0.3842	0.05583	2.3499	0.5299	9.7155	349.8224
45-45.5	0.3773	0.0561	2.3046	0.5506	9.6987	3492817.

Table 10 – Effect of θ_i's on performance measures in Model II when the arrival process is MZC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12	18.0958	17.1635	11.0909	0.8490	10.1654	256.5463
13	13.8212	12.9044	10.6417	0.9602	10.2618	263.5676
14	10.7643	9.8647	10.2140	1.0681	10.2973	267.9635
15	8.5599	7.6789	9.8059	1.1719	10.2821	270.1769
16	6.9502	6.0887	9.4169	1.2710	10.2259	**270.6121**
17	5.7566	4.9151	9.0473	1.3654	10.1376	269.6305
18	4.8564	4.0352	8.6972	1.4550	10.0253	267.5456
19	4.1654	3.3645	8.3669	1.5400	9.8956	264.6201
20	3.6258	2.8447	8.0563	1.6208	9.7540	261.0680
21	3.1973	2.4354	7.7647	1.6977	9.6047	257.0590
22	2.8514	2.1063	7.4914	1.7708	9.4510	252.7250
23	2.5682	1.8432	7.2352	1.8406	9.2950	248.1673
24	2.3331	1.6255	6.9951	1.9071	9.1387	243.4630
25	2.1355	1.4446	6.7698	1.9708	8.9831	238.6709

Table 11 – Effect of θ_i's on performance measures in Model I when the arrival process is MZC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12	1.3046	0.7347	6.3168	0.0197	7.2919	215.4835
14	1.0974	0.5670	5.7222	0.0324	7.8457	244.5159
16	0.9525	0.3558	5.2278	0.0487	8.2889	267.8495
18	0.8454	0.2863	4.8108	0.0686	8.6446	286.6823
20	0.7630	0.2361	4.4546	0.0918	8.9298	301.9020
22	0.6974	0.1987	4.1471	0.1180	9.1576	314.1811
24	0.6438	0.1699	3.8789	0.1469	9.3381	324.0394
26	0.5992	0.1472	3.6430	0.1782	9.4792	331.8862
28	0.5614	0.1290	3.4341	0.2117	9.5873	338.0488
30	0.5289	0.1142	3.2476	0.2469	9.6675	342.7921
32	0.5006	0.1020	3.0803	0.2837	9.7243	346.3337
34	0.4757	0.0917	2.9293	0.3217	9.7610	348.8537
36	0.4535	0.0829	2.7924	0.3607	9.7806	350.5009
37	0.4434	0.0791	2.7286	0.3805	9.7848	351.0379
38	0.4337	0.0755	2.6677	0.4006	9.7856	351.4027
39	0.4246	0.0721	2.6094	0.4207	9.7843	351.6079
40	0.4159	0.0690	2.5536	0.4411	9.7782	**351.6654**
41	0.4076	0.0661	2.5001	0.4615	9.7704	351.5859
42	0.3997	0.0643	2.4488	0.4820	9.7601	351.3796
43	0.3921	0.0609	2.3996	0.5026	9.7476	351.0556
44	0.3849	0.0585	2.3523	0.5233	9.7329	350.6227
45	0.3780	0.0563	2.3068	0.5440	9.7163	350.0888

Table 12 – Effect of θ_i's on performance measures in Model II when the arrival process is MZC

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10.6 MAP with zero correlation (MZC)and the clock follows Erlang distribution

Tables 11 and 12 show the effect of θ on various performance measures and the revenue function, when the arrival process is MZC and the clock, is an Erlang clock. ER is maximum at $\theta = 16$ and the maximum revenue is 270.6121 in Model I and ER is maximum when $\theta = 40$ and the maximum revenue is 351.6654 in Model II. Also, the values of the rate of perfect service (RP) in Model II are greater than the corresponding values in Model I.

From Tables 1-12, we can conclude that in all cases, the values of ER and RP in Model II is greater than the corresponding values of ER and RP in Model I. Moreover the values of RK_1 and RK_2 in Model II are less than the corresponding values of RK_1 and RK_2 in Model I.

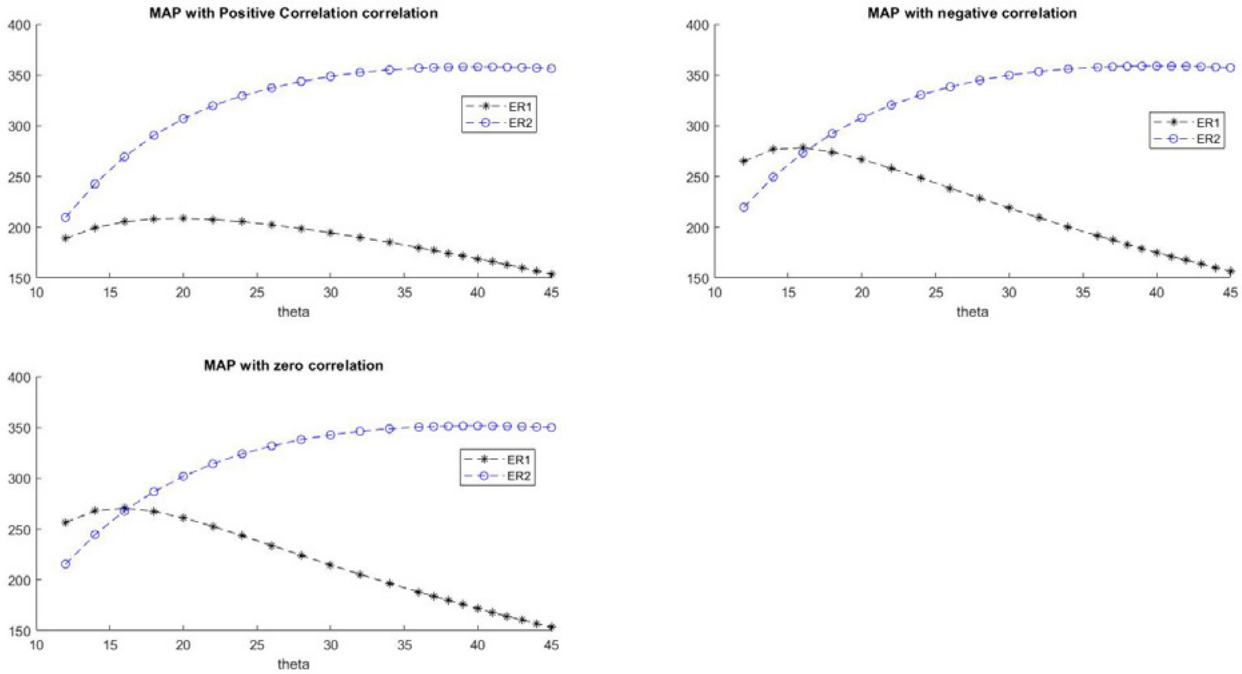

Figure 1 – Graph of Revenue Function

11 CONCLUSIONS

In this paper, we considered a $MAP/PH/1$ queue. We analysed this model by using the matrix-analytic method. We obtained the expected service time of a customer and also found the waiting time of a tagged customer. Also, we constructed a revenue function and other performance measures. To increase revenue, in Model II we consider the service time as the phase-type distribution $(\gamma^{'}, L)$ of order $n = n_1 + n_2$, which is the convolution of the two phase-type distributions $(\alpha^{'}, T^{'})$ of order n_1 and $(\beta^{'}, S^{'})$ of order n_2. We also performed some numerical experiments to evaluate some performance measures and also found that the revenue is maximum in Model II.

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/09/

ON A PRODUCTION-INVENTORY SYSTEM WITH DEFECTIVE ITEMS AND LOST SALES

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ABSTRACT

A production-inventory system with the item produced being admitted (added to the inventory) with probability δ as well as an item from the inventory supplied to the customer with probability γ at the end of a service, is considered in this paper. The (s, S) control policy is followed. We obtain the joint distribution of the number of customers and the number of items in the inventory as the product of their marginals under the assumption that customers do not join when the inventory level is zero. Performance measures that impact the system are obtained. A few level-crossing results are derived. In particular optimal pairs (s, S) are obtained through numerical procedures for values of (γ, δ) on the set $\{0.1, 0.2, \ldots, 1\} \times \{0.1, 0.2, \ldots, 1\}$. A comparison of the performance measures for a few (γ, δ) pair values is provided. Finally, we discuss the first emptiness time distribution for the $M/M/1/1$ production-inventory system.

KEYWORDS

Production-inventory, Positive Service Time, Stochastic Decomposition, Defective Items, Lost Sales

1 INTRODUCTION

Sigman and Levi (1992) and Melikov and Molchanov (1992) were the first to introduce inventory with positive service time, and now such systems are popularly known as queueing-inventory systems. They assumed arbitrarily distributed service time and exponentially distributed replenishment lead time with customer arrival forming a Poisson process. Under the condition of stability of the system, they investigate several performance characteristics. In the context of arbitrarily distributed lead time, the reader's attention is invited to a very recent paper by Saffari et al. (2013) where the authors provide a product form solution for system steady-state probability distribution under the assumption that *no customer joins the system when inventory level is zero*.

Schwarz et al. (2006a) requires special mention as the first piece of work to establish asymptotic independence of the number of customers in the system and the number of items in the inventory under exponentially distributed service time as well as lead time and Poisson input of customers. Nevertheless, this is achieved under the assumption that customers do not join when the inventory level is zero (of course, Saffari et al. (2013) is the extension of this to arbitrary distributed lead time). This is despite the strong correlation between the number of customers joining the system and the lead time. Subsequently, several authors made the above assumption in their models to develop product-form solutions, the details of which can be seen below. Their work is subsumed in Krishnamoorthy and Viswanath (2013), wherein the authors have reduced the Schwarz et al. (2006a) model to a production inventory system with a single batch, bulk production for the quantum of inventory required. Krishnamoorthy and Viswanath (2010), Deepak et al. (2008), Schwarz and Daduna (2006b) and Schwarz et al. (2007) are a few other significant contributions to queueing-inventory systems. A detailed survey on queueing-inventory models is given by Krishnamoorthy et al. (2021) to summarise the contributions until 2019.

A classical queue with inventoried items for service is also studied by Saffari et al. (2011). The control policy is (s, Q), and lead time is the mixed exponential distribution. Arrivals, when inventory is out of stock, are lost to the system. This leads to a product-form solution for the system state probability. Schwarz et al. (2007) consider queueing networks with an attached inventory. They consider rerouting the customers served from a particular station when it has zero inventory. Thus no customer is lost to the system. The authors derive the joint stationary distribution of queue length and on-hand inventory at various stations in explicit product form. Using dynamic programming Ning Zhao and Zhanotong Lian (2011) obtained the necessary and sufficient conditions for a priority queueing-inventory system to be stable. A contribution of interest to inventory with positive service time involving a random environment is by Krenzler and Daduna (2014), where again, they establish a stochastic decomposition of the system. They prove a necessary and sufficient condition for a product from steady-state distribution of the joint queueing-environment process to exist. Krenzler and Daduna (2013) investigate inventory with positive service time in a random environment embedded in a Markov chain. They provide a counter-example to show that the steady-state distribution of an $M/G/1/\infty$ system with (s, S) policy and lost sales does not have a product form. Nevertheless, in general, loss systems in a random environment have a product from steady-state distribution.

Apart from the above-mentioned papers, the contributions to production-inventory models with positive service time is worth mentioning; in this context, a recent paper by Yue *et al.* (2019a) considered a production-inventory system with positive service time and vacations of a production facility. Wherein they consider that the system has a single production facility that produces one type of product with (s, S) policy. The production facility takes a vacation of random duration once the inventory level becomes S. The authors obtained the product form of the stationary joint distribution of the queue length and the on-hand inventory level. Again another paper by Yue *et al.* (2019b) discussed a production-inventory system with a service facility, production interruptions, and (s, S) control policy. In this model, also they obtained the product form solution for the stationary joint distribution of the number of customers and the on-hand inventory level. Krishnamoorthy *et al.* (2019) studied an (s, S) production inventory model with lead time involving local purchases to ensure customer satisfaction and goodwill. The problem was modelled as a Continuous Time Markov Chain and obtained stochastic decomposition of system states. Analysis of this model has high social relevance as it helps reduce the total expected cost of the production process, which improves the profit in the cottage industry.

Otten *et al.* (2019) investigated a class of separable systems consisting of parallel production systems at several locations associated with local inventories under a base-stock policy connected with a supplier network. The production system manufactures according to customers' demands on a make-to-order basis. They studied two lost sales based on local inventory or available inventory. They obtained the product-form steady-state distribution. A recent contribution to a discrete-time production inventory system with positive service time and (s, S) order policy is by Anilkumar and Jose (2021). The authors assumed that the customer arrival follows a Bernoulli process and service time follows a geometric distribution. A supply chain consisting of production-inventory systems at several locations, common supplier couples were considered in Otten (2022). The item routing depends on the inventory to obtain "load balancing."Under a constant review base stock policy, the supplier produces raw materials to replenish local inventories. The service starts immediately If the server is prepared to serve a customer ahead of the line and the inventory has not been depleted. Otherwise, the service begins when the next replenishment arrives at the local inventory. They showed that the stationary distribution is a product of the marginal distributions of the production and inventory-replenishment subsystem. Also, they derived an explicit solution for some special cases for the marginal distribution of the inventory-replenishment subsystem.

Product-form solution for the production-inventory systems is not always possible due to the complexity of the models. However, such problems can be analysed by adopting numerical techniques. Barron (2019) considered a continuous-review storage system under the generalized order-up-to-level policy. They derived the explicit cost components of the resulting costs by taking a simple probability approach and applying stopping time theory to fluid processes and martingales. In another recent work by Beena An (s, S) production inventory model with varying production rates and multiple server vacations was analyzed by Beena, and Jose (2020). The model considered customer arrival as MAP and service time as PH distribution. A production inventory system under (s, S) policy with unit items produced at a time, heterogeneous servers, vacationing servers, and retrial customers is analyzed by Jose and Beena (2020). Another recent work by Jose and Reshmi (2021) discussed a continuous review perishable inventory system with a production unit and retrial facility, where customers arrive in a homogeneous Poisson distribution with (s, S) ordering policy and perishable items. A recent contribution by Noblesse *et al.* (2022) studied a continuous review finite capacity production-inventory system with two products in inventory. The model reflects a supply chain that operates in an environment with high levels of volatility. They considered the production facility a multitype $MMAP[K]/PH[K]/1$ queue. Another recent work by Otten and Daduna (2022) studied a production-inventory system with two classes of customers with different priorities admitted into the system via a flexible admission control scheme. The service time is exponentially distributed with parameter $\mu > 0$ for both types of customers. An arriving demand that finds the inventory depleted is lost because the inventory management follows a base stock policy (lost sales). To find the equilibrium behaviour of the system, they examined the global balance equations of the related Markov process and deduced structural features of the steady-state distribution. The authors derived a sufficient condition for ergodicity for both customer classes in the case of unbounded queues using the Foster-Lyapunov stability criterion.

In all work quoted above, customers are provided with an item from the inventory on completion of service. Nevertheless, there are several situations where a customer may not be served the item with probability one at the end of his service. The service time can be regarded as describing the features of the inventoried item. At the end of this, the customer decides to buy an item with probability γ or leaves the system with complementary probability $1 - \gamma$ without buying the item. In this connection, one may refer to Krishnamoorthy et al. (2015) for some recent developments. In this paper, we analyze such types of situations under Poisson demand, exponentially distributed service time, and the time for producing an item has exponential distribution. We further impose the condition that no customer joins when the on-hand inventory is zero (those who are already present stay back in the system until served). On the production side, a manufactured non-defective item is produced with a certain positive probability δ; hence it goes to the shelf for sale and with complementary probability $1 - \delta$, the item is defective and hence rejected. Thus, this paper further generalizes the work reported by Krishnamoorthy and Vishwanath (2013).

We arrange the presentation in this paper as indicated below: section 2 provides the mathematical

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formulation of the problem under study. The analysis of the system is carried out in section 3. In particular, we derive the long-run stability of the system. Then under this condition, we show that the system steady-state probability distribution can be decomposed: that is to say, we get the system steady-state probability distribution as the product of the marginal distribution of the components. Next, we compute system performance measures that have a significant impact. Further, to construct an appropriate cost function, we compute the expected length of a production cycle in section 4. A few results on up and down crossings of level s during a production cycle are also discussed in that section. Having achieved that, we construct a cost function. Then we look for the optimal pair (s, S) values that would result in cost minimization for different pairs of values of γ and δ and a comparison of the performance measures for a few (γ, δ) pair values is provided. This is reported in section 5. Emptiness time distribution for the $M/M/1/1$ production inventory system is discussed in Section 6. Finally, a few remarks in the conclusion are made. **Notations** used in the sequel are:

- $\mathcal{N}(t)$: number of customers in the system at time t.
- $\mathcal{I}(t)$: inventory level in the system at time t.
- $\mathcal{P}(t)$: status of the production process at time t.
  That is, $\mathcal{P}(t) = \begin{cases} 0, \text{ if production is off at time t.} \\ 1, \text{ if production is on at time t.} \end{cases}$
- $\mathcal{C}(t)$: status of the server is idle/ busy at time t.
  That is, $\mathcal{C}(t) = \begin{cases} 0, \text{ if server is idle at time t.} \\ 1, \text{ if server is busy at time t.} \end{cases}$
- I_k : identity matrix of order k.
- $\boldsymbol{e}$: $(1, 1, ..., 1)'$ a column vector of 1's of appropriate order.
- $\boldsymbol{e_1}$: $(1, 0, ..., 0)$ a row vector having 1 in the first element and 0's of appropriate order.
- CTMC: Continuous-time Markov chain.
- LIQBD: Level independent Quasi birth and death process.

2 DESCRIPTION OF THE MODEL

We consider an (s, S) production inventory system with a single server. Demands by customers for the item occur according to a Poisson process of rate λ. Processing of the customer request requires a random amount of time, which is exponentially distributed with the parameter μ. However, it is not essential that the item from inventory is provided to the customer at the end of a service. More precisely, an item from inventory is provided to a customer with probability γ at the end of his service, and with complement probability $1-\gamma$, the customer leaves the system empty-handed. When the inventory level depletes to s, the production process is immediately switched on. Each production is of 1 unit, and the production process is kept in the on mode until the inventory level becomes S. To produce an item, it takes a random amount of time which follows exponentially distributed with parameter β. A produced item is not necessarily added to the inventory due to manufacturing defect: with probability δ, it is accepted, and with probability $1-\delta$, the item is rejected. We assume that no customer is allowed to join the queue when the inventory level is zero; such demands are considered lost. It is assumed that the amount of time for the item produced to reach the retail shop is negligible. Thus the

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system is a CTMC $\{\mathcal{X}(t); t \geq 0\} = \{(\mathcal{N}(t), \mathcal{I}(t), \mathcal{P}(t)); t \geq 0\}$. The production process is in on mode if $0 \leq \mathcal{I}(t) \leq s$ and it is in off mode if $\mathcal{I}(t) = S$; but when the inventory level lies between $s+1$ and $S-1$, $\mathcal{P}(t)$ is either 0 or 1 according as the production is in off or in on mode, respectively. Thus to describe the status of the process we need to write $\mathcal{P}(t) = 0$ or 1 only when $\mathcal{I}(t)$ takes values $s+1, \ldots, S-1$. Thus the state space of the CTMC is $\mathbf{\Omega} = \bigcup_{i=0}^{\infty} \mathcal{L}(i)$, where $\mathcal{L}(i)$, called level i of the CTMC, is given by, $\{(i,j,1); 0 \leq j \leq s\} \bigcup \{(i,j,k); s+1 \leq j \leq S-1, k = 0, 1\} \bigcup \{(i,S,0)\}, \forall\, i \geq 0$. The number of states (called phases in that level) within i^{th} level is $2S - s$. The infinitesimal generator of this CTMC is,

$$\boldsymbol{\mathcal{Q}} = \begin{bmatrix} B & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \ldots \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

where B contains transition rates within $\mathcal{L}(0)$; A_0 represents the transition from level i to level $i+1,\ i \geq 0$; A_1 represents the transitions within $\mathcal{L}(i)$ for $i \geq 1$ and A_2 represents transitions from $\mathcal{L}(i)$ to $\mathcal{L}(i-1),\ i \geq 1$. The rates of transitions of the process $\{\mathcal{X}(t); t \geq 0\}$ are

$$[B]_{(j,k)(l,m)} = \begin{cases} -\delta\beta, & \text{for } l = j = 0; m = k = 1, \\ -(\lambda + \delta\beta), & \text{for } l = j+1; j = 0, 1, ..., S-1; m = k = 1, \\ -\lambda, & \text{for } l = j; j = s+1, s+2, ..., S; m = k = 0, \\ \delta\beta, & \text{for } l = j+1; j = 0, 1, 2, ..., s; m = k = 1, \\ \delta\beta, & \text{for } l = j+1; j = s+1, s+2, ..., S-1; k = 0; m = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_0]_{(j,k)(l,m)} = \begin{cases} \lambda, & \text{for } l = j; j = 1, 2, ..., S; m = k; k = 0 \text{ and } 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_1]_{(j,k)(l,m)} = \begin{cases} -(\delta\beta + \mu), & \text{for } l = j = 0; m = k = 1, \\ -(\lambda + \delta\beta + \mu), & \text{for } l = j+1; j = 0, 1, ..., S-1; m = k = 1, \\ -(\lambda + \mu), & \text{for } l = j; j = s+1, s+2, ..., S; m = k = 0, \\ \delta\beta, & \text{for } l = j+1; j = 0, 1, 2, ..., s; m = k = 1, \\ \delta\beta, & \text{for } l = j+1; j = s+1, s+2, ..., S-1; k = 0; m = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_2]_{(j,k)(l,m)} = \begin{cases} \gamma\mu, & \text{for } l = j-1; j = 1, 2, ..., s; m = k = 1, \\ \gamma\mu, & \text{for } l = j-1; j = s+1, s+2, ..., S; m = k = 0, \\ (1-\gamma)\mu & \text{for } l = j; j = 1, 2, ..., S; m = k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that all entries (block matrices) in $\boldsymbol{\mathcal{Q}}$ are of the same order, namely, $2S+1$. These matrices contain transition rates within the level (in the case of diagonal entries) and between levels (in the case of off-diagonal entries).

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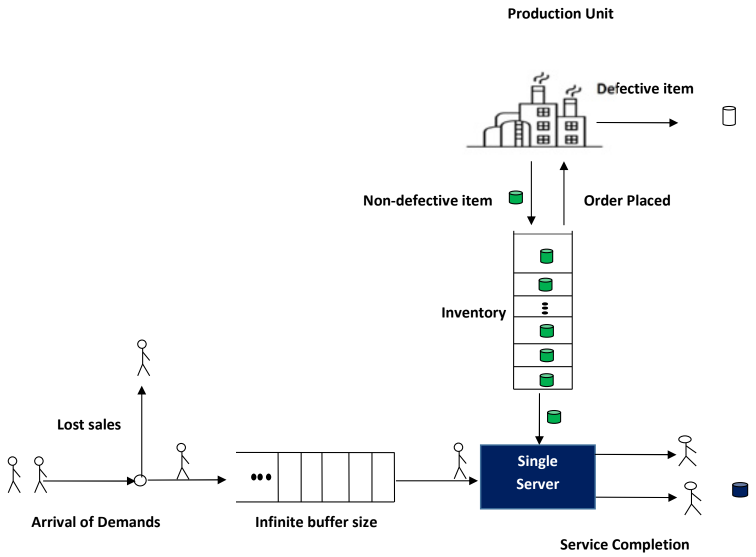

Figure 1 – $M/M/1$ Production-inventory system with defective items and lost sales

3 ANALYSIS OF THE SYSTEM

In this section, we perform the steady-state analysis of the (s, S) production inventory model under study by first establishing the stability condition of the system. Define $A{=}A_0{+}A_1{+}A_2$. This is the infinitesimal generator of the finite state CTMC corresponding to the inventory level $\{(j,1); 0 \le j \le s$ with production on$\}\cup$ $\{(j,k); s+1 \le j \le S-1, k = 0,1\} \cup \{(S,0)$ with production Off$\}$. Let $\boldsymbol{\varphi}$ denote the steady-state probability vector of A. That is $\boldsymbol{\varphi}$ satisfies

$$\boldsymbol{\varphi} A = 0,\ \ \boldsymbol{\varphi}\boldsymbol{e} = 1. \tag{1}$$

For convenience of notation we write $\varphi\,(j,1) = \varphi\,(j)$ for $0 \le j \le s$ and $\varphi\,(S,0)$ as$\varphi\,(S)$. By using the above relation (1), we get the components of the probability vector $\boldsymbol{\varphi}$ (note that, $\gamma\mu < \delta\beta$) explicitly as:

$$\varphi\,(s-j) = \varphi\,(S)\,\frac{\gamma\mu}{\delta\beta-\gamma\mu}\left(1-\left(\frac{\gamma\mu}{\delta\beta}\right)^{S-s}\right)\left(\frac{\gamma\mu}{\delta\beta}\right)^{j}, 0 \le j \le s,$$

$$\varphi\,(j,0) = \varphi\,(S)\,, s+1 \le j \le S-1,$$

$$\varphi\,(j,1) = \varphi\,(S)\,\frac{\gamma\mu}{\delta\beta-\gamma\mu}\left(1-\left(\frac{\gamma\mu}{\delta\beta}\right)^{S-j}\right), s+1 \le j \le S-1,$$

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and the unknown probability

$$\varphi(S)=\frac{\left(\frac{\gamma\mu}{\delta\beta}-1\right)^{2}}{\left(\frac{\gamma\mu}{\delta\beta}\right)^{S+2}-\left(\frac{\gamma\mu}{\delta\beta}\right)^{s+2}-(S-s)\left(\frac{\gamma\mu}{\delta\beta}-1\right)}.$$

Since the Markov chain under study is an LIQBD process, it is stable if and only if the left drift rate exceeds the right drift rate. That is,

$$\boldsymbol{\varphi}A_0\boldsymbol{e}<\boldsymbol{\varphi}A_2\boldsymbol{e}. \tag{2}$$

We have the following lemma:

Lemma 3.1. *The CTMC $\{\mathcal{X}(t);t\geq 0\}$ is stable if and only if $\lambda<\mu$.*

*Proof.*From the well known result in Neuts (1994) on the positive recurrence of A, we have $\boldsymbol{\varphi}A_0\boldsymbol{e}<\boldsymbol{\varphi}A_2\boldsymbol{e}$. With a bit of computation, this simplifies to the result $\lambda<\mu$. For future reference we define ρ as

$$\rho=\frac{\lambda}{\mu}. \tag{3}$$

3.1 STEADY-STATE ANALYSIS

For computing the steady-state distribution of the process $\{\mathcal{X}(t);t\geq 0\}$, we first consider a production inventory system with negligible service time where no backlog of customers is allowed (that is when inventory level is zero, no demand joins the system). The rest of the assumptions such as those on the arrival process and lead time are the same as given earlier. Designate the Markov chain so obtained as $\{\widetilde{\mathcal{X}}(t);t\geq 0\}=\{(\mathcal{I}(t),\mathcal{P}(t))\,;t\geq 0\}$. Transitions of the generator matrix, $\widetilde{\boldsymbol{\mathcal{Q}}}$ is given by,

$$[\widetilde{\mathcal{Q}}]_{(j,k)(l,m)}=\begin{cases}-\delta\beta, & \text{for } l=j=0;k=m=1,\\ -(\gamma\lambda+\delta\beta), & \text{for } l=j+1;j=0,1,...,S-1;k=m=1,\\ -\gamma\lambda, & \text{for } l=j;j=s+1,s+2,...,S;k=m=0,\\ \delta\beta, & \text{for } l=j+1;j=0,1,2,...,s;k=m=1,\\ \delta\beta, & \text{for } l=j+1;j=s+1,s+2,...,S-2;k=0;m=1,\\ \delta\beta, & \text{for } l=j+1;j=S-1;k=1;m=0,\\ \gamma\lambda, & \text{for } l=j-1;j=s+1;k=0 \text{ or } 1;m=1,\\ \gamma\lambda, & \text{for } l=j-1;j=1,2,\ldots,s;k=m=1,\\ \gamma\lambda, & \text{for } l=j-1;j=s+2,s+3,\ldots,S-1;k=m=0 \text{ or } 1,\\ \gamma\lambda, & \text{for } l=j-1;j=S;k=m=0,\\ 0, & \text{otherwise.}\end{cases}$$

The stationary probability vector $\boldsymbol{\pi}$ of $\widetilde{\boldsymbol{\mathcal{Q}}}$ satisfies

$$\boldsymbol{\pi}\widetilde{\boldsymbol{\mathcal{Q}}}=0,\ \ \boldsymbol{\pi}\boldsymbol{e}=1 \tag{4}$$

Thus the components of $\boldsymbol{\pi}$ are given by:

$$\pi\left(s-j\right)=\pi\left(S\right)\frac{\gamma\lambda}{\delta\beta-\gamma\lambda}\left(1-\left(\frac{\gamma\lambda}{\delta\beta}\right)^{S-s}\right)\left(\frac{\gamma\lambda}{\delta\beta}\right)^{j},0\leq j\leq s,$$

$$\pi\left(j,0\right)=\pi\left(S\right),s+1\leq j\leq S-1,$$

$$\pi\left(j,1\right)=\pi\left(S\right)\frac{\gamma\lambda}{\delta\beta-\gamma\lambda}\left(1-\left(\frac{\gamma\lambda}{\delta\beta}\right)^{S-j}\right),s+1\leq j\leq S-1.$$

and the unknown probability

$$\pi(S)=\frac{\left(\frac{\gamma\lambda}{\delta\beta}-1\right)^{2}}{\left(\frac{\gamma\lambda}{\delta\beta}\right)^{S+2}-\left(\frac{\gamma\lambda}{\delta\beta}\right)^{s+2}-(S-s)\left(\frac{\gamma\lambda}{\delta\beta}-1\right)}.$$

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Using the components of the probability vector $\boldsymbol{\pi}$, we shall find the steady-state probability vector of the CTMC $\{\mathcal{X}(t); t \geq 0\}$. For this, let $\boldsymbol{x}$ be the steady-state probability vector of the original system. Then the steady-state vector must satisfy the set of equations

$$\boldsymbol{x}\boldsymbol{\mathcal{Q}} = 0, \ \ \boldsymbol{x}\boldsymbol{e} = 1. \tag{5}$$

partition $\boldsymbol{x}$ by levels as

$$\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2}, \ldots) \tag{6}$$

where the sub-vectors of $\boldsymbol{x}$ are further partitioned as, $\boldsymbol{x_i} = (x_i(0), x_i(1), \ldots, x_i(s),$ $x_i(s+1,1), \ldots x_i(S-1,1), x_i(s+1,0), \ldots x_i(S-1,0), x_i(S)), i \geq 0$. Then the above system of equations reduces to

$$\boldsymbol{x_0}B + \boldsymbol{x_1}A_2 = 0 \tag{7}$$

$$\boldsymbol{x_i}A_0 + \boldsymbol{x_{i+1}}A_1 + \boldsymbol{x_{i+2}}A_2 = 0, i \geq 0 \tag{8}$$

Now we assume that

$$\boldsymbol{x_i} = \xi \left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\pi}, i \geq 0 \tag{9}$$

where ξ is a constant to be determined. It can be easily verified that (7) and (8) are satisfied by (9):

$$\boldsymbol{x_0}B + \boldsymbol{x_1}A_2 = \xi\boldsymbol{\pi}\left(B + \frac{\lambda}{\mu}A_2\right) = \xi\boldsymbol{\pi}\widetilde{\boldsymbol{\mathcal{Q}}} = 0, \tag{10}$$

$$\boldsymbol{x_i}A_0 + \boldsymbol{x_{i+1}}A_1 + \boldsymbol{x_{i+2}}A_2 = \xi\left(\frac{\lambda}{\mu}\right)^{i+1}\boldsymbol{\pi}\left(B + \frac{\lambda}{\mu}A_2\right) = \xi\left(\frac{\lambda}{\mu}\right)^{i+1}\boldsymbol{\pi}\widetilde{\boldsymbol{\mathcal{Q}}} = 0. \tag{11}$$

Now applying the normalizing condition $\boldsymbol{x}\boldsymbol{e}$=1, we get

$$\xi\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \cdots\right] = 1$$

Hence under the condition that $\lambda < \mu$, we have

$$\xi = 1 - \frac{\lambda}{\mu}. \tag{12}$$

We write $\lim_{t\to\infty} \mathcal{X}(t) = \mathcal{X}$. Thus we arrive at the following decomposition of the process $\{\mathcal{X}\}$ in the long run:

Theorem 1. *Under the necessary and sufficient condition $\lambda < \mu$ for stability, the steady-state probability vector of the process $\{\mathcal{X}(t); t \geq 0\}$ has stochastic decomposition: That is, $\boldsymbol{x_i} = (1-\rho)\rho^i\boldsymbol{\pi}, i \geq 0$, where ρ is as defined in* (3) *and $\boldsymbol{\pi}$ is the inventory level probability vector.*

3.2 PERFORMANCE MEASURES

We enumerate below the long-run system performance characteristics that are useful in formulating an optimization problem.

- Mean number of customers in the system, $L_s = \frac{\lambda}{\mu-\lambda}$.
- Mean number of customers waiting in the system during the stock-out period, $W_s = L_s\pi(0)$.
- Mean number of customers waiting in the system when inventory is available, $\widetilde{W_s} = L_s\left(1 - \pi(0)\right)$.

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- Mean number of items in the inventory,
  $E_{inv} = \sum_{i=0}^{s} i\pi(i) + \sum_{i=s+1}^{S-1} i(\pi(i,0) + \pi(i,1)).$
- Mean rate at which the production process is *switched on*,
  $E_{on} = \gamma\mu \left(\sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \pi(s+1,0) \right).$
- Expected rate at which items are added to the inventory,
  $E_{rp} = \delta\beta \left(\sum_{i=0}^{s} \pi(i) + \sum_{i=s+1}^{S-1} \pi(i,1) \right).$
- Expected *loss* rate of the *manufactured item* due to rejection,
  $M_{loss} = (1-\delta)\beta \left(\sum_{i=0}^{s} \pi(i) + \sum_{i=s+1}^{S-1} \pi(i,1) \right).$
- Expected *loss* rate of *customers* (customers not joining the system for want of inventory), $C_{loss} = \lambda\pi(0).$

Following Krishnamoorthy and Viswanath (2013) we have the expected production cycle time as given,

Lemma 3.2. *The expected length of a production cycle is given by,*
$E_{cycle} = \frac{1}{\delta\beta} \left((S-s) \sum_{j=0}^{s} \left(\frac{\gamma\lambda}{\delta\beta}\right)^j + \sum_{j=s+1}^{S-1} (S-j) \left(\frac{\gamma\lambda}{\delta\beta}\right)^j \right) = \frac{1}{\gamma\lambda} \left(\frac{1}{\pi(S)} - (S-s) \right).$

Corollary 1. *The expected number of production up-crossings of level s is given by,*
$\overline{E} = \left[x_0(s) \frac{\delta\beta}{\lambda+\delta\beta} + \frac{\delta\beta}{\lambda+\mu+\delta\beta} \sum_{i=1}^{\infty} x_i(s) \right] .E_{cycle}$
$= (1-(S-s)\pi(S)) \left(\frac{\delta\beta}{\delta\beta-\gamma\lambda}\right) \left(1 - \left(\frac{\gamma\lambda}{\delta\beta}\right)^{S-s}\right) \left(\frac{1-\rho}{\lambda+\delta\beta} + \frac{\rho}{\lambda+\mu+\delta\beta}\right).$

Corollary 2. *The expected number of production down crossings of level s is given by,*
$\underline{E} = (1-(S-s)\pi(S)) \left(\frac{\gamma\lambda}{(\delta\beta-\gamma\lambda)(\lambda+\gamma\mu+\delta\beta)}\right) \left(1 - \left(\frac{\gamma\lambda}{\delta\beta}\right)^{S-s}\right).$

Some of the above down and/ up-crossings of s may not go below/above s. The expected number of such crossings are given in the following corollaries

Corollary 3. *The expected number of production down crossings that goes below s in a production cycle, $P_{down} = \underline{E}*$Probability of a service completion before addition of an inventoried item. That is,*

$$P_{down} = \underline{E}. \left(\sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \left(\frac{\gamma\mu}{\delta\beta+\gamma\mu}\right) + \xi \int_{t=0}^{\infty} \int_{v=0}^{t} \lambda e^{-\lambda v} \gamma\mu e^{-\mu(t-v)} \delta e^{-\beta t} dv dt \right)$$

$$= \underline{E}. \left(\sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \left(\frac{\gamma\mu}{\delta\beta+\gamma\mu}\right) + \frac{\xi\delta\lambda\gamma\mu}{(\lambda+\beta)(\mu+\beta)} \right).$$

Corollary 4. *The expected number of production up-crossings that go above s in a production cycle, $P_{up} = \overline{E}*$Probability of a unit produced before a service completion. That is,*

$$P_{up} = \overline{E}. \left(\begin{array}{l} \sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \left(\frac{\delta\beta}{\delta\beta+\gamma\mu}\right) + \left(\frac{\delta\beta}{\delta\beta+\lambda}\right) \xi \\ + \xi \int_{t=0}^{\infty} \int_{v=0}^{t} \lambda e^{-\lambda v} e^{-\mu(t-v)} \delta(1 - e^{-\beta t} - e^{-\beta v}) dv dt \end{array} \right)$$

$$= \overline{E}. \left(\sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \left(\frac{\delta\beta}{\delta\beta+\gamma\mu}\right) + \left(\frac{\delta\beta}{\delta\beta+\lambda}\right) \xi + \frac{\xi\delta}{(\lambda+\beta)} \left[\frac{\beta(\mu+\beta)+\lambda\mu}{\mu(\mu+\beta)}\right] \right).$$

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Having obtained the expected length of a production cycle we turn to compute the optimal pair (s, S) values and the corresponding minimum costs. Lemma 3.2 provides us the rate at which the production process is switched on in unit time.

4 COMPUTING OPTIMAL (s, S) PAIRS AND THE MINIMUM COST

We look for the optimal values of s (the level, reaching at which the production process is switched on) and the maximum inventory level S of the production inventory model under discussion. Now for checking the optimality of s and S, the following cost function is constructed. Define $\mathcal{F}(s, S)$ as the expected cost per unit time in the long run. Then

$$\mathcal{F}(s, S) = K.E_{on} + h_{inv}.E_{inv} + c_1.C_{loss} + c_2.M_{loss} + c_3.E_{rp} + c_4.W_s + c_5.\widetilde{W}_s$$

where K is the fixed cost for starting a production run, h_{inv} is the cost per unit time per inventory towards holding, c_1 is the cost incurred due to loss per customer when the inventory is out of stock, c_2 is the cost incurred due to rejection per unit manufactured item, c_3 is the cost of production per unit time, c_4 is the waiting cost per unit time per customer during the stock out period and c_5 is the waiting cost per unit time per customer when inventory is available. Though we are not able to compute explicitly the optimal values of s and S, due to the highly complex form of the cost function, we arrive at these using numerical techniques.

For the following input values $\lambda = 2, \mu = 3, \beta = 2.5, K = \$5000, h_{inv} = \$20, c_1 = \$400, c_2 = \$100, c_3 = \$200, c_4 = \$300, c_5 = \$100$ and varying δ and γ we arrive at Table 1. δ and γ are given values from 0.1 to 1 at 0.1 spacing. Note that the case of $\gamma = \delta = 1$ is what is discussed in Krishnamoorthy and Vishwanath (2013). The pair of values given in each cell of Table 1 indicates the optimal (s, S) pair and the value at the bottom of each cell corresponds to the minimum cost (in Dollars). As γ and δ are varied we get distinct optimal pairs of (s, S) and the corresponding minimum cost. We observe that the minimum cost is a decreasing function of δ, or at first decreasing and then starts growing with δ. This can be attributed to the fact that for fixed γ, and for δ increasing, initially the loss of manufactured items get reduced; but subsequently from a point on, the holding cost factor dominates the gain from acceptance of produced item. The optimal (s, S) pair first decreases with δ increasing, comes to a minimum and then starts rising up. Same is the trend shown by the minimum cost values. The explanation for this trend is that with γ increasing, customers are provided the item at the end of their service with increasing probability, so shortage is bound to occur with higher probability. To some extend, increasing δ value can cope with this, since produced items are accepted with higher probability. Nevertheless, increase in δ results in increase in the holding cost. For the given input parameters the “best” among the optimal pair is $(1, 11)$ and the minimum cost is \$461.02 which correspond to $\delta = 1$ and $\gamma = 0.1$.

Now by using the same input values of Table 1 and with $s = 5$ and $S = 11$ we provide a comparison of the performance measures for a few (γ, δ) pair values in Table 2. For example we observe from Table 2 that the production cycle length and loss rate of customers are largest for the (γ, δ) pair values $(1, 0.5)$ and least for $(0.5, 1)$ among the three pairs of values indicated in that table. Similarly expected inventory held is least for (γ, δ) pair value $(1, 0.5)$ and the highest for $(0.5, 1)$.

5 CONCLUSIONS

This paper generalizes a few of the existing works by introducing positive service time in a production-inventory model. It provides a stochastic decomposition of the system’s steady state. The expected length of a production cycle is derived. A few level-crossing results are presented. The findings provide the management with the minimum cost for each pair of values of (γ, δ) and the corresponding optimal

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Table 1 – Optimal **(s, S)** values and minimum cost

δ \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	(3,11) 605.4	(1,26) 958.33	(1,12) 1189.3	(1,9) 1309.1	(1,8) 1381.7	(1,7) 1430.3	(1,7) 1465.1	(1,7) 1491.2	(1,6) 1511.6	(1,6) 1527.9
0.2	(1,10) 515.24	(2,13) 649.96	(6,20) 793.76	(1,27) 983.33	(1,15) 1120	(1,13) 1214.3	(1,13) 1282.5	(1,10) 1334.1	(1,9) 1374.4	(1,9) 1406.7
0.3	(1,10) 490.34	(1,12) 610.1	(2,14) 689.76	(4,18) 765.15	(7,25) 804.83	(1,23) 1008.3	(1,16) 1105.2	(1,19) 1180.1	(1,13) 1239.3	((1,12) 1287
0.5	(1,10) 472.89	(1,13) 584.32	(1,15) 664.66	(1,15) 722.9	(1,16) 763.58	2,18) 795.24	(4,21) 838.8	(6,26) 908.47	(1,29) 987.12	(1,24) 1058.3
0.6	(1,10) 468.89	(1,13) 578.74	(1,16) 660.13	(1,16) 721.23	(1,16) 766.65	(1,17) 797.98	(2,18) 821.01	(3,20) 849.05	(5,24) 896.26	(4,29) 959.93
0.7	(1,11) 466.11	(1,14) 574.69	(1,16) 656.36	(1,17) 720.28	(1,17) 769.82	(1,17) 806.51	(1,18) 831.65	(2,18) 849.35	(2,20) 867.81	(4,23) 899.16
0.9	(1,11) 462.32	(1,14) 569.49	(1,16) 651.76	(1,18) 732.47	(1,18) 773.71	(1,19) 818.36	(1,19) 853.53	(1,19) 879.47	(1,19) 896.85	(1,19) 907.9
1	**(1,11) 461.02**	(1,14) 567.74	(1,16) 650.26	(1,18) 717.95	(1,19) 774.64	(1,20) 822.1	(1,20) 860.79	(1,20) 891.35	(1,20) 913.86	(1,20) 928.76

Table 2 – Effect of γ and δ on various performance measures

Performance Measures	$\gamma = 1$ and $\delta = 0.5$	$\gamma = 0.5$ and $\delta = 1$	$\gamma = \delta = 1$
L_s	0.00085731	0.10005	0.038268
W_s	0.75643	0.0013604	0.07402
$\widetilde{W}_s$	1.2436	1.9986	1.926
E_{inv}	1.5852	7.8376	5.9064
E_{rp}	1.2436	0.99932	1.926
E_{cycle}	580.22	3.9955	10.066
C_{loss}	0.75643	0.0013604	0.07402

(s, S) pair. First emptiness time distribution of the inventory is computed for the case when the waiting room capacity is restricted to one. We propose to study the transient behaviour of such a system. In addition, we analyze the case in which the transfer time from the production plant to the retail shop is a positive valued random variable. Also, the case of arbitrarily distributed lead time and/or service time is being investigated.

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PROBLEMS OF ONLINE MATHEMATICS TEACHING AND LEARNING DURING THE PANDEMIC: A REVERBERATION IN TO THE PERCEPTION OF PROSPECTIVE TEACHERS

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ABSTRACT

Mathematics teaching and learning has been a special focus in the academic discourse due to its felt complexity in teaching and learning. This felt complexity originate from the teacher preparation process itself. Covid 19 pandemic had given a drastic blow to all the educational activities. This has affected the curricular practices in an unimaginable way. An attempt is made here to examine the various problems faced by prospective teachers in online mathematics teaching and learning during the pandemic period. Samples were selected from government/aided/unaided colleges using stratified random sampling technique with the help of a questionnaire developed for the purpose. Analysis of the data has been carried out by using techniques such as percentage, t test and ANOVA. It is concluded from the study that the several hurdles and difficulties has been there specific to mathematics teaching and learning and majority of them were unhappy with the online mathematics teaching and learning.

KEYWORDS

Perception, Online Learning, Mathematics Learning, Online Teaching

1. INTRODUCTION

All teaching procedures centred around three pivotal factors: the students, the teacher, and the subject. Out of these three factors, the teacher is the medium of communication between the other two. Pandemic has brought out drastic necessities in paradigms and operations to meet with the changing demands. Teachers of all levels were trying to adjust with the new normal. However, it is interesting to note that during the pandemic period, prospective teachers insisted on adopting "online mode" teaching to cope up with the existing practice. Here the important question raises: what is effectiveness output of online teaching and learning? Since it is a very new experience for prospective teachers, the effectiveness of online mode of teaching and learning needs to be considered as an important study to improve to prospective teachers who teach and learn very fundamental subjects, like Mathematics. Thus, "online mode" needs to be considered as a fourth important factor in addition to the above mentioned three facts; because of the new experience of handling classes by the prospective teacher without any prior knowledge and experience or training; as well as the learning experience gained by the students.

Online Mathematics teaching and learning, how much time should be allotted for mathematical education for helping them gain higher level competency and accuracy in basic computational procedural skills need to be strictly evaluated. Similarly, how much time needs to be allotted for learning higher order skills such as problem solving, problem presentation, solving complex problems, and transforming mathematical knowledge and skills to problems in non-mathematical disciplines is also to be considered. Mathematics as a subject is difficult to learn through online medium. Mathematics contains anumber of concepts which require interaction, continuous support, monitoring from the teacher. These aspects were lacking in the online mode of teaching and learning. In the learning process the prospective teachers faced general problems such as lack of internet facility, problems with the learner's understanding level, problems with the absence of mind and less student-teacher interaction and they are losing the learning capacity

In March 2020, the COVID-19 pandemic forced many countries to close schools and universities. To help students continue learning, a common policy evolved was to provide instructions in the online context. Online teaching and learning are an unprecedented experience for most teachers and students; consequently, they have a limited experience with it. Mathematics is a subject which needs a lot of illustration for its effectiveness in the teaching and learning context. Hence it is interesting to analyse how mathematics teaching and learning customised to suit with the change in paradigm. In this context, this paper, tried to understand the varied problems and perspectives of online Mathematics Teaching and learning among Prospective Teachers During the Pandemic as perceived by the prospective teachers at secondary level

2. EMPIRICS AND LEARNING FROM LITERATURE

Substantive number of studies were done concerning problems faced by online mathematics teaching and learning in various forms by many researchers. However, only a few have come up with the form and pattern required for this study. Nevertheless, a review of the studies shown-below has thrown some light on the fact that problems faced by online mathematics teaching and learning are more or less related and dependent on each other.

The first reported work on the impact of online teaching and learning is by Li (2003) provides the information which can be useful in implementing rational changes to mathematics teacher education. The data analysis is concentrated on three areas: the math phobia issue, the equity issue and teachers' beliefs about the instructional use of technology.

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Borba (2012) investigated the online mathematics education in Brazil within the context of research on digital technology over the past 25 years. Wherein, the researcher argued that Brazilian research on technology in mathematics education can be divided into four phases and then the author discussed in detail with an example that "blends" aspects of the second and third phases.

Larkin et al. (2015) examined the impact of a series of design changes to an online mathematics education course in terms of transactional distance between learner and teachers, pre-service education students' attitudes towards mathematics, and their development of mathematical pedagogical knowledge.

In a study by Bansilal, S. (2015) on the rapid global technological developments and that affected all facets of life, including the teaching and learning of mathematics. This qualitative study designed to identify the ways in which technology used and to explore the nature of this use by a group of 52 mathematics student teachers It is recommended that the education department prioritise the provision of specialist mathematics software that can be used to improve learning outcomes in mathematics.

Adnan, M., & Yaman, B. B. (2015) considered a qualitative review of the perspectives of faculty members teaching mathematics about teaching mathematics online. Data analysis shows that the most important concern for mathematicians in teaching mathematics through online was the nature of mathematics as a discipline, while for mathematics educators it was the nature of the methodology in teaching mathematics. Istenic et al. (2016) explored an integrative approach in applying ICT in learning with specific reference to the formation of mathematics teaching capability in preservice teachers. The effect of online problem posing on students' problem-solving ability in mathematics were studied by Suarsana et al. (2019).

Recent contribution of interest by Aliyyah et al. (2020) provides an overview on the effect of the online collaborative learning between teachers, parents, and schools that impact student success. Broadly, the success of online learning in Indonesia during the COVID-19 Pandemic was determined by the readiness of technology in line with the national humanist curriculum, support, and collaboration from all stakeholders, including government, schools, teachers, parents and the community. This study explores the perceptions of primary school teachers of online learning in a program developed in Indonesia called School from Home during the COVID-19 Pandemic.

Fakhrunisa, F., & Prabawanto, S (2020) revealed the issues on the mathematics teacher's perception of online mathematics learning challenges and possibilities during the COVID-19 pandemic. This survey results indicate that educational experience is one of the significant factors that will vary the ability of teachers in presenting online learning. The analysis of survey results reveals that mathematics teachers had a positive perception of online learning implementation

Cao et al. (2021) investigated how teachers in China perceived the effects of online instruction on mathematics learning and examined the challenges they encountered when the country shifted to online instruction during the COVID-19 pandemic. Results showed that the teachers believed that the effectiveness of online teaching largely depends on student self-discipline. Analysis suggested a need to expand technology use during instruction, reshape the way teachers interact with students, and reorganize teaching methods in face-to-face classroom instruction.

Radmer, F., & Goodchild, S (2021) studied to explore lecturers' and students' experiences of online mathematics teaching and learning and to enable sharing of solutions to the challenges encountered. The study findings show that many students missed the social contact, being physically present at the university, and face to face interaction with their lecturers.

3. SIGNIFICANCE OF THE STUDY

Unlike the other sectors of life in India the progression of online culture has witnessed a slow pacing in education. There has been no concerted effort in a scientific and methodical manner to augment the teachers in this regard. The teachers are not trained to use online learning and teaching mathematics, and there is a lack of teaching aids and materials and technological tools. A sudden shift from the site bound approach to online multiple line of approach has made the system more complex. The purpose of this study was to determine the problems of online learning and teaching about prospective teachers 'mathematical communication skills during this pandemic period and the problems of online learning about students' mathematical problem-solving skills during this pandemic.

The study aims to identify the issues that prospective teachers face during their onlineclasses during the pandemic period. Mathematics teaching and learning faced more difficulty than the other subjects due to a lack of student participation and internet issues. In addition, various literature studies show that mathematics teaching and learning faced more difficulty than the other subjects. The study expected to explore problems experienced by prospective teachers in an online teaching and learning environment and how the epidemic influenced their online teaching and learning experience. This study also focused on the teaching, logistics, social, technological and psychosocial online teaching and learning problems experienced by prospective teachers in the wake of the pandemic.

4.OBJECTIVES OF THE STUDY

- To examine various problems faced by prospective teachers in online teaching and learning of mathematics during the pandemic period.
- To examine the different concerns of prospective teachers about the online mathematics classes.
- To examine whether there exist any significant differences of the perceived problems on the basis of the categorical variables.

  Gender: Boys and Girls

  Institution: Govt., Aided, Unaided

  Locality: Urban and rural.

5. HYPOTHESIS

- There is no significant difference between the perceived problems on the basis of locality of the institution
- There is no significant difference between the perceived problems of online teaching and learning of mathematics among prospective teachers in terms of gender (girls andboys).
- There is no significant difference between the perceived problems on the basis of aided, government and unaided colleges.

6. RESEARCH METHOD

The study utilised a quantitative normative survey type research.

6.1 SAMPLE SELECTED

The prospective teachers who are the students of various colleges of education at secondary level forms the population of the study. Out of these 120 prospective teachers of different aided, government and unaided colleges were selected. The samples were selected using stratified random sampling technique giving the representation to gender, locale and types of institution of the study.

6.2 TOOL USED

A questionnaire has been prepared which comprehensively covers the different aspects of the problem under consideration. Both closed and open-ended questions were used. The items covered all the necessary dimensions of the variable i.e., Perceived problems and issues of online teaching and learning.

7. ANALYSIS AND INTERPRETATION

The Statistical techniques used for analysing the data were percentage, student's t-test, ANOVA. More specifically, students t-test used for the values of table 1 and 2; and ANOVA is used for the computational results in table 3.

Table 1: Significant difference between the perceived problems on the basis of male and female students. Note: *Indicate not significant at 0.05 level.

Subsample	Mean	Significant Difference	Mean Difference	Degrees of Freedom	't' Value
Female (N=108)	37.81	8.032	-1.611	118	-0.644*
Male (N=12)	39.42	9.802	-1.611	12.698	

The table 1, shows that the mean difference of female and male prospective teachers is -1.611. Itcan be seen that the 't' value is less than the table value of 1.96 at 0.05 level of significance. Hence it is not significant at 0.05 level. That is, it shows that there is no significant difference betweenfemale and male prospective teachers in their responses. The result is, the hypothesis is not rejected. It reveals that there is no significant difference in the mean score of prospective teachers' responses of female and male.

Table 2: Significant difference between the perceived problems on the basis of rural and urban areas. Note: *Indicate not significant at 0.05 level.

Subsample	Mean	Significant Difference	Mean Difference	Degrees of Freedom	't value
Rural (N=99)	38.11	8.515	0.825	118	0.491*
Urban (N=21)	37.29	6.634	0.825	35.534	

The table 2, shows that the mean difference of rural and urban prospective teachers is 0.825. The 't' value is less than the table value of 1.96 at 0.05 level of significance. Hence it is not significant at 0.05 level. It means that there is no significant difference between rural and urban in their responses. Thus, the result is hypothesis is not rejected and it shows that there is no significant difference in the mean score of prospective teachers' responses in rural and urban areas.

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Table 3: Significant difference between the perceived problems on the basis of aided unaided, aided colleges. *Indicate not significant at 0.05 level

Type of Institution	Mean	Significant Difference	F Value	P Value
Aided College	37.37	8.527		
Government College	39.90	9.544	0.738	0.481*
Unaided College	37.77	7.175		

The mean scores of aided colleges have 37.37 and government colleges have 39.90 and unaided colleges have 37.77. Looking at the mean scores, it is clear that there is no significant differencebetween the aided, government and unaided prospective teachers. Since P value is greater than levels, the above hypothesis is not rejected at 5% level of significance.

8. PERCENTAGE ANALYSIS

Of the total sample, 11.7 percent of the prospective teachers are low level. 75.0 percent are moderate level and 13.3 percent are high level problems.

Out of the 108 female students, 12 percent in low level, 76.9 percent in moderate level and 11.1 percent in high level of problems faced byteaching and learning mathematics in online. The percentage of responses from prospective teachers of male, 8.3 percent low level, 58.3 percent moderate level and 33.3percent in high level of problems faced by teaching and learning mathematics in online.

From the 99 rural area is 12.1 percent in low level, 71.7 percent in moderate level and16.2 percent in high level of problems faced by teaching and learning in online mathematics. The percentage of responses from prospective teachers of urban areas is 9.5 percent low level, 38.1 percent moderate leveland 16.2. percent in high level.

The percentage of responses from prospective teachers of aided colleges is 11.5 percent in low level, 76.9 percent in moderate level and 11.5 percent in high level of problems faced by teaching and learning in online mathematics. The percentage of responses from prospective teachers of government college is 1 14.3 percent low level 61.9 percent moderate level and 23.8 percent in high level. The percentage of responses from prospective teachers of unaided is 10.6 percent low level,78.7 percent moderate level and 10.6 percent in high level of problems faced by teaching and learning mathematics in online

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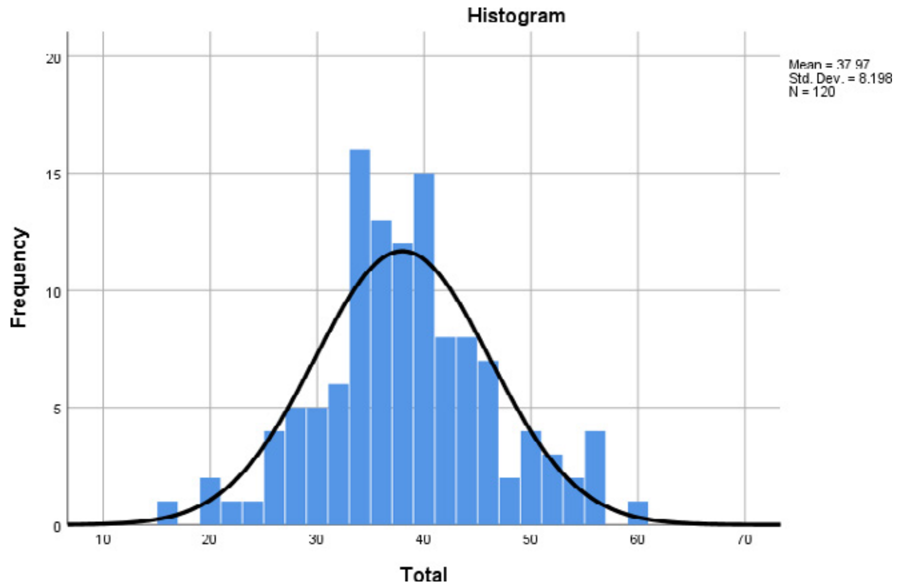

Figure 1. Histogram & normal distribution of the sample

9.MAJOR FINDINGS OF THE STUDY

From the study, the following issues have been identified in the perception of prospective teachers related to the online teaching and learning process: lack of internet facility, less student-teacher interaction, problems with the learner's understanding level, problems with the absence of mind, lack of student teacher satisfaction. In addition to the above issues, the prospective teachers faced the most challenging problem in teaching/learning process is problem-solving sessions in mathematics. In this study, we observed that the subject like mathematics, face to face classes are very much essential and it is more effective than the virtual classes and it is mostly not possible in the virtual learning/teaching process. To makeover these problems, the teachers must adopt more effective tools and gain adequate knowledge in the latest technologies. However, it may not be successful if the teachers were not provided special training for the same.

10. CONCLUSION

This paper, mainly focused the problems of prospective teachers and students experienced in teaching and learning of mathematics during the pandemic. The findings of this study revealed that most of the respondents (prospective teachers) were unhappy with online mathematics classes and prospective teachers faced several hurdles and difficulties while attending mathematics classes in virtual mode.

From the data, it can be revealed that teaching and learning mathematics subject through online classrooms are different and more challenging than learning the other subjects/disciplines and their understanding, enthusiasm for learning and understanding of mathematics are decreased during the online classes.

The prospective teacher also revealed that it was a difficult situation for them especially in the initial days, because for them the online classes are a first-time experience but gradually, they start to adapt to the new methods, and they start to consider this as an opportunity for innovation and experimentation. Most of the prospective teachers faced a lot of challenges during online classes due to less student-teacher interaction, internet issues, and lack of participation from the student side. They also found difficulties in problem-solving sessions which need good participation from both teacher and the student. Most of the prospective teachers agreed that it is more difficult to teach mathematics than the other disciplines.

The major hindrances evolved out for the effectiveness of online teaching and learning are: lack of internet, less student-teacher interaction, problems with the learner's understanding level, problems with the absence of mind, lack of student teacher satisfaction, solving a problem in mathematics that requires a lot of help from the prospective teacher, which is difficult to do through online classes.

The findings of the study made the authors to yield some implications. Some of the educational implications are as follows; provide internet facility, provide adequate training to teachers to integrate various ICT tools in online classes, provide support for the teachers to ensure the presence of the student in the class, use frequent assessment in online classes, to integrate interactive tools in online classes and to integrate different augmented reality in online classes.

11.RECOMMENDATION AND SUGGESTIONS

The following implications can be implemented by the government to improve the effectiveness of online classes: Provide free internet facility for all the prospective teachers in the state.

The following strategies can be implemented by the institutions to improve the effectiveness of online classes:

- Provide adequate training to teachers to integrate various ICT tools inonline classes.
- Provide a smart classroom for handling online classes for the teachers.
- The following implications can be implemented by the institutions to improve the effectiveness of online classes:
    - Use frequent assessment in online classes.
    - Integrate interactive tools in online classes.
    - Integrate the new technologies like, augmented reality for more effective online classes.

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/11/

TENSOR PRODUCTS OF WEAK HYPERRIGID SETS

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ABSTRACT

In this article, we show that concerning the spatial tensor product of W^*-algebras, the tensor product of two weak hyperrigid operator systems is weak hyperrigid. We prove this result by demonstrating unital completely positive maps have unique extension property for operator systems if and only if the tensor product of two unital completely positive maps has unique extension property for the tensor product of operator systems. Consequently, we prove as a corollary that the tensor product of two boundary representations for operator systems is boundary representation for the tensor product of operator systems. The corollary is an analogue result of Hopenwasser's [9] in the setting of W^*-algebras.

KEYWORDS

Operator system, W^*-algebra, Weak Korovkin set, Boundary representation.

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1 INTRODUCTION

Positive approximation processes play a fundamental role in approximation theory and appear naturally in many problems. In 1953, Korovkin [11] discovered the most powerful and, at the same time, the simplest criterion to decide whether a given sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of positive linear operators on the space of complex-valued continuous functions $C(X)$, where X is a compact Hausdorff space is an approximation process. That is, $\phi_n(f) \to f$ uniformly on X for every $f \in C(X)$. In fact it is sufficient to verify that $\phi_n(f) \to f$ uniformly on X only for $f \in \{1, x, x^2\}$. This set is called a Korovkin set. Starting with this result, during the last thirty years, many mathematicians have extended Korovkin's theorem to other function spaces or, more generally, to abstract spaces such as Banach algebras, Banach spaces, C^*-algebras and so on. At the same time, strong and fruitful connections of this theory have been revealed with classical approximation theory and other fields such as Choquet boundaries, convexity theory, uniqueness of extensions of positive linear maps, and so on.

Here, we provide an expository review of the non-commutative analogue of Korovkin's theorems with weak operator convergence and norm convergence. The notion of *boundary representation* of a C^*-algebra for an *operator system* introduced by Arveson [2] greatly influenced the theory of noncommutative approximation theory and other related areas such as Korovkin type properties for completely positive maps, peaking phenomena for operator systems and noncommutative convexity, etc. Arveson [4] introduced the notion of *hyperrigid set* as a noncommutative analogue of classical Korovkin set and studied the relation between hyperrigid operator systems and boundary representations extensively.

In 1984, Limaye and Namboodiri [13] studied the non-commutative Korovkin sets on $B(H)$ using weak operator convergence, which they named weak Korovkin sets. Limaye and Namboodiri [13] proved an exciting result using a famous boundary theorem of Arveson [3] that is as follows: An irreducible subset of $B(H)$ containing identity and a nonzero compact operator is weak Korovkin in $B(H)$ if and only if the identity representation of the C^*-algebra generated by the irreducible set has a unique completely positive linear extension to the C^*-algebra when restricted to the irreducible set. Limaye and Namboodiri gave many examples to establish these notions and theorems.

Namboodiri, inspired by Arveson's paper [4] on hyperrigidity, modified [15] the notion of weak Korovkin set on $B(H)$ to weak hyperrigid set in the context of W^*-algebras using weak operator convergence. He generalized the theorem in [13], characterizing the weak Korovkin set without assuming the presence of compact operators, and explored all nondegenerate representations. The result is as follows: An operator system is weak hyperrigid in the W^*-algebra generated by it if and only if every nondegenerate representation has a unique completely positive linear extension to the W^*-algebra when restricted to the operator system. Using this theorem, he established the partial answer to the non-commutative analogue of Saskin's theorem [18] relating weak hyperrigidity and Choquet boundary. Namboodiri gave a brief survey of the developments in the 'non-commutative Korovkin-type theory' in [14]. Namboodiri, Pramod, Shankar, and Vijayarajan [16] studied the non-commutative analogue of Saskin's theorem using the notions quasi hyperrigidity and weak boundary representations. Shankar and Vijayarajan [21] proved that the tensor product of two hyperrigid operator systems is hyperrigid in the spatial tensor product of C^*-algebras. Arunkumar and Vijayarajan [1] studied the tensor products of quasi hyperrigid operator systems introduced in [16]. Shankar [19] established hyperrigid generators for certain C^*-algebras.

In this article, we study the weak hyperrigidity of operator systems in W^*-algebras in the context of tensor products of W^*-algebras. It is interesting to investigate whether the tensor product of weak hyperrigid operator systems is weak hyperrigid. As a result of Hopenwasser [9], the tensor product of boundary representations of C^*-algebras for operator systems is a boundary representation if one of the constituent C^*-algebras is a GCR algebra. Since weak hyperrigidity implies that all irreducible representations are boundary representations for W^*-algebra, we will be able to deduce Hopenwasser's result for W^*-algebras as a spacial case. We achieve this by establishing first that unique extension property for unital completely positive maps on operator systems carry over to the tensor product of those maps defined on the tensor product of operator systems in the spatial tensor product of W^*-algebras.

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2 PRELIMINARIES

To fix our notation and terminology, we recall the fundamental notions. Let H be a complex Hilbert space and let $B(H)$ be the bounded linear operators on H. An *operator system* S in a W^*-algebra $\mathcal{M}$ is a self-adjoint linear subspace of $\mathcal{M}$ containing the identity of $\mathcal{M}$. An *operator algebra* A in a W^*-algebra $\mathcal{M}$ is a subalgebra of $\mathcal{M}$ containing the identity of $\mathcal{M}$.

Let ϕ be a linear map from a W^*-algebra $\mathcal{M}$ into a W^*-algebra $\mathcal{N}$, we can define a family of maps $\phi_n : M_n(\mathcal{M}) \to M_n(\mathcal{N})$ given by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. We say that ϕ is *completely bounded* (CB) if $||\phi||_{\text{CB}} = \sup_{n\geq 1} ||\phi_n|| < \infty$. We say that ϕ is *completely contractive* (CC) if $||\phi||_{\text{CB}} \leq 1$ and that ϕ is *completely isometric* if ϕ_n is isometric for all $n \geq 1$. We say that ϕ is *completely positive* (CP) if ϕ_n is positive for all $n \geq 1$, and that ϕ is *unital completely positive* (UCP) if in addition $\phi(1) = 1$.

Definition 1. *[2] Let S be an operator system in a W^*-algebra $\mathcal{M}$. A nondegenerate representation $\pi : \mathcal{M} \to B(H)$ has a unique extension property (UEP) for S if $\pi_{|_S}$ has a unique completely positive extension, namely π itself to $\mathcal{M}$. If π is an irreducible representation, then π is said to be a boundary representation for S.*

Definition 2. *[15] A set G of generators of a W^*-algebra $\mathcal{M}$ containing the identity $1_{\mathcal{M}}$ is said to be weak hyperrigid if for every faithful representation $\mathcal{M} \subseteq B(H)$ of $\mathcal{M}$ on a separable Hilbert space H and every net $\{\phi_\alpha\}_{\alpha\in I}$ of contractive completely positive maps from $B(H)$ to itself.*

$$\lim_{\alpha} \phi_\alpha(g) = g \text{ weakly } \forall\ g \in G \Longrightarrow \lim_{\alpha} \phi_\alpha(a) = a \text{ weakly } \forall\ a \in \mathcal{M}.$$

Theorem 1. *[15] For every separable operator system S, that generates a W^*-algebra $\mathcal{M}$, the following are equivalent.*

(i) S is weak hyperrigid.

(ii) For every nondegenerate representation $\pi : \mathcal{M} \to B(H)$, on a separable Hilbert space H and every net $\{\phi_\alpha\}_{\alpha\in I}$ of contractive completely positive maps from $\mathcal{M}$ to $B(H)$.
$\lim_\alpha \phi_\alpha(s) = \pi(s)$ *weakly* $\forall\ s \in S \Longrightarrow \lim_\alpha \phi_\alpha(a) = \pi(a)$ *weakly* $\forall\ a \in \mathcal{M}$.

(iii) For every nondegenerate representation $\pi : \mathcal{M} \to B(H)$ on a separable Hilbert space H, $\pi_{|_S}$ has a unique extension property.

(iv) For every W^*-algebra $\mathcal{N}$, every homomorphism $\theta : \mathcal{M} \to \mathcal{N}$ such that $\theta(1_{\mathcal{M}}) = 1_{\mathcal{N}}$ and every contractive completely positive map $\phi : \mathcal{N} \to \mathcal{N}$,
$\phi(x) = x\ \forall\ x \in \theta(S) \Longrightarrow \phi(x) = x\ \forall\ x \in \theta(\mathcal{M})$.

In this context, mentioning the 'hyperrigidity conjecture' posed by Arveson [4] is relevant. The hyperrigidity conjecture states that if every irreducible representation of a C^*-algebra A is a boundary representation for a separable operator system $S \subseteq A$ and $A = C^*(S)$, then S is hyperrigid. Arveson [4] proved the conjecture for C^*-algebras having a countable spectrum, while Kleski [10] established the conjecture for all type-I C^*-algebras with some additional assumptions. Recently Davidson and Kennedy [6] proved the conjecture for function systems.

Using the apparent correspondence between representations and modules, one can translate many aspects of the above notions into Hilbert modules. Muhly and Solel [12] gave an algebraic characterization of boundary representations in terms of Hilbert modules. Following Muhly and Solel, Shankar and Vijayarajan [20, 22] established a Hilbert module characterization for hyperrigidity (weak hyperrigidity) of specific operator systems in a C^*-algebra (W^*-algebras).

We need to consider tensor products of W^*-algebras in this article. Let $A_1 \otimes A_2$ denote the algebraic tensor product of A_1 and A_2. Let $A_1 \otimes_s A_2$ denote the closure of $A_1 \otimes A_2$ provided with the spatial norm, which is the minimal C^*-norm on the tensor product of W^*-algebras. In what follows, we will consider the spatial norm for the tensor product of W^*-algebras. We know that if representations π_1 is nondegenerate on A_1 and π_2 is nondegenerate on A_2, then the representation $\pi_1 \otimes \pi_2$ is nondegenerate

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on $A_1 \otimes A_2$. Conversely, from [5, Theorem II.9.2.1] and [17, Proposition 1.22.11] we can see that if π is a nondegenerate representation of $A_1 \otimes A_2$, then there are unique nondegenerate representations π_1 of A_1 and π_2 of A_2 such that $\pi = \pi_1 \otimes \pi_2$.

Tensor products of operator spaces (linear subspaces) of C^*-algebras and operator spaces of tensor product of C^*-algebras were explored by Hopenwasser earlier in [8], and [9] to study boundary representations. In [8], it was shown that boundary representations of an operator subspace of a C^*-algebra $A \otimes M_n(\mathbb{C})$ under certain conditions are parameterized by the boundary representations of an operator subspace of the C^*-algebra A which is given by the operator subspace in $A \otimes M_n(\mathbb{C})$. In [9], it was proved that if one of the C^*-algebras of the tensor product is a GCR algebra, then the boundary representations of the tensor product of C^*-algebras correspond to products of boundary representations.

3 MAIN RESULTS

In our main result, we investigate the relationship between the weak hyperrigidity of the tensor product of two operator systems in the tensor product W^*-algebra and the weak hyperrigidity of the individual operator systems in the respective W^*-algebras. The following result shows that the unique extension property of completely positive maps on operator systems carries over to the tensor product of those maps defined on the tensor product of operator systems.

Theorem 2. *Let S_1 and S_2 be operator systems generating W^*-algebras A_1 and A_2 respectively. Let $\pi_i : S_i \to B(H_i), i = 1, 2$ be unital completely positive maps. Then π_1 and π_2 have unique extension property if and only if the unital completely positive map $\pi_1 \otimes \pi_2 : S_1 \otimes S_2 \to B(H_1 \otimes H_2)$ has unique extension property for $S_1 \otimes S_2 \subseteq A_1 \otimes_s A_2$.*

Proof. Assume that $\pi_1 \otimes \pi_2$ has unique extension property, that is $\pi_1 \otimes \pi_2$ has unique completely positive extension $\widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2 : A_1 \otimes_s A_2 \to B(H_1 \otimes H_2)$ which is a representation of $A_1 \otimes_s A_2$. We will show that π_1 and π_2 have unique extension property. On the contrary, assume that one of the factors, say π_1 does not have unique extension property. This means that there exist at least two extensions of π_1, a completely positive map $\phi_1 : A_1 \to B(H_1)$ and the representation $\widetilde{\pi}_1 : A_1 \to B(H_1)$ such that $\phi_1 \neq \widetilde{\pi}_1$ on A_1, but $\phi_1 = \widetilde{\pi}_1 = \pi_1$ on S_1. Using [5, II.9.7], we can see that the tensor product of two completely positive maps is completely positive. We have $\phi_1 \otimes_s \widetilde{\pi}_2$ is a completely positive extension of $\pi_1 \otimes \pi_2$ on $S_1 \otimes S_2$, where $\widetilde{\pi}_2$ is a unique completely positive extension of π_2 on S_2. Hence $\phi_1 \otimes_s \widetilde{\pi}_2 \neq \widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2$ on $A_1 \otimes_s A_2$. This contradicts our assumption.

Conversely, assume that π_1 and π_2 have the unique extension property, that is π_1 and π_2 have unique completely positive extensions $\widetilde{\pi}_1 : A_1 \to B(H_1)$ and $\widetilde{\pi}_2 : A_2 \to B(H_2)$ respectively where $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ are representations of A_1 and A_2 respectively. We will show that $\pi_1 \otimes \pi_2$ has the unique extension property. We have $\widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2 : A_1 \otimes_s A_2 \to B(H_1 \otimes H_2)$ is a representation and an extension of $\pi_1 \otimes \pi_2$ on $S_1 \otimes S_2$. It is enough to show that if $\phi : A_1 \otimes_s A_2 \to B(H_1 \otimes H_2)$ is a completely positive extension of $\pi_1 \otimes \pi_2$ on $S_1 \otimes S_2$ then $\phi = \widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2$ on $A_1 \otimes A_2$.

Let P be any rank one projection in $B(H_2)$. The map $a \to (1 \otimes P)\phi(a \otimes 1)(1 \otimes P)$ is completely positive on A_1, since the map is a composition of three completely positive maps. Let v be a unit vector in the range of P and let K be the range of $1 \otimes P$. Define $U : H_1 \to K$ by $U(x) = x \otimes v,\ x \in H_1$, U is a unitary map. Let $\hat{\pi} = U\widetilde{\pi}_1(a)U^*,\ a \in A_1$ and $\hat{\pi}(a)$ is the restriction to K of $\widetilde{\pi}_1(a) \otimes P = (1 \otimes P)(\widetilde{\pi}_1(a) \otimes 1)(1 \otimes P)$. Since $\hat{\pi}$ is unitarily equivalent to $\widetilde{\pi}_1$, the representation $\hat{\pi}_{|S_1}$ has unique extension property. Let $\psi(a)$ be the restriction to K of $(1 \otimes P)\phi(a \otimes 1)(1 \otimes P)$ which implies that ψ is a completely positive map that agrees with $\hat{\pi}$ on S_1, hence on all of A_1.

Let $x, y \in H_1$ and $r \in H_2$. From the above paragraph we have, for any $a \in A_1$, $\langle \phi(a \otimes 1)(x \otimes r), y \otimes r \rangle = \langle (\widetilde{\pi}_1(a) \otimes 1)(x \otimes r), y \otimes r \rangle$. (Letting P be the rank one projection on the subspace spanned by r.) Let $D = \phi(a \otimes 1) - \widetilde{\pi}_1 \otimes 1$. Then we have $\langle D(x \otimes r), y \otimes r \rangle = 0$, for all $x, y \in H_1$, $r \in H_2$. Using polarization formula

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$$\begin{aligned} 4\langle D(x\otimes r), y\otimes s\rangle &= \langle D(x\otimes (r+s)), y\otimes (r+s)\rangle \\ &\quad -\langle D(x\otimes (r-s)), y\otimes (r-s)\rangle \\ &\quad +i\langle D(x\otimes (r+is)), y\otimes (r+is)\rangle \\ &\quad -i\langle D(x\otimes (r-is)), y\otimes (r-is)\rangle . \end{aligned}$$

We have $\langle D(x\otimes r), y\otimes s\rangle = 0$, for all $x, y \in H_1$ and for all $r, s \in H_2$. Consequently, if $z_1 = \sum_{i=1}^{n} x_i \otimes r_i$ and $z_2 = \sum_{i=1}^{m} y_i \otimes s_i$, then $\langle Dz_1, z_2\rangle = 0$. Since z_1, z_2 run through a dense subset of $H_1 \otimes H_2$ and D is bounded, $D = 0$. Therefore $\phi(a\otimes 1) = \widetilde{\pi}_1(a)\otimes 1$, for all $a \in A_1$. In the same way we can obtain $\phi(1\otimes b) = 1\otimes\widetilde{\pi}_2(b)$, for all $b \in A_2$. Since ϕ is a completely positive map on $A_1\otimes A_2$ and $\phi(1\otimes b) = 1\otimes\widetilde{\pi}_2(b)$, for all $b \in A_2$, using a multiplicative domain argument, e.g., see [9, Lemma 2] we have

$$\phi(a\otimes b) = \phi(a\otimes 1)(1\otimes\widetilde{\pi}_2(b)) = (1\otimes\widetilde{\pi}_2(b))\phi(a\otimes 1)$$

for all $a \in A_1$, $b \in A_2$. Also $\phi(a\otimes 1) = \widetilde{\pi}_1(a)\otimes 1$, for all $a \in A_1$. Hence $\phi = \widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2$ on $A_1 \otimes_s A_2$.

Corollary 1. *Let S_1 and S_2 be separable operator systems generating W^*-algebras A_1 and A_2 respectively. Assume that either A_1 or A_2 is a GCR algebra. Then S_1 and S_2 are weak hyperrigid in A_1 and A_2 respectively if and only if $S_1 \otimes S_2$ is weak hyperrigid in $A_1 \otimes_s A_2$.*

Proof. Assume that $S_1 \otimes S_2$ is weak hyperrigid in the W^*-algebra $A_1 \otimes_s A_2$. By theorem 1, every unital representation $\pi : A_1 \otimes_s A_2 \to B(H_1 \otimes H_2)$, $\pi_{|S_1\otimes S_2}$ has unique extension property. We have if π is a unital representation of $A_1 \otimes_s A_2$, since one of the W^*-algebras is GCR then by [7, Proposition 2] there are unique unital representations π_1 of A_1 and π_2 of A_2 such that $\pi = \pi_1 \otimes_s \pi_2$. Using theorem 2, we can see that $\pi_{1|S_1}$ and $\pi_{2|S_2}$ have unique extension property. This implies that S_1 and S_2 are weak hyperrigid in A_1 and A_2 respectively again by theorem 1.

Conversely, assume that S_1 is weak hyperrigid in A_1 and S_2 is weak hyperrigid in A_2. By theorem 1, for every unital representations $\pi_1 : A_1 \to B(H_1)$ and $\pi_2 : A_2 \to B(H_2)$, $\pi_{1|S_1}$ and $\pi_{2|S_2}$ have unique extension property. We have, if π_1 and π_2 are unital representations of A_1 and A_2 respectively, then $\pi_1 \otimes_s \pi_2$ is an unital representation of $A_1 \otimes_s A_2$. Using theorem 2, we can see that $\pi_1 \otimes_s \pi_{2|S_1\otimes S_2}$ has unique extension property. Now, by theorem 1 $S_1 \otimes S_2$ is weak hyperrigid in $A_1 \otimes_s A_2$.

Let $A_1 \otimes_m A_2$ denote the closure of $A_1 \otimes A_2$ provided with maximal C^*-norm. There are C^*-algebras A_1 for which the minimal and the maximal norm on $A_1 \otimes A_2$ coincide for all C^*-algebras A_2 and consequently the C^*-norm on $A_1 \otimes A_2$ is unique. Such C^*-algebras are called nuclear. The spatial norm assumption in the above results is redundant if the C^*-algebras are nuclear. But general C^*-algebras with the lack of injectivity associated with other C^*-norms, including the maximal one, will require additional assumptions.

Let A_1 and A_2 be W^*-algebras and γ is any C^*-cross norm on $A_1 \otimes A_2$. If π_1 and π_2 are irreducible representations of A_1 and A_2 respectively, then $\pi_1 \otimes_\gamma \pi_2$ is an irreducible representation of $A_1 \otimes_\gamma A_2$. Conversely, every irreducible representation π on $A_1 \otimes_\gamma A_2$ need not factor as a product $\pi_1 \otimes_\gamma \pi_2$ of irreducible representations. If we assume, one of the W^*-algebra is a GCR algebra, and then by [7, Proposition 2], every irreducible representation does factor. Since GCR algebras are nuclear, there is a unique C^*-cross norm on $A_1 \otimes A_2$, which we denote by $A_1 \otimes_\gamma A_2$.

Using the above facts, the result by Hopenwasser [9] relating boundary representations of tensor products of C^*-algebras will become a corollary to our theorem 2.

Corollary 2. *Let S_1 and S_2 be unital operator subspaces of generating W^*-algebras A_1 and A_2 respectively. Assume that either A_1 or A_2 is a GCR algebra. Then the representation $\pi_1 \otimes_\gamma \pi_2$ of $A_1 \otimes_\gamma A_2$ is a boundary representation for $S_1 \otimes S_2$ if and only if the representations π_1 of A_1 and π_2 of A_2 are boundary representations for S_1 and S_2 respectively.*

Now, we will provide some examples which illustrate the results above.

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Example 1. *Let $G = linear\ span(I, S, S^*)$, where S is the unilateral right shift in $B(H)$ and I is the identity operator. Let $A = C^*(G)$ be the C^*-algebra generated by G. We have $K(H) \subseteq A$, $A/K(H) \cong C(\mathbb{T})$ is commutative, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. Let Id denotes the identity representation of the C^*-algebra A. Let $S^*Id(\cdot)S$ be a completely positive map on the C^*-algebra A such that $S^*IdS_{|_G} = Id_{|_G}$, it is easy to see that $S^*IdS_{|_A} \neq Id_{|_A}$. Therefore the unital representation $Id_{|_G}$ does not have unique extension property. Using [15, Theorem 3.1], we conclude that G is not a weak hyperrigid operator system in a W^*-algebra $B(H)$.*

Let $G_1 = G$, $A_1 = A$ and Id_1 denotes the identity representation of A_1. Let $G_2 = A_2 = M_n(\mathbb{C})$ and Id_2 denotes the identity representation of the C^*-algebra A_2. The completely positive map $S^*Id_1S \otimes Id_2$ on the C^*-algebra $A_1 \otimes A_2$ is such that $S^*Id_1S \otimes Id_2 = Id_1 \otimes Id_2$ on operator system $G_1 \otimes G_2$. By the above conclusion we see that $S^*Id_1S \otimes Id_2 \neq Id_1 \otimes Id_2$ on the C^*-algebra $A_1 \otimes A_2$. Therefore the unital representation $Id_1 \otimes Id_2$ does not have unique extension property for $G_1 \otimes G_2$. Hence by theorem [15, Theorem 3.1], $G_1 \otimes G_2$ is not a weak hyperrigid operator system in a W^*-algebra $B(H) \otimes M_n(\mathbb{C})$.

Example 2. *Let the Volterra integration operator V acting on the Hilbert space $H = L^2[0, 1]$ be given by*

$$Vf(x) = \int_0^x f(t)dt, \qquad f \in L^2[0, 1].$$

V generates the C^*-algebra $K = K(H)$ of all compact operators. Let $S = linear\ span(V, V^*, V^2, V^{2*})$ and S is weak hyperrigid [4, Theorem 1.7] and [15, Theorem 3.1] in W^*-algebra $B(H)$. Let $S_1 = S_2 = S$ and $A_1 = A_2 = B(H)$. We know that S_1 and S_2 are weak hyperrigid operator systems in the W^*-algebra A_1 and A_2 respectively. By corollary 1 we conclude that $S_1 \otimes S_2$ is weak hyperrigid operator system in the W^*-algebra $A_1 \otimes A_2$.

Example 3. *Let $G = linear\ span(I, S, S^*, SS^*)$, where S is the unilateral right shift in $B(H)$ and I is the identity operator. Let $A = C^*(G)$ be the C^*-algebra generated by the operator system G. We have, $K(H) \subseteq A$. $A/K(H) \cong C(\mathbb{T})$ is commutative, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. Since S is an isometry, G is a weak hyperrigid operator system in the W^*-algebra $B(H)$ [15, Theorem 3.1]. Let $G_1 = G$, $A_1 = B(H)$ and $G_2 = A_2 = M_n(\mathbb{C})$. It is clear that G_2 is a weak hyperrigid operator system in the W^*-algebra $A_2 = M_n(\mathbb{C})$. By corollary 1, $G \otimes M_n(\mathbb{C})$ is a weak hyperrigid operator system in $B(H) \otimes M_n(\mathbb{C})$.*

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/12/

$\mathcal{Z}$-HYPERRIGIDITY AND $\mathcal{Z}$-BOUNDARY REPRESENTATIONS

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ABSTRACT

In this article, we introduce the notions of $\mathcal{Z}$-finite representations and $\mathcal{Z}$-separation property of representations for operator $\mathcal{Z}$-systems generating C^*-algebras. We use these notions to characterize the $\mathcal{Z}$-boundary representations for operator $\mathcal{Z}$-systems. We introduce $\mathcal{Z}$-hyperrigidity of operator $\mathcal{Z}$-systems. We investigate an analogue version of Saskin's theorem in the setting of operator $\mathcal{Z}$-systems generating C^*-algebras.

KEYWORDS

Completely positive maps, Operator systems, Representations of C^*-algebras, Hilbert C^*-modules.

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1 INTRODUCTION

Let S be a subspace (subalgebra) of $C(X)$, the set of continuous functions on compact metric space X. The *Choquet boundary* of S consists of the points $x \in X$ with the property that there is a unique probability measure μ on X, such that $f(x) = \int_X f d\mu$, $f \in S$. In other words, the points $x \in X$ lie in the Choquet boundary of S if the point evaluation functional $f \mapsto f(x)$, $f \in S$ extends to a unique state on the C^*-algebra $C(X)$. The Choquet boundary is a significant object to study for at least two reasons. The Choquet boundary of S is dense in the *Shilov boundary* of S. Shilov boundary is the smallest closed subset of X on which every function in S attains its maximum modulus. Choquet boundary supplies a tool to identify the "minimal" representations of the elements of S as functions on some compact metric space. For more details, refer to [5].

Korovkin theorem [14] deals with the convergence of positive linear maps on function algebras. The classical Korovkin theorem is as follows: for each $n \in \mathbb{N}$, let $\phi_n : C[0,1] \to C[0,1]$ be a positive linear map. If $\lim_{n\to\infty} ||\phi_n(f) - f||$ for every $f \in \{1, x, x^2\}$, then $\lim_{n\to\infty} ||\phi_n(f) - f||$ for every $f \in C[0,1]$. The set $\{1, x, x^2\}$ is called a Korovkin set in $C[0,1]$. There is a close connection between Korovkin sets and Choquet boundaries. Saskin [5, 23] proved that G is a Korovkin set in $C[0,1]$ if and only if the Choquet boundary of G is $[0,1]$.

Arveson [2] initiated a non-commutative analogue of the Choquet boundary in the context of unital operator algebras and operator systems in C^*-algebra. The central objects in his approach are the so-called boundary representations. Certain unital completely positive linear maps have unique extension property, almost in the spirit of defining property for points to lie in the classical Choquet boundary. The conjecture of Arveson states that every operator system and every unital operator algebra has sufficiently many boundary representations to norm it completely. Hamana [11] constructed the C^*-envelope of the operator system using a different method. Arveson [3] proved the conjecture for separable C^*-algebras. Davidson and Kennedy [8] completely settled conjecture on boundary representations. Fuller, Hartz, and Lupini [10] introduced the notion of boundary representations for operator spaces in ternary rings of operators. They established the natural operator space analogue of Arveson's conjecture on boundary representations. Magajna [17] introduced $\mathcal{Z}$-boundary representations for operator $\mathcal{Z}$-system generating a C^*-algebra on self-dual Hilbert $\mathcal{Z}$-modules, where $\mathcal{Z}$ is abelian von Neumann algebra. Magajna [17] proved analogue of Arveson's conjecture for $\mathcal{Z}$-boundary representations of C^*-algebra generated by operator $\mathcal{Z}$-systems on self-dual Hilbert $\mathcal{Z}$-modules over abelian von Neumann algebra $\mathcal{Z}$.

Arveson [4] introduced the notion of hyperrigid set, which is a non-commutative analogue of the Korovkin set. Arveson studied hyperrigidity in the setting of operator systems in C^*-algebras, and he tried to prove an analogue version of Saskin's theorem using hyperrigidity and boundary representations. Arveson [4] proved if every operator system is hyperrigid in generating C^*-algebra, then every irreducible representation of C^*-algebra is a boundary representation for the operator system. But he could not be able to prove the converse in generality. The converse of the above result is called *Arveson's hyperrigidity conjecture*. Hyperrigidity conjecture is as follows: for an operator system S and the generated C^*-algebra A, if every irreducible representation of A is a boundary representation for S, then an operator system S is hyperrigid. Arveson [4] showed that the hyperrigidity conjecture is valid for C^*-algebras with a countable spectrum.

Davidson and Kennedy [9] established a dilation-theoretic characterization of the Choquet order on the space of measures on a compact convex set using ideas from the theory of operator algebras. This yields an extension of Cartier's dilation theorem to the non-separable case and a non-separable version of Šaškin's theorem from approximation theory. They showed that a slight variant of this order characterizes the representations of commutative C^*-algebras with the unique extension property relative to a set of generators. This reduces the commutative case of Arveson's hyperrigidity conjecture to whether measures that are maximal concerning the classical Choquet order are also maximal concerning this new order.

Kleski [13] established the hyperrigidity conjecture for all type-I C^*-algebras with additional assumptions on the co-domain. The hyperrigidity conjecture is still open for general C^*-algebras. The hyperrigidity conjecture inspired several studies in recent years [6, 7, 12, 21]. Arunkumar, Shankar, and

Vijayarajan [1] introduced rectangular hyperrigidity in setting operator spaces in a ternary ring of operators. They established an analogue version of Saskin's theorem in the case of a finite-dimensional ternary ring of operators, and they gave some partial answers analogue to results in the papers [4, 13].

This paper is divided into three sections besides the introduction. In Section 2, we gather the necessary background material and required results. In section 3, we introduce the notions of $\mathcal{Z}$-finite representations for operator $\mathcal{Z}$-systems and $\mathcal{Z}$-separation property of operator $\mathcal{Z}$-systems. These notions are generalizations of finite representation and separating property for representations introduced by Arveson [2]. We use these notions to characterize the $\mathcal{Z}$-boundary representations for operator $\mathcal{Z}$-systems. In section 4, We introduce $\mathcal{Z}$-hyperrigidity of operator $\mathcal{Z}$-systems in generating C^*-algebras which is a generalization of hyperrigidity introduced by Arveson [4]. We investigate an analogue version of Saskin's theorem in the setting of operator $\mathcal{Z}$-systems generating C^*-algebra.

2 PRELIMINARIES

A *representation* of a unital C^*-algebra A on a Hilbert space $\mathcal{H}$ makes $\mathcal{H}$ a Hilbert A-module. Let $\mathbb{B}_A(\mathcal{H})$ denote the set of all bounded A-module maps on $\mathcal{H}$. We will denote von Neumann algebras by $\mathcal{A}, \mathcal{B}, ..., \mathcal{Z}$ and general C^*-algebras by $A, B, ...$ Let A be C^*-algebra and A is faithfully represented on a Hilbert space $\mathcal{H}$. $X \subseteq \mathbb{B}(\mathcal{H})$ is said to be a *faithful operator C-system* if X is a norm closed self-adjoint C-subbimodule of $\mathbb{B}(\mathcal{H})$ (for more details and abstract characterization refer [22]).

Let $\mathcal{H}$ be a Hilbert A-module. Let $\mathrm{CCP}_A(X, \mathbb{B}(\mathcal{H}))$ denote the set of all contractive completely positive A-bimodule maps form X into $\mathbb{B}(\mathcal{H})$. Let $\mathrm{UCP}_A(X, \mathbb{B}(\mathcal{H}))$ denote the set of all unital completely positive A-bimodule maps form X into $\mathbb{B}(\mathcal{H})$. Let X be a faithful operator A-system contained in a C^*-algebra B so that A and B have the same unit 1. By the well-known multiplicative domain argument [22, 3.18] any completely positive extension to B of a map $\varphi \in \mathrm{UCP}_A(X, \mathbb{B}(\mathcal{H}))$ must be a A-bimodule map since φ extends the representation $\varphi|_A$.

The motive of this article is to extend the main results of the papers [2], and [4] in the context of Hilbert spaces are replaced by Hilbert C^*-modules over abelian von Neumann algebra $\mathcal{Z}$. For a theory of Hilbert C^*-modules, we refer to [15, 19]. Hilbert C^*-modules over von Neumann algebras $\mathcal{Z}$ are like Hilbert spaces, except that the inner product takes values in $\mathcal{Z}$. Let $\mathcal{E}$ be Hilbert $\mathcal{Z}$-module, we denote $\langle \cdot, \cdot \rangle$ the $\mathcal{Z}$-valued inner product on $\mathcal{E}$ and let $|e| := \sqrt{\langle e, e \rangle}$ the corresponding $\mathcal{Z}$-valued norm. For $e \in \mathcal{E}$, the scalar-valued norm is denoted by $||e|| := ||\langle x, x \rangle||^{\frac{1}{2}}$. A Hilbert $\mathcal{Z}$-module is said to *self-dual* if each $\mathcal{Z}$-module map ϕ from $\mathcal{E}$ to $\mathcal{Z}$ has the form $\phi(e) = \langle e, f \rangle$ for an $f \in \mathcal{E}$. Let $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ denote the set of all bounded $\mathcal{Z}$-module endomorphisms of $\mathcal{E}$. If $\mathcal{E}$ is self-dual then $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is adjointable. If $\mathcal{E} \subseteq \mathcal{F}$ are self-dual C^*-modules over $\mathcal{Z}$ then $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}^\perp$.

The following definitions and results are due to Magajna [17]. A map $\psi \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ is called $\mathcal{Z}$- dilation of $\varphi \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ for self-dual C^*-module $\mathcal{F} \supseteq \mathcal{E}$ over $\mathcal{Z}$ if $p\psi(x)|_{\mathcal{E}} = \varphi(x)\ \forall\ x \in X$, where $p : \mathcal{F} \to \mathcal{E}$ is the orthogonal projection. We write $\psi \succeq_{\mathcal{Z}} \varphi$ if ψ is a $\mathcal{Z}$-dilation of φ. A map $\varphi \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ is said to be $\mathcal{Z}$-maximal if every $\psi \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$, where $\mathcal{F}$ is a self-dual C^*-module over $\mathcal{Z}$, satisfying $\psi \succeq_{\mathcal{Z}} \varphi$, decomposes as $\psi = \varphi \oplus \theta$ for some $\theta \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}^\perp))$.

Remark 1. *From [17, Remark 4.12] observe that, if an operator $\mathcal{Z}$-system X is contained in a C^*-algebra B generated by X and containing $\mathcal{Z}$ in its center, any map $\varphi \in UCP_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ can be extended to a map $\tilde{\varphi} \in UCP_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$. An analogue version of Stinespring's dilation theorem for $\tilde{\varphi}$ can be represented as follows:*

$$\tilde{\varphi}(b) = V^*\pi(b)V \quad \forall \quad b \in B,$$

where $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ is a representation on a self-dual C^*-module $\mathcal{F}$ over $\mathcal{Z}$ and $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$ is an isometry such that $[\pi(B)V\mathcal{E}] = \mathcal{F}$. Observe that $[\pi(B)V\mathcal{E}] = \mathcal{F}$ is the minimality condition for an analogue version of Stinespring's decomposition. For more details see [15, Theorem 5.6] and [20, Corollary 5.3]. Paschke [20, Proposition 5.4] proved the analogue of Arveson's [2, Theorem 1.4.2] affine order isomorphism theorem.

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Definition 1. *[17] A map $\varphi \in UCP_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ is said to have a $\mathcal{Z}$-unique extension property ($\mathcal{Z}$-u.e.p) if φ has a unique completely positive $\mathcal{Z}$-bimodule extension $\tilde{\varphi} : C^*(X) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ and $\tilde{\varphi}$ is a representation of $C^*(X)$ on $\mathcal{E}$.*

Arveson [3, Proposition 2.4] proved that maximality is equivalent to the notion of unique extension property in the Hilbert space setting. Similar arguments from [3, Proposition 2.4] imply that the idea of $\mathcal{Z}$-maximality is equivalent to the notion of $\mathcal{Z}$-unique extension property in Hilbert $\mathcal{Z}$-module setting.

A representation (i.e., a homomorphism of C^*-algebras) $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is said to be $\mathcal{Z}$-irreducible if $\pi(B)' = \pi(\mathcal{Z})$.

Definition 2. *[17] A map $\varphi \in UCP_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ is said to be $\mathcal{Z}$-pure if every $\psi \in UCP_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$, $\psi \leq \varphi$ implies that $\psi = c\varphi$, where $c \in \mathcal{Z}$.*

Remark 2. *We can observe that an analogue of [2, Corollary 1.4.3] follows from [17, Remark 4.12 and Remark 4.14]. A non zero pure map in $UCP_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ are precisely those of the form $\tilde{\varphi}(b) = V^*\pi(b)V \quad \forall \quad b \in B$, where π is an $\mathcal{Z}$-irreducible representation of B on some self-dual Hilbert C^*-module $\mathcal{F}$ over $\mathcal{Z}$ and $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$, $V \neq 0$.*

Definition 3. *[17] A $\mathcal{Z}$-irreducible representation $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ (for some self-dual $\mathcal{E}$) is called $\mathcal{Z}$-boundary representation of B for X if $\pi|_X$ has the $\mathcal{Z}$-unique extension property.*

Magajna [17] proved analogue of Arveson's conjecture on $\mathcal{Z}$-boundary representations as follows:

Theorem 1. *If X is a central operator $\mathcal{Z}$-system generating a C^*-algebra A, then $\mathcal{Z}$-boundary representation of A for X on self-dual Hilbert C^*-modules over $\mathcal{Z}$ completely norm X.*

3 Z-BOUNDARY REPRESENTATION

This section establishes the characterization theorem for $\mathcal{Z}$-boundary representations. This characterization theorem is an analogue version of [2, Theorem 2.4.5]. In general, checking the given representation is $\mathcal{Z}$-boundary representation is not easy. Using this characterization theorem, at least we can detect the representations that are not $\mathcal{Z}$-boundary representations.

Proposition 1. *Let X be a operator $\mathcal{Z}$-system and B be a C^*-algebra generated by X. If π is a $\mathcal{Z}$-boundary representation of B for X then $\pi|_X$ is $\mathcal{Z}$-pure.*

Proof. Let $\mathcal{E}$ self-dual Hilbert C^*-module over $\mathcal{Z}$ on which π acts. Let $\varphi_1, \varphi_2 \in \mathrm{CP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ be such that $\pi|_X = \varphi_1 + \varphi_2$. By [17, Remark 4.12] each φ_i can be extended to unital completely positive $\mathcal{Z}$-bimodule map $\tilde{\varphi}_i : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ such that $\tilde{\varphi}_i|_X = \varphi$ for $i = 1, 2$. Observe that $\tilde{\varphi}_1 + \tilde{\varphi}_2 : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is a completely positive $\mathcal{Z}$-bimodule extension of $\pi|_X$. Since π is a $\mathcal{Z}$-boundary representation for X, thus $\tilde{\varphi}_1(b) + \tilde{\varphi}_2(b) = \pi(b)$ for all $b \in B$. Also, π is an $\mathcal{Z}$-irreducible representation of B so by Remark 2 $\tilde{\varphi}_1 + \tilde{\varphi}_2$ is a $\mathcal{Z}$-pure map in $\mathrm{CP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$. Thus, there are $c_i \in \mathcal{Z}$ such that $\tilde{\varphi}_i = c_i\pi$ on B for $i = 1, 2$. Restricting to X we have $\varphi_i = c_i\pi|_X$ for $i = 1, 2$. Hence $\pi|_X$ is $\mathcal{Z}$-pure.

Magajna [16, 18] studied an analogue of C^*-convexity and C^*-extreme points of operators on Hilbert C^*-modules. He introduced A-convexity and A-extreme points as follows: Let $\mathcal{K}$ be a Hilbert module over a C^*-algebra A. A subset $K \subseteq \mathbb{B}_A(\mathcal{K})$ is called *A-convex* if $\sum_{j=1}^{n} a_j^* y_j a_j \in K$ whenever $y_j \in K$, $a_j \in A$ and $\sum_{j=1}^{n} a_j^* a_j = 1$. A point x in an A-convex set K is called an *A-extreme point* of K if the condition $x = \sum_{j=1}^{n} a_j^* y_j a_j$, where $x_j \in K$, $a_j \in A$, $\sum_{j=1}^{n} a_j^* a_j = 1$ (n finite) and a_j are invertible, implies that there exist unitary elements $u_j \in A$ such that $x_j = u_j^* x u_j$. By [16, Lemma 5.5], it is enough to check the A-extreme point condition for the case $n = 2$.

Proposition 2. *Let X be a operator $\mathcal{Z}$-system and B be a C^*-algebra generated by X. Let π be a $\mathcal{Z}$-irreducible representation of B such that $\mathfrak{K} = \{\varphi \in UCP_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E})) : \varphi|_X = \pi|_X\}$. If π is a $\mathcal{Z}$-boundary representation of B for X then every $\varphi \in \mathfrak{K}$ is a $\mathcal{Z}$-extreme point of $\mathfrak{K}$.*

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Proof. Let φ be in $\mathfrak{K}$. Suppose $\varphi = \sum_{i=1}^{n} V_i^* \varphi_i V_i$, where $\varphi_i \in \mathfrak{K}$, $V_i \in \mathcal{Z}$, $\sum_{i=1}^{n} V_i^* V_i = 1$ (n finite) and V_i are invertible. Since π is a $\mathcal{Z}$-boundary representation of B for X and $\pi|_X = \varphi|_X$, we have $\pi = \varphi$ on B. An analogue version of minimal Stinespring decomposition of φ is trivial. Thus, by [20, Proposition 5.4] the inequality $V_i^* \varphi_i V_i \leq \varphi$ implies that there exist positive contractions $S_i \in \varphi(B)' = \varphi(\mathcal{Z})$ such that $V_i^* \varphi_i V_i = S_i \varphi$. Therefore $\varphi_i = (S_i^{\frac{1}{2}} V_i^{-1})^* \varphi (S_i^{\frac{1}{2}} V_i^{-1})$ and $\varphi_i(1) = \varphi(1) = 1$. Thus $S_i^{\frac{1}{2}} V_i^{-1}$ is an isometry. Again, using an analogue of minimal Stinespring decomposition of φ is trivial. We conclude that φ_i is unitarily equivalent to φ for every i. Hence φ is a $\mathcal{Z}$-extreme point of $\mathcal{K}$.

We introduce a $\mathcal{Z}$-finite representation, an analogue version of finite representation from [2].

Definition 4. *Let X be an operator $\mathcal{Z}$-system and B be a C^*-algebra generated by X. Let $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ be a representation of B. π is called $\mathcal{Z}$-finite representation for X if for every isometry $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ the condition $V^* \pi(x) V = \pi(x)$ for all $x \in X$, implies that V is unitary.*

Proposition 3. *Let X be an operator $\mathcal{Z}$-system in a C^*-algebra B such that $B = C^*(X)$ and let π be an $\mathcal{Z}$-irreducible representation of B. If π is a $\mathcal{Z}$-boundary representation for X then π is a $\mathcal{Z}$-finite representation for X.*

Proof. Let π acts on the self-dual Hilbert C^*-module $\mathcal{E}$ and let V be an isometry in $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ such that $V^* \pi(x) V = \pi(x)$ for all $x \in X$. Then $V^* \pi(\cdot) V : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is a completely positive $\mathcal{Z}$-bimodule extension of $\pi|_X$. Since π is a $\mathcal{Z}$-boundary representation implies that $V^* \pi(b) V = \pi(b)$ for all $b \in B$. We have V is isometry and $[\pi(B) V \mathcal{E}] = \mathcal{E}$ implies $V \mathcal{E}$ is a reducing subspace for $\pi(B)$. Also, π is $\mathcal{Z}$-irreducible implies $V \mathcal{E} = \mathcal{E}$. Therefore V is unitary. Hence π is a $\mathcal{Z}$-finite representation for X.

We introduce separating operator $\mathcal{Z}$-system, an analogue version of separating operator system from [2].

Definition 5. *Let X be an operator $\mathcal{Z}$-system and B be a C^*-algebra generated by X. Let $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ be a $\mathcal{Z}$-irreducible representation of B. We say that X $\mathcal{Z}$-separates π if for every $\mathcal{Z}$-irreducible representation σ of B on self-dual Hilbert $\mathcal{Z}$-module $\mathcal{F}$ and for every isometry $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$, the condition $V^* \sigma(x) V = \pi(x)$ for all $x \in X$ implies that σ and π are unitarily equivalent representations of B.*

Proposition 4. *Let X be an operator $\mathcal{Z}$-system in a C^*-algebra B such that $B = C^*(X)$. If $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is a $\mathcal{Z}$-boundary representation of B for X then X $\mathcal{Z}$-separates π.*

Proof. Let $\sigma : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ be a $\mathcal{Z}$-irreducible representation of B, where $\mathcal{F}$ is self-dual Hilbert $\mathcal{Z}$-module and $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$ is an isometry such that $V^* \sigma(x) V = \pi(x)$ for all $x \in X$. Since π is a $\mathcal{Z}$-boundary representation for X and $V^* \sigma(\cdot) V$ is a completely positive $\mathcal{Z}$-bimodule extension of $\pi|_X$ implies $V^* \sigma(b) V = \pi(b)$ for all $b \in B$. We have V is isometry and $[\pi(B) V \mathcal{E}] = \mathcal{F}$ implies $V \mathcal{E}$ is a reducing subspace for $\sigma(B)$. Also, σ is $\mathcal{Z}$-irreducible implies $V \mathcal{E} = \mathcal{F}$. Thus V is unitary, showing that σ and π are unitarily equivalent representations. Hence X $\mathcal{Z}$-separates π.

The characterization theorem for $\mathcal{Z}$-boundary representations as follows:

Theorem 2. *Let X be an operator $\mathcal{Z}$-systems in a C^*-algebra B such that $B = C^*(X)$. Let π be an $\mathcal{Z}$-irreducible representation of B. Then π is a $\mathcal{Z}$-boundary representation for X if and only if the following conditions are satisfied:*

(i) *π is a $\mathcal{Z}$-finite representation for X.*

(ii) *$\pi|_X$ is $\mathcal{Z}$-pure.*

(iii) *X $\mathcal{Z}$-separates π.*

(iv) *Let $\mathfrak{K} = \{\varphi \in CP_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E})) : \varphi|_X = \pi|_X\}$ and every φ in $\mathfrak{K}$ is $\mathcal{Z}$-extreme point of $\mathfrak{K}$.*

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Proof. Suppose π is a $\mathcal{Z}$-boundary representation of B for X then conditions (i), (ii),(iii) and (iv) follows from Proposition 3, Proposition 1, Proposition 4 and Proposition 2.

Conversely, Suppose the $\mathcal{Z}$-irreducible representation satisfies all four conditions (i), (ii), (iii), and (iv). Let $\mathfrak{K} = \{\varphi \in \mathrm{CP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E})) : \varphi|_X = \pi|_X\}$. To show π is a $\mathcal{Z}$-boundary representation for X, it is enough to show that $\mathfrak{K}$ is $\{\pi\}$. Using (iv), let φ be $\mathcal{Z}$-extreme point of $\mathfrak{K}$. Now, we prove that φ is a $\mathcal{Z}$-pure in $\mathrm{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$. Let φ_1, φ_2 be in $\mathfrak{K}$ such that $\varphi_1(b) + \varphi_2(b) = \varphi(b)$ for all $b \in B$. In particular, $\varphi_1(x) + \varphi_2(x) = \varphi(x)$ for all $x \in X$. Our assumption, $\varphi|_X = \pi|_X$ is pure implies there exists $c_i \geq 0$ in $\mathcal{Z}$ such that $\varphi_i(x) = c_i\varphi(x)$ for all $x \in X$. If $c_1 = 0$ and $1 \in X$ then $\varphi_1(1) = 0$, thus $\varphi_1 = 0$. This contracts to the choice of φ_1, therefore $c_1 > 0$ and similarly $c_2 > 0$. Using [16, Definition 5.1 and Proposition 5.2], and every $\mathcal{Z}$-extreme points are Choquet $\mathcal{Z}$-points, we have $\varphi_i = (c_i^{\frac{1}{2}})^*\varphi(c_i^{\frac{1}{2}})$ and $(c_1^{\frac{1}{2}})^*c_1^{\frac{1}{2}} + (c_2^{\frac{1}{2}})^*c_2^{\frac{1}{2}} = 1$. Now put $\psi_i = (c_i^{-\frac{1}{2}})^*\varphi_i(c_i^{-\frac{1}{2}})$ for $i = 1, 2$. Then $\psi_i \in \mathfrak{K}$ and $(c_1^{\frac{1}{2}})^*\psi_1(c_1^{\frac{1}{2}}) + (c_2^{\frac{1}{2}})^*\psi_2(c_1^{\frac{1}{2}}) = \varphi$. Since φ is $\mathcal{Z}$-extreme point of $\mathfrak{K}$ then there exists unitary elements $u_i \in \mathcal{Z}$ for $i = 1, 2$ such that $\psi_i = u_i^*\varphi u_i$ for $i = 1, 2$. We have $(c_i^{-\frac{1}{2}})^*\varphi_i(c_i^{-\frac{1}{2}}) = u_i^*\varphi u_i$ for $i = 1, 2$ and $c_i^{-1}\varphi_i = u_i^*\varphi u_i$ for $i = 1, 2$. By [16, Definition 5.1 and Proposition 5.2], thus $\varphi_i = c_i\varphi$ for $i = 1, 2$. Hence φ is a $\mathcal{Z}$-pure in $\mathrm{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$.

By Remark 2, there is a $\mathcal{Z}$-irreducible representation σ of B on a self-dual Hilbert $\mathcal{Z}$-module $\mathcal{F}$ and an isometry $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$ such that $\varphi = V^*\sigma V$. In particular, $\pi(x) = \varphi(x) = V^*\sigma(x)V$ for all $x \in X$. The assumption (iii), X $\mathcal{Z}$-separates π implies that σ is unitarily equivalent to π. Thus there exists a unitary $U \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$ such that $\sigma = U^*\pi U$. Therefore we have $\pi(x) = (UV)^*\pi(x)UV$ for all $x \in X$. UV is isometry in $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$. The assumption (i), π is a $\mathcal{Z}$-finite representation for X implies UV is unitary. Thus $V = U^*UV$ is a unitary in $\mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$. Now $\pi|_X = V^*\sigma V|_X$ becomes $\pi(x) = V^{-1}\sigma(x)V$ for all $x \in X$. $V^{-1}\sigma V$ is a representation of B which agrees with π on X. Therefore $\pi(b) = V^{-1}\sigma(b)V$ for all $b \in B = C^*(X)$. Hence $\varphi = \pi$ on B.

4 Z-HYPERRIGIDITY

In this section, we introduce the notion of $\mathcal{Z}$-hyperrigidity in the operator $\mathcal{Z}$-system. $\mathcal{Z}$-hyperrigidity is an analogue version of Arveson's [4] notion of hyperrigidity. We define $\mathcal{Z}$-hyperrigidity as follows:

Definition 6. *Let A be a C^*-algebra, and let $G \subseteq A$(finite or countably infinite) be a set of generators of A (i.e., $A = C^*(G)$). Then G is said to be $\mathcal{Z}$-hyperrigid if for every faithful representation $A \subseteq \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ of A on a self-dual Hilbert $\mathcal{Z}$-module $\mathcal{E}$ and every sequence of unital completely positive $\mathcal{Z}$-bimodule maps $\varphi_n : \mathbb{B}_{\mathcal{Z}}(\mathcal{E}) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$, $n = 1, 2, ...,$*

$$\lim_{n\to\infty} ||\varphi_n(g) - g|| = 0, \ \forall\ g \in G \Longrightarrow \lim_{n\to\infty} ||\varphi_n(a) - a|| = 0, \ \forall\ a \in A. \tag{1}$$

We have lightened notation in the above definition by identifying the C^*-algebra A with its image $\pi(A)$ in a faithful nondegenerate representation $\pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ on a self-dual Hilbert $\mathcal{Z}$-module $\mathcal{E}$. Notably, $\mathcal{Z}$-hyperrigidity of operator $\mathcal{Z}$-system on a self-dual Hilbert $\mathcal{Z}$-module implies not only that (1) should hold for sequences of UCP $\mathcal{Z}$-bimodule maps φ_n defined on $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$, but also that the property should hold for every other faithful representation of A. If $\mathcal{Z} = \mathbb{C}$, then the definition of $\mathcal{Z}$-hyperrigity is the same as the definition of hyperrigidity in [4, definition 1.1].

Proposition 5. *Let A be a C^*-algebra and G be a generating subset of A. Then G is $\mathcal{Z}$-hyperrigid if and only if the operator $\mathcal{Z}$-system generated by G is $\mathcal{Z}$-hyperrigid.*

Proof. The proof follows directly from the definition of $\mathcal{Z}$-hyperrigidity.

Now we prove the characterization theorem for $\mathcal{Z}$-hyperrigid operator $\mathcal{Z}$-systems.

Theorem 3. *Let X be a separable operator $\mathcal{Z}$-system and X generates a C^*-algebra A (i.e., $A = C^*(X)$). The following are equivalent:*

(i) X is $\mathcal{Z}$-hyperrigid.

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(ii) *For every nondegenerate representation* $\pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ *on a separable self-dual Hilbert* $\mathcal{Z}$*-module and every sequence* $\varphi_n : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ *of UCP* $\mathcal{Z}$*-bimodule maps,*

$$\lim_{n\to\infty} ||\varphi_n(x) - \pi(x)|| = 0 \ \forall \ x \in X \Longrightarrow \lim_{n\to\infty} ||\varphi_n(a) - \pi(a)|| = 0 \ \forall \ a \in A.$$

(iii) *For every nondegenerate representation* $\pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ *on a separable self-dual Hilbert* $\mathcal{Z}$*-module,* $\pi|_X$ *has the* $\mathcal{Z}$*-unique extension property.*

(iv) *For every unital* C^**-algebra* B*, every unital homomorphism of* C^**-algebra* $\theta : A \to B$ *and every UCP* $\mathcal{Z}$*-module map* $\varphi : B \to B$*,*

$$\varphi(x) = x \ \forall \ x \in \theta(X) \Longrightarrow \varphi(x) = x \ \forall \ x \in \theta(A).$$

Proof. *(i)* $\Longrightarrow$ *(ii): Let* $\pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ *be a nondegenerate representation on a separable self-dual Hilbert* $\mathcal{Z}$*-module and let* $\varphi_n : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ *be a sequence of UCP* $\mathcal{Z}$*-module maps such that* $||\varphi_n(x) - \pi(x)|| \to 0$ *for all* $x \in X$*.*

Let $\sigma : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ *be a faithful representation of* A *on another separable self-dual Hilbert* $\mathcal{Z}$*-module* $\mathcal{F}$*. Then* $\sigma \oplus \pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F} \oplus \mathcal{E})$ *is a faithful representation, so that each of the* $\mathcal{Z}$*-module maps* $\psi_n : (\sigma \oplus \pi)(A) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F} \oplus \mathcal{E})$

$$\psi_n : \sigma(a) \oplus \pi(a) \mapsto \sigma(a) \oplus \varphi_n(a), \quad a \in A,$$

is unital completely positive $\mathcal{Z}$*-module map. By [17, Remark 4.12]* ψ_n *can be extended to a UCP* $\mathcal{Z}$*-module map* $\tilde{\psi}_n : \mathbb{B}_{\mathcal{Z}}(\mathcal{F} \oplus \mathcal{E}) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F} \oplus \mathcal{E})$*. By our assumption* $\varphi_n|_X$ *converges to* $\pi|_X$ *in pointwise norm. Thus* $\tilde{\psi}_n$ *converges in pointwise norm to the identity map on* $(\sigma \oplus \pi)(X)$*. Since* X *is* $\mathcal{Z}$*-hyperrigid, we have* $\tilde{\psi}_n$ *converges in pointwise norm to the identity map on* $(\sigma \oplus \pi)(A)$*. Therefore for every* $a \in A$*,*

$$\begin{aligned} \limsup_{n\to\infty} ||\varphi_n(a) - \pi(a)|| &\leq \limsup_{n\to\infty} ||\sigma(a) \oplus \varphi_n(a) - \sigma(a) \oplus \pi(a)|| \\ &= \lim_{n\to\infty} ||\tilde{\psi}_n(\sigma(a) \oplus \pi(a)) - \sigma(a) \oplus \pi(a)|| = 0. \end{aligned}$$

Hence φ_n *converges in pointwise norm to* π *on* A*.*

(ii) $\Longrightarrow$ *(iii): Let* $\varphi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ *be a unital completely positive* $\mathcal{Z}$*-module map such that* $\varphi|_X = \pi|_X$*. Take* $\varphi_n(X) = \varphi(X)$ *for all* $n \in \mathbb{N}$*, so by hypothesis (ii),* $\varphi(A) = \pi(A)$*. Thus* $\pi|_X$ *has the* $\mathcal{Z}$*-unique extension property.*

(iii) $\Longrightarrow$ *(iv): Let* ρ *be a unital* $*$*-homomorphism from* C^**-algebra* A *to* C^**-algebra* B*. Let* $\varphi : B \to B$ *be a UCP* $\mathcal{Z}$*-module map.* φ *satisfies* $\varphi(\rho(x)) = \rho(x) \ \forall \ x \in X$*. We claim that* $\varphi(\rho(a)) = \rho(a) \ \forall \ a \in A$*.*

Let B_0 *be the separable* C^**-sub algebra of* B *generated by*

$$\rho(A) \cup \varphi(\rho(A)) \cup \varphi^2(\rho(A)) \cup \cdots .$$

Observe that $\varphi(B_0) \subseteq B_0$*. Since* B_0 *is separable, we can faithfully represent* $B_0 \subseteq \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ *for some separable self-dual Hilbert* $\mathcal{Z}$*-module* $\mathcal{F}$*. By [17, Remark 4.12], there is a UCP* $\mathcal{Z}$*-module map* $\tilde{\varphi} : \mathbb{B}_{\mathcal{Z}}(\mathcal{F}) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ *such that* $\tilde{\varphi}|_{B_0} = \varphi$ *and in particular* $\tilde{\varphi}(\rho(x)) = \rho(x)$ *for* $x \in X$*. Since* $a \in A \mapsto \rho(a) \in \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ *is a representation on a separable self-dual Hilbert* $\mathcal{Z}$*-module. Our assumption (iii) implies that* $\tilde{\varphi}$ *must fix* $\rho(A)$ *elementwise. Therefore* $\varphi(\rho(a)) = \tilde{\varphi}(\rho(a)) = \rho(a) \ \forall \ a \in A$*.*

(iv)$\Longrightarrow$ *(i): Suppose that* $A \subseteq \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ *is faithfully represented on some self-dual Hilbert* $\mathcal{Z}$*-module* $\mathcal{F}$*, and* $\varphi_1, \varphi_2, \cdots : \mathbb{B}_{\mathcal{Z}}(\mathcal{F}) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ *is a sequence of UCP* $\mathcal{Z}$*-module maps satisfying* $\lim_{n\to\infty} ||\varphi_n(x) - x|| = 0 \ \forall \ x \in X$*. We claim that*

$$\lim_{n\to\infty} ||\varphi_n(a) - a|| = 0, \quad \forall \ a \in A.$$

Let $\ell^\infty(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ *denote the set of all bounded sequences with components in* $\mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ *such that* $\ell^\infty(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ *is a* C^**-algebra. Let* $c_0(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ *denote the set of all sequences in* $\ell^\infty(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ *that converges to zero in norm and* $c_0(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ *is ideal in* $\ell^\infty(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$*.*

 https://doi.org/10.17993/3cemp.2022.110250.173-184

Define the UCP $\mathcal{Z}$-module map $\varphi_0 : \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F})) \to \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$ as follows:

$$\varphi_0(a_1, a_2, a_3, ...) = (\varphi_1(a_1), \varphi_2(a_2), \varphi_3(a_3), ...).$$

Thus the map φ_0 carries the ideal $c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$ into itself. Hence we can define the UCP $\mathcal{Z}$-module map of the quotient $\varphi : \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})) \to \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$ by

$$\varphi(x + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))) = \varphi_0(x) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})), \quad x \in \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F})).$$

Now consider the natural embedding $\rho : A \to \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$,

$$\rho(a) = (a, a, a, ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})).$$

By our assumption, $||\varphi_n(x) - x|| \to 0$ as $n \to \infty$ for $x \in X$, and thus

$$\varphi(\rho(x)) = (\varphi_1(x), \varphi_2(x), ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})) = (x, x, ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})) = \rho(x).$$

Therefore φ restricts the identity map on $\rho(X)$.

Applying assumption (iv) to the inclusions

$$\rho(X) \subseteq \rho(A) \subseteq \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$$

and the UCP $\mathcal{Z}$-module map $\varphi : \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})) \to \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$, implies that φ must fix every element of $\rho(A)$. Since $\rho(a) = (a, a, ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$ and

$$\varphi(\rho(a)) = (\varphi_1(a), \varphi_2(a), ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})),$$

hence we have $(\varphi_1(a) - a, \varphi_2(a) - a, ...) \in c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$. This proves our claim.

Now we discuss the examples of $\mathcal{Z}$-hyperrigidity. Let $V_1, V_2, ..., V_n$ be an arbitrary set of isometries acting on some self-dual Hilbert $\mathcal{Z}$-module. We exhibit a $\mathcal{Z}$-hyperrigid generator for a C^*-algebra generated by the isometries $V_1, V_2, ..., V_n$.

Theorem 4. *Let $V_1, V_2, ..., V_n$ be a set of isometries on some self-dual Hilbert $\mathcal{Z}$-module and generate a C^*-algebra A. Let*

$$G = \{V_1, V_2, ..., V_n, V_1V_1^* + V_2V_2^* + \cdots + V_nV_n^*\}$$

then G is a $\mathcal{Z}$-hyperrigid generator for A.

Proof. Let X be the operator $\mathcal{Z}$-systems generated by G. By Corollary 5, G is hyperrigid if and only if X is hyperrigid. Using Theorem 3, it is enough to prove that for every nondegenerate representation π of A, $\pi|_X$ has the $\mathcal{Z}$-unique extension property.

Consider a representation $\pi : A \to \mathbb{B}_\mathcal{Z}(\mathcal{E})$ and let $W_1, W_2, ..., W_n$ be isometries such that $W_i = \pi(V_i),\ i = 1, 2, ..., n$. Let $\varphi : A \to \mathbb{B}_\mathcal{Z}(\mathcal{E})$ be a UCP $\mathcal{Z}$-module map satisfying

$$\varphi(V_i) = W_i,\ 1 \le i \le n,$$

and

$$\varphi(V_1V_1^* + V_2V_2^* + \cdots + V_nV_n^*) = W_1W_1^* + W_2W_2^* + \cdots + W_nW_n^*.$$

Thus, $\varphi(x) = \pi(x)\ \forall\ x \in X$. We claim that $\varphi = \pi$ on A.

From Remark 1, Using an analogue version of the Stinespring's dilation theorem. We can express the UCP $\mathcal{Z}$-module map φ as follows:

$$\varphi(a) = W^*\sigma(a)W, \qquad \forall\ a \in A.$$

Where σ is a representation of A on a self-dual Hilbert $\mathcal{Z}$-module $\mathcal{F}$, $W : \mathcal{E} \to \mathcal{F}$ is an isometry, and which is minimal in the sense that span closure of $\sigma(A)W\mathcal{E}$ is $\mathcal{F}$.

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First we prove that $\sigma(V_i)W = WW_i$, $1 \le i \le n$. For $i = 1, 2, ...n$ we have

$$W^*\sigma(V_i)WW^*\sigma(V_i)V = \varphi(V_i)^*\varphi(V_i) = W_i^*W_i = \mathbf{1}_{\mathcal{E}},$$

thus $W^*\sigma(V_i)(\mathbf{1} - WW^*)\sigma(V_i)W = 0$, therefore $W\mathcal{E}$ is invariant under $\sigma(V_i)$. Hence we get $\sigma(V_i)W = WW^*\sigma(V_i)W = W\varphi(V_i) = WW_i$.

Next, since $\sum_{i=1}^{n} W_iW_i^* = \pi(\sum_{i=1}^{n} V_iV_i^*) = \varphi(\sum_{i=1}^{n} V_iV_i^*)$, we get

$$\begin{aligned}\sum_{i=1}^{n} \sigma(V_i)WW^*\sigma(V_i)^* &= \sum_{i=1}^{n} WW_iW_i^*W^* \\ &= W\varphi(\sum_{i=1}^{n} V_iV_i^*)W^* \\ &= WW^*\sum_{i=1}^{n}\sigma(V_iV_i^*)WW^* \\ &= \sum_{i=1}^{n} WW^*\sigma(V_i)\sigma(V_i^*)WW^*.\end{aligned}$$

We know that $\sigma(V_i)W = WW^*\sigma(V_i)W$ for all i. In the above equations, subtract the left side from the right, and we have

$$\sum_{i=1}^{n} WW^*\sigma(V_i)(\mathbf{1}_{\mathcal{F}} - WW^*)\sigma(V_i)^*WW^* = 0.$$

Thus $(\mathbf{1}_{\mathcal{F}} - WW^*)\sigma(V_i)^*WW^* = 0$ for all $i = 1, 2, ..., n$. Therefore $W\mathcal{E}$ is invariant under both $\sigma(V_i)$ and $\sigma(V_i)^*$ for all $i = 1, 2, ..., n$. Since the C^*-algebra A is generated by the V_i, we have $\sigma(A)W\mathcal{E} \subseteq W\mathcal{E}$. By the minimality condition, we have $W\mathcal{E} = \mathcal{F}$. Thus W is unitary. Therefore $\varphi(a) = W^{-1}\sigma(a)W$ is a representation on A. By our assumption, φ agrees with π on a generating set. Hence $\varphi = \pi$ on C^*-algebra A.

The Cuntz algebras $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $V_1, V_2, ..., V_n$ such that $V_1V_1^* + V_2V_2^* + \cdots + V_nV_n^* = \mathbf{1}$. We can discard the identity operator from the generating set G to conclude the above result.

Corollary 1. *The set $G = \{V_1, V_2, ..., V_n\}$ of generators of the Cuntz algebra $\mathcal{O}_n$ is $\mathcal{Z}$-hyperrigid.*

Theorem 5. *Let X be a separable operator $\mathcal{Z}$-system generating a C^*-algebra A. If X is $\mathcal{Z}$-hyperrigid then every $\mathcal{Z}$-irreducible representation of A is a $\mathcal{Z}$-boundary representation for X.*

Proof. Suppose X is an operator $\mathcal{Z}$-system in a C^*-algebra A. Then by Theorem 3, every nondegenerate representation of A on separable self-dual Hilbert $\mathcal{Z}$-module has the $\mathcal{Z}$-unique extension property when nondegenerate representation restricted to X. Since every $\mathcal{Z}$-irreducible representation of a C^*-algebra A is a nondegenerate representation of A. Therefore, every $\mathcal{Z}$-irreducible representation of A on separable self-dual Hilbert $\mathcal{Z}$-module has the $\mathcal{Z}$-unique extension property when $\mathcal{Z}$-irreducible representation restricted to X. Hence, every $\mathcal{Z}$-irreducible representation of A is a $\mathcal{Z}$-boundary representation for X.

Problem 1. *Let X be a separable operator $\mathcal{Z}$-system generating a C^*-algebra A. If every $\mathcal{Z}$-irreducible representation of C^*-algebra A is a $\mathcal{Z}$-boundary representation for a separable operator $\mathcal{Z}$-system $X \subseteq A$. Then X is $\mathcal{Z}$-hyperrigid.*

Proposition 6. *Let X be an operator $\mathcal{Z}$-system generating a C^*-algebra $A = C^*(X)$. Let $\pi_i : A \to \mathbb{B}(\mathcal{E}_i)$ be a representation on a self-dual Hilbert $\mathcal{Z}$-module such that $\pi_i|_X$ has the $\mathcal{Z}$-unique extension property for each i in an index set I. Then the direct sum of UCP $\mathcal{Z}$-module maps*

$$\pi = \oplus_{i\in I}\pi_i|_X : X \to \mathbb{B}(\oplus_{i\in I}\mathcal{E}_i)$$

has the $\mathcal{Z}$-unique extension property.

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Proof. Let $\varphi : A \to \mathbb{B}(\oplus_{i\in I}\mathcal{E}_i)$ be a UCP $\mathcal{Z}$-module map such that $\pi|_X = \varphi|_X$. For each $i \in I$, let $\varphi_i : A \to \mathbb{B}(\mathcal{E}_i)$ be the UCP $\mathcal{Z}$-module map such that

$$\varphi_i(a) = P_i\varphi(a)|_{\mathcal{E}_i}, \quad a \in A$$

where P_i is the projection from $\oplus_{i\in I}\mathcal{E}_i$ onto $\mathcal{E}_i$. Observe that $\varphi|_X = \pi|_X$. Our assumption $\pi_i|_X$ has $\mathcal{Z}$-unique extension property implies that $\varphi_i(a) = \pi_i(a)$ for all $a \in A$. Equivalently, we have $P_i\varphi(a)P_i = \pi(a)P_i$ for all $a \in A$. Using the Schwarz inequality of φ, we have

$$\begin{aligned} P_i\varphi(a)^*(\mathbf{1} - P_i)\varphi(a)P_i &= P_i\varphi(a)^*\varphi(a)P_i - P_i\varphi(a)^*P_i\varphi(a)P_i \\ &\leq P_i\varphi(a^*a)P_i - P_i\varphi(a)^*P_i\varphi(a)P_i \\ &= \pi(a^*a)P_i - \pi(a)^*\pi(a)P_i = 0. \end{aligned}$$

Therefore, $|(\mathbf{1} - P_i)\varphi(a)P_i|^2 = 0$. Thus it follows that P_i commutes with the self-adjoint family of operators $\varphi(A)$. Hence for every $a \in A$, we have

$$\varphi(a) = \sum_{i\in I}\varphi(a)P_i = \sum_{i\in I}P_i\varphi(a)P_i = \sum_{i\in I}\pi(a)P_i = \pi(a).$$

Let A be a separable C^*-algebra. The set of unitary equivalence classes of $\mathcal{Z}$-irreducible representations of A is said to be a spectrum of A.

Theorem 6. *Let X be a separable operator $\mathcal{Z}$-system generating a C^*-algebra A and let A have a countable spectrum. If every $\mathcal{Z}$-irreducible representation of A is a $\mathcal{Z}$-boundary representation for X then X is $\mathcal{Z}$-hyperrigid.*

Proof. By the Theorem 3, it is enough to prove that for every representation $\pi : A \to \mathbb{B}(\mathcal{E})$ of A on a separable self-dual Hilbert $\mathcal{Z}$-module, the UCP Z-module map $\pi|_X$ has the $\mathcal{Z}$-unique extension property. Since the spectrum A is countable, A is the type I C^*-algebra. Therefore π can be decomposed uniquely into a direct integral of mutually disjoint type I factor representation. Because the spectrum A is countable, the direct integral must be a countable direct sum. Hence π can be decomposed into a direct sum of subrepresentations $\pi_n : A \to \mathbb{B}(\mathcal{E}_n)$ of A on a separable self-dual Hilbert $\mathcal{Z}$-modules. Thus

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \qquad \pi = \pi_1 \oplus \pi_2 \oplus \cdots$$

With the property that each π_n is unitarily equivalent to a finite or countable direct sum of copies of a single $\mathcal{Z}$-irreducible representations $\sigma_n : A \to \mathbb{B}(\mathcal{F}_n)$ of A on a separable self-dual Hilbert $\mathcal{Z}$-modules.

By our assumption, each UCP $\mathcal{Z}$-module map $\sigma_n|_X$ has the $\mathcal{Z}$-unique extension property. Therefore the above decomposition of $\pi|_X$ can be expressed as a double direct sum of UCP $\mathcal{Z}$-module maps with the $\mathcal{Z}$-unique extension property. Using Proposition 6, we have $\pi|_X$ has the $\mathcal{Z}$-unique extension property.

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/13/

ADDITIVE NUMBER THEORY: NOTES AND SOME PROBLEMS

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ABSTRACT

A brief account of some of the major results in additive number theory is given along with a small list of problems.

KEYWORDS

Number theory

1 INTRODUCTION

Additive number theory is a relatively young discipline and has seen some spectacular progress in the last few decades. The aim of this short article is to give a brief description of some of the major results and also list a few problems that are easy to state and undertstand but not clear how they can (or will) be solved. The goal is to stimulate and kindle interest in the subject. This is a review paper and most of the material is well known. The papers of Melvyn B. Nathanson [3], [4], Kannan Soundararajan [5], Andrew Granville [2], and H V Chu et al [1] are especially informative and detailed. We begin with a well known and popular result, the Friendship theorem. **Friendship Theorem**
At any party with at least six people, there are three people who are all either mutual acquaintances (each one knows the other two) or mutual strangers.

Ramsey Theory
For any given integer c, any given integers $n_1, \dots, n_c$, there is a number, $R(n_1, \dots, n_c)$, such that if the edges of a complete graph of order $R(n_1, \dots, n_c)$ are coloured with c different colours, then for some i between 1 and c, it must contain a complete subgraph of order n_i whose edges are all colour i.
The special case above has $c = 2$ and $n_1 = n_2 = 3$.

Broad Philosophy Can find order in disorder. Are there such analogs in Number theory? Answer is YES. Order is Arithmetic progressions (APs). 3-AP means 3 consecutive terms in AP and k-AP means k consecutive terms in AP. One can ask under what conditions will a set A of integers contain atleast one 3-AP? No 3-AP at all? Infinitely many 3-APs?
Broadly, given a sufficiently large set of integers A we interested in understanding additive patterns that appear in A. An important example is whether A contains non-trivial arithmetic progressions of some given length k.
Reason: They are quite indestructible structures. They are preserved under translations and dilations of A, and they cannot be excluded for trivial congruence reasons.
For example the pattern a, b and $a+b$ all being in the set seems quite close to the arithmetic progression case $a, b, (a + b)/2$, but the former case can never occur in any subset of the odd integers (and such subsets can be very large).
Other questions: Whether all numbers can be written as a sum of s elements from a given set A. For example, all numbers are sums of four squares, nine cubes etc. Waring's problem and the Goldbach conjectures are two classical examples. In the same spirit, given a set A of N integers we may ask for information about the sumset $A + A := \{a + b : a, b \in A\}$. If there are not too many coincidences, then we may expect $|A + A| \gg N^2$. But when A is an AP, $|A + A| \leq 2|A| - 1$. The subject may be said to begin with a beautiful result of van der Waerden (1927).

Theorem 1. ***(van der Waerden)****. Let k and r be given. There exists a number $N = N(k, r)$ such that if the integers in $[1, N]$ are colored using r colors, then there is a non-trivial monochromatic k term arithmetic progression.*

van der Waerden's proof was by an ingenious elementary induction argument on k and r. The proof does not give any good bound on how large $N(k, r)$ needs to be. A more general result was subsequently found by Hales and Jewett (1963), with a nice refinement of Shelah (1988), but again the bounds for the van der Waerden numbers are quite poor.

Theorem 2. ***(Schur)****. Given any positive integer $r > 1$, if $N \geq N(r)$ and the integers in $[1, N]$ are colored using r colors then there is a monochromatic solution to $x + y = z$.*

Lemma 1. *Suppose that the edges of the complete graph K_N are colored using r colors. If $N \geq N(r)$ then there is a monochromatic triangle.*

Proof: We will use induction on r. It is very well known that if $r = 2$ and $N \geq 6$ then there is a monochromatic triangle. Suppose we know the result for $r - 1$ colorings, and we need $N \geq N(r - 1)$ for that result. Pick a vertex. There are $N - 1$ edges coming out of it. So for some color there are $[(N - 1)/r] + 1$ edges starting from this vertex having the same color. Now the complete graph on the

 https://doi.org/10.17993/3cemp.2022.110250.186-196

other vertices of these edges must be colored using only $r-1$ colors. Thus if $N \geq rN(r-1)-r+2$ we are done.

Proof of Schur's Theorem: Consider the complete graph on N vertices labeled 1 through N. Color the edge joining a to b using the color of $|a-b|$. By our lemma, if N is large then there is a monochromatic triangle. Suppose its vertices are $a<b<c$, then $(c-a)=(c-b)+(b-a)$ is a solution, proving Schur's theorem.

Theorem 3. ***(Hales-Jewett)****. Let k and r be given. There exists a number $N=N(k,r)$ such that if the points in $[1,k]^N$ are colored using r colors then there is a monochromatic "combinatorial line". Here a combinatorial line is a collection of k points of the following type: certain of the coordinates are fixed, and a certain non-empty set of coordinates are designated as "wildcards" taking all the values from 1 to k.*

A picturesque way of describing the Hales-Jewett theorem is that a "tic-tac-toe" game of getting k in a row, played by r players, always has a result in sufficiently high dimensions. Since there is obviously no disadvantage to going first, the first player wins; but no constructive strategy solving the game is known. One can recover van der Waerden's theorem by thinking of $[1,k]^N$ as giving the base k digits(shifted by 1) of numbers in $[0,k^N-1]$.

Erdos and Turan proposed a stronger form of the van der Waerden, partly in the hope that the solution to the stronger problem would lead to a better version of van der Waerden's theorem.

The Erdos-Turan conjecture: Let δ and k be given. There is a number $N=N(k,\delta)$ such that any set $A \subset [1,N]$ with $|A| \geq \delta N$ contains a non-trivial arithmetic progression of lenghth k.

In 1953, Roth proved the Erdos - Turan conjecture in the case $k=3$.

Theorem 4. ***(Roth)****. There exists a positive constant C such that if $A \subset [1,N]$ with $|A| \geq CN/\log\log N$ then A has a non-trivial three term AP. In other words, $N(\delta,3) \leq exp(exp(C/\delta))$ for some positive constant C.*

This stronger result does in fact give a good bound on the van der Waerden numbers for $k=3$. We know now(Bourgain), that $|A| \gg N(\log\log N/\log N)^{1/2}$ suffices. Thus the double exponential bound can be replaced by a single exponential.

Let $r_3(N)$ denote the size of the largest subset of $[1,N]$ having no non-trivial three term APs. Then as mentioned above, $r_3(N) \ll N\sqrt{\log\log N/\log N}$. What is the true nature of $r_3(N)$? If we pick a random set A in $[1,N]$ we may expect that it has about $|A|^3/N$ three term APs. This suggests that $r_3(N)$ is perhaps of size $N^{\frac{1}{3}}$. However, in 1946 Behrend found an ingenious construction that does much much better.

Theorem 5. ***(Behrend)****. There exists a set $A \subset [1,N]$ with $|A| \gg Bexp(-c\sqrt{logN})$ containing no non-trivial three term arithmetic progressions. In other words $r_3(N) \gg Nexp(-c\sqrt{\log N})$.*

Roth's proof is based on Fourier analysis. It falls naturally into two parts: either the set A looks random in which case we may easily count the number of three term progressions, or the set has some structure which can be exploited to find a subset with increased density. The crucial point is that the idea of randomness here can be made precise in terms of the size of the Fourier coefficients of the set. This argument is quite hard to generalize to four term progressions (or longer), and was only extended recently with the spectacular work of Gowers.

Returning to the Erdős-Turan conjecture, the next big breakthrough was made by Szemeredi who in 1969 established the case $k=4$, and in 1975 dealt with the general case $k \geq 5$. His proof was a tour-de-force of extremely ingenious and difficult combinatorics. One of his ingredients was van der Waerden's theorem, and so this did not lead to a good bound there.

Theorem 6. ***(Szemeredi)****. Given k and $\delta > 0$, there exists $N=N(k,\delta)$ such that any set $A \subset [1,N]$ with $|A| \geq \delta N$ contains a non-trivial k term arithmetic progression.*

 https://doi.org/10.17993/3cemp.2022.110250.186-196

An entirely different approach was opened by the work of Furstenberg (1977) who used ergodic theoretic methods to obtain a new proof of Szemeredi's theorem. The ergodic theoretic approach also did not lead to any good bounds, but was useful in proving other results previously inaccessible. For example, it led to a multi-dimensional version of Szemeredi's theorem, also a density version of the Hales-Jewett theorem (due to Katznelson and Ornstein), and also allowed for the common difference of the APs to have special shapes (e.g. squares).

In 1998-2001 Gowers made a major breakthrough by extending Roth's harmonic analysis techniques to prove Szemeredi's theorem. This approach finally gave good bounds for the van der Waerden numbers.

Theorem 7. ***(Gowers)****. There exists a positive constant c_k such that any subset A in $[1,N]$ with $|A| \gg N/(\log\log N)^{c_k}$ contains a non-trivial k term arithmetic progression.*

One of the major insights of Gowers is the development of a "quadratic theory of Fourier analysis" which substitutes for the "linear Fourier analysis" used in Roth's theorem. Gowers's ideas have transformed the field, opening the door to many spectacular results, most notably the work of Green and Tao.

The Green-Tao Theorem (2003). The primes contain arbitrarily long non-trivial arithmetic progressions.

By the Prime Number Theorem, upto N there are about $N/\log N$ primes. This density is much smaller than what would be covered by Gowers theorem; even in the case $k = 3$ it is not covered by the best known results on $r_3(N)$. Another result is the celebrated three primes theorem.

Theorem 8. ***(Vinogradov, 1937)****. Every large odd number is the sum of three primes.*

Another brilliant result of Green and Tao, developing Gowers ideas, is that $r_4(N) \ll N(\log N)^{-c}$ where $r_4(N)$ denotes the largest cardinality of a set in $[1,N]$ containing no four term progressions. Another theme is Freiman's theorem on sumsets. If A is a set of N integers then $A+A$ is bounded above by $N(N+1)/2$, and below by $2N-1$. The lower bound is attained only when A is highly structured, and is an arithmetic progression of length N. Clearly if A is a subset of an arithmetic progression of length CN then $|A+A| \leq 2C|A|$. More generally suppose $d_1, ..., d_k$ are given numbers, and consider the set $\{a_0 + a_1 d_1 + ... + a_k d_k : 1 \leq a_i \leq N_i \text{ for } 1 \leq i \leq k\}$. We may think of this as a generalized arithmetic progression (GAP) of dimension k. Note that this GAP has cardinality at most $N_1...N_k$. If these sums are all distinct (so that the cardinality equals $N_1...N_k$) we call the GAP **proper**. Note that if A is contained in a GAP of dimension k and size $\leq CN$ then $|A+A| \leq 2^k CN$. Freiman's theorem provides a converse to this showing that all sets with small sumsets must arise in this fashion.

Theorem 9. ***(Freiman)****. If A is a set with $|A+A| \leq C|A|$ then there exists a proper GAP of dimension k (bounded in terms of C) and size $\leq C_1|A|$ for some constant C_1 depending only on C.*

Qualitatively Freiman's theorem says that any set with a small sumset looks like an arithmetic progression. Similarly we may expect that a set with a small product set should look like a geometric progression. But of course no set looks simultaneously like an arithmetic and a geometric progression! This led to the following conjecture which says either the sumset or the product set must be large.

Erdős-Szemeredi Conjecture. If A is a set of N integers then $|A+A| + |A.A| \gg N^{2-\epsilon}$, for any $\epsilon > 0$.

This is currently known for $\epsilon > 3/4$ The sum-product theory (and its generalizations) is another very active problem in additive combinatorics, and has led to many important applications (bounding exponential sums etc).

Poincare recurrence: Let X be a probability space with measure μ, and let T be a measure preserving transformation (so $\mu(T^{-1}A) = \mu(A)$). For any set V with positive measure there exists a point $x \in V$ such that for some natural number n, $T^n x$ also is in V.

Proof: This is very simple: note that the sets $V, T^{-1}V, T^{-2}V, ...$ cannot all be disjoint. Therefore $T^{-m}V \cap T^{-m-n}V \neq \phi$ for some natural numbers m and n. But this gives readily that $V \cap T^nV \neq \phi$ as needed.

It is clear from the proof that the number n in Poincare's result may be found below $1/\mu(V)$. As an example, we may take X to be the circle R/Z, and take V to be the interval $[-1/2Q, 1/2Q]$, and T to be the map $x \to x + \theta$ for some fixed number θ. We thus obtain:

Theorem 10. ***(Dirichlet)****. For any real number θ, and any $Q \geq 1$ there exists $1 \leq q \leq Q$ such that $||q\theta|| \leq 1/Q$. Here $||x||$ denotes the distance between x and its nearest integer.*

If X happens also to be a separable (covered by countably many open sets) metric space, then we can divide X into countably many balls of radius $\epsilon/2$. Then it follows that almost every point of X returns to within ϵ of itself. That is, almost every point is recurrent.

We don't really need a probability space to find recurrence. Birkhoff realized that this can be achieved purely topologically and holds for compact metric spaces.

Theorem 11. ***(Birkhoff's Recurrence)****. Let X be a compact metric space, and T be a continuous map. Then there exists a recurrent point in X; namely, a point x such that there is a sequence $n_k \to \infty$ with $T^{n_k}x \to x$.*

Proof: Since X is compact, any nested sequence of non-empty closed sets $Y_1 \supset Y_2 \supset Y_3...$ has a non-empty intersection. Consider T-invariant closed subsets of X; that is, Y with $TY \subset Y$. By Zorn's lemma and our observation above, there exists a non-empty minimal closed invariant set Y. Let y be any point in Y and consider the closure of $y, Ty, T^2y, ...$ This set is plainly a closed invariant subset of Y, and by minimality equals Y. Therefore y is recurrent.

These are some basic simple results, of the same depth as Dirichlet's pigeonhole principle and its application to Diophantine approximation. In the example of Diophantine approximation, we see that if $||n\theta||$ is small then so are $||2n\theta||, ||3n\theta||$ etc. This suggests the possibility of multiple recurrence.

Topological Multiple Recurrence. Let X be a compact metric space, and T be a continuous map. For any integer $k \geq 1$ there exists a point $x \in X$ and a sequence $n_l \to \infty$ with $T^{jn_l}x \to x$ for each $1 \leq j \leq k$.

This theorem is analogous to van der Waerden's theorem, and indeed implies it. To see this, let $\Lambda = \{1, ..., r\}$ represent r colors, and consider $\Omega = \Lambda^{\mathbb{Z}}$. Thus Ω is the space of all r colorings of the integers, and by $x \in \Omega$ we understand a particular r coloring of the integers. We make Ω into a compact metric space (check using sequential compactness), by taking as the metric $d(x, y) = 0$ if $x = y$ and $d(x, y) = 2^{-l}$ where l is the least magnitude for which either $x(l) \neq y(l)$ or $x(-l) \neq y(-l)$. We define the shift map T by $Tx(n) = x(n + 1)$.

Now suppose we are given a coloring ξ of the integers. Take X to be the closure of $T^n\xi$ where n ranges over all integers. By definition this is a closed invariant compact metric space, and so by the Topological Multiple Recurrence Theorem there is a $x \in X$ and some $n \in \mathbb{Z}$ with $x(0) = x(n) = x(2n) = ... = x(kn)$. But from the definition of the space X we may find an $m \in \mathbb{Z}$ such that $T^m\xi$ and x agree on the interval $[-kn, kn]$. Then it follows that $\xi(m) = \xi(m + n) = ... = \xi(m + kn)$ producing a $k + 1$ term AP.

The above argument gives an infinitary version of the van der Waerden theorem where we color all the integers. But from it we may deduce the finite version. Suppose not, and there are r colorings of $[-N, N]$ with no monochromatic k-APs for each natural number N . Extend each of these colorings arbitrarily to $\mathbb{Z}$, obtaining an element in Ω. By compactness we may find a limit point in Ω of these elements. That limit point defines a coloring of $\mathbb{Z}$ containing no monochromatic k-APs, and this is a contradiction.

The ergodic theoretic analog of Szemeredi's theorem is Furstenberg's multiple recurrence theorem for measure preserving transformations, and this implies Szemeredi by an argument similar to the one above.

https://doi.org/10.17993/3cemp.2022.110250.186-196

Theorem 12. ***(Furstenberg)***. *Let X be a probability measure space and let T be a measure preserving transformation. If V is a set of positive measure, then there exists a natural number n such that $V \cap T^{-n}V \cap T^{-2n}V \cap ... \cap T^{-kn}V$ has positive measure.*

Behrend's Example. Behrend constructed a surprisingly large set in $[1, N]$ with no 3-APs.

Behrend's Theorem A. There is a set A in $[1, N]$ which is free of 3 APs and satisfies $|A| \gg Nexp(-c\sqrt{\log N})$. Here c is an absolute positive constant.

Behrend's Theorem B. There exists a set A in $[1, N]$ with $|A| \geq \delta N$ which has $\ll \delta^{c \log (1/\delta)} N^2$ three term progressions. Here c is an absolute positive constant, and $\delta > 0$.

Theorem 13. ***(Varnavides)***. *For every $\delta > 0$ there exists $C(\delta) > 0$ such that if $A \subset [1, N]$ with $|A| \geq \delta N$ then A contains at least $C(\delta)N^2$ three term progressions.*

Erdos and Szemeredi: Expect that additive and multiplicative structures are independent. Hence one of the two sets, i.e., sumset and product set, must be large.

Theorem 14. ***(Solymosi)***. *Let A, B and C be finite sets of real numbers, each having at least two elements. Then*

$$|A + B| \times |A.C| \gg (|A|^3|B||C|)^{1/2}$$

In particular, if A, B and C all have cardinality N then either $A + B$ or $A.C$ has cardinality $\gg N^{5/4}$.

Note: Both, **Erdos-Turan conjecture** and **Szemeredi's Theorem** assume that the set A has a positive density (Schnirelman density). However, the primes have zero density because of the **prime number theorem**. Hence **Green - Tao theorem** is much harder.

Now we consider **finite sets:** For any set A of integers, we define the **sumset**

$$A + A = \{a + a' : a, a' \in A\}$$

and the **difference set**

$$A - A = \{a - a' : a, a' \in A\}.$$

We consider finite sets of integers, and the relative sizes of their sumsets and difference sets. If A is a finite set of integers and $x, y \in Z$, then the **translation** of A by x is the set $x + A = \{x + a : a \in A\}$ and the **dilation** of A by y is $y * A = \{ya : a \in A\}$. We have

$$(x + A) + (x + A) = 2x + 2A$$

and

$$(x + A) - (x + A) = A - A.$$

Similarly,

$$y * A + y * A = y * (A + A)$$

and

$$y * A - y * A = y * (A - A).$$

It follows that

$$|(x + y * A) + (x + y * A)| = |2A|$$

and

$$|(x + y * A) - (x + y * A)| = |A - A|$$

https://doi.org/10.17993/3cemp.2022.110250.186-196

So the cardinalities of the sum and difference sets of a finite set of integers are invariant under affine transformations of the set. Easy to see that if $|A| = N$, then $|A + A|$ is bounded above by $N(N + 1)/2$ and below by $2N - 1$. The latter occurs when A is an AP or symmetric.

The set A is symmetric with respect to the integer z if $A = z - A$ or, equivalently, if $a \in A$ if and only if $z - a \in A$. For example, the set $\{4, 6, 7, 9\}$ is symmetric with $z = 13$. If A is symmetric, then

$$A + A = A + (z - A) = z + (A - A)$$

and so $|A + A| = |A - A|$. Equality also holds if A is an AP. Examples of equality exists even if A is neither symmetric nor an AP. $A = \{0, 1, 3, 4, 5, 8\}$ is neither symmetric nor an AP but $|A+A| = |A-A|$.
If $A = \{a, b, c\}$ with $a < b < c$ and $a + c \neq 2b$, then $|A + A| = 6 < 7 = |A - A|$.
If $A = \{0, 2, 3, 4, 7\}$ then $A + A = [0, 14] \setminus \{1, 12, 13\}$, $A - A = [-7, 7] \setminus \{-6, 6\}$ and $|A + A| = 12 < 13 = |A - A|$.

This is the typical situation. Since $2 + 7 = 7 + 2$ but $2 - 7 \neq 7 - 2$
It is natural to expect that in any finite set of integers there are always at least as many differences as sums. There had been a conjecture, often ascribed incorrectly to John Conway, that asserted that $|A + A| \leq |A - A|$ for every finite set A of integers.

This conjecture is **false**, and a counterexample is the set $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$, for which
$A + A = [0, 28] \setminus \{1, 20, 27\}$
$A - A = [-14, 14] \setminus \{\pm 6, \pm 13\}$ and $|A + A| = 26 > 25 = |A - A|$.

Given the existence of such aberrant sets, we can ask for the smallest one. The set A above satisfies $|A| = 8$.

Problem 1.
What is the smallest such A?
i.e. find $min\{|A| : A \subseteq Z \quad \text{and} \quad |A + A| > |A - A|\}$?

Note: Hegarty has proved that this minimum is indeed equal to 8 and is affinely equivalent to the set $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$. One may also ask:

Problem 2.
What is the structure of finite sets satisfying $|A + A| > |A - A|$?

If A is a finite set of integers and m is a sufficiently large positive integer (for example, $m > 2 \max(\{|a| : a \in A\})$, then the set

$$A_t = \left\{ \sum_{i=0}^{t-1} a_i m^i : a_i \in A \quad \text{for} \quad i = 0, 1, \ldots, t - 1 \right\}$$

has the property that $|A_t + A_t| = |A + A|^t$ and $|A_t - A_t| = |A - A|^t$. This can be seen as follows: the elements of $A_t + A_t$ can be thought of as having a base-m expansion with 'digits' coming from the set $A + A$.

It follows that if $|A + A| > |A - A|$, then $A_t + A - t| > |A_t - A_t|$ and, moreover,

$$\lim_{t \to \infty} \frac{|A_t + A_t}{|A_t - A_t|} = \lim_{t \to \infty} \left(\frac{|A + A|}{|A - A|} \right)^t = \infty$$

The sequence of sets $\{A_t\}_{t=1}^{\infty}$ is the standard parametrized family of sets with more sums than differences (called MSTD sets).

Problem 3.
Are there other parametrized families of sets satisfying $|A + A| > |A - A|$?

Even though there exist sets A that have more sums than differences, such sets should be rare, and it must be true with the right way of counting that the vast majority of sets satisfies $|A - A| > |A + A|$.

Problem 4.
Let $f(n)$ denote the number sets $A \subseteq [0, n - 1]$ such that $|A - A| < |A + A|$, and let $f(n, k)$ denote the

https://doi.org/10.17993/3cemp.2022.110250.186-196

number of **such sets** $A \subseteq [0, n-1]$ with $|A| = k$. Compute

$$\lim_{n\to\infty} \frac{f(n)}{2^n}$$

and

$$\lim_{n\to\infty} \frac{f(n,k)}{\binom{n}{k}}$$

What about other functions that count finite sets of nonnegative integers with respect to sums and differences?

Problem 5.
Prove that $|A - A| > |A + A|$ for almost all sets A with respect to other appropriate counting functions.

Binary linear forms: The problem of sums and differences can be considered a special case of a more general problem about binary linear forms

$$f(x, y) = ux + vy$$

where u and v are nonzero integers. For every finite set A of integers, let

$$f(A) = \{f(a, a') : a, a' \in A\}.$$

We are interested in the cardinality of the sets $f(A)$. For example, the sets associated to the binary linear forms

$$s(x, y) = x + y$$

and

$$d(x, y) = x - y$$

are the sumset $s(A) = A + A$ and the difference set $d(A) = A - A$.

To every binary linear form there is a unique normalized binary linear form $f(x, y) = ux + vy$ such that

$$u \geq |v| \geq 1 \quad \text{and} \quad (u, v) = 1.$$

The natural question is: If $f(x, y)$ and $g(x, y)$ are two distinct normalized binary linear forms, do there exist finite sets A and B of integers such that $|f(A)| > |g(A)|$ and $|f(B)| < |g(B)|$, and, if so, is there an algorithm to construct A and B?

Brooke Orosz gave constructive solutions to this problem in some important cases. For example, she proved the following: Let $u > v \geq 1$ and $(u, v) = 1$, and consider the normalized binary linear forms

$$f(x, y) = ux + vy \quad \text{and} \quad g(x, y) = ux - vy.$$

For $u \geq 3$, the sets

$$A = \{0, u^2 - v^2, u^2, u^2 + uv\}$$

and

$$B = \{0, u^2 - uv, u^2 - v^2, u^2\}$$

satisfy the inequalities

$$|f(A)| = 14 > 13 = |g(A)|$$

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and

$$f(B) = 13 < 14 = |g(B)|.$$

For $u = 2$, $v = 1$ we have $f(x, y) = 2x + y$ and $g(x, y) = 2x - y$. The sets $A = \{0, 3, 4, 6\}$ and $B = \{0, 4, 6, 7\}$ satisfy the inequalities $|f(A)| = 13 > 12 = |g(A)|$ and $|f(B)| = 13 < 14 = |g(B)|$. The problem of pairs of binary linear forms has been completely solved by Nathanson, O'Bryant, Orosz, Ruzsa, and Silva.

Theorem 15. *Let $f(x, y)$ and $g(x, y)$ be distinct normalized binary linear forms. There exist finite sets A, B, C with $|C| \geq 2$ such that $|f(A)| > |g(A)|$, $|f(B)| < |g(B)|$ and $|f(C)| = |g(C)|$.*

Problem 6.
Let $f(x, y)$ and $g(x, y)$ be distinct normalized binary linear forms. Determine if $|f(A)| > |g(A)|$ for most or for almost all finite sets of integers A.

These results should be extended to linear forms in three or more variables.

Problem 7.
Let $f(x_1, , \ldots, x_n) = u_1x_1 + \ldots + u_nx_n$ and $g(x_1, \ldots, x_n) = v_1x_1 + \ldots + v_nx_n$ be linear forms with integer coefficients. Does there exist a finite set A of integers such that $|f(A)| > |g(A)|$?

Polynomials over finite sets of integers and congruence classes
An integer-valued function is a function $f(x_1, x_2, \ldots, x_n)$ such that if $x_1, x_2, \ldots, x_n \in Z$, then $f(x_1, x_2, \ldots, x_n) \in Z$. The binomial polynomial

$$\binom{x}{k} = \frac{x(x-1)(x-2)\ldots(x-k+1)}{k!}$$

is integer-valued, and every integer-valued polynomial is a linear combination with integer coefficients of the polynomials $\binom{x}{k}$, (**George Polya**). For any set $A \subseteq Z$, we define

$$f(A) = \{f(a_1, a_2, \ldots, a_n) : a_i \in A \quad \text{for} \quad i = 1, 2, \ldots, n\} \subseteq Z.$$

Problem 8.
Let $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ be integer-valued polynomials. Determine if there exist finite sets A, B, C of positive integers with $|C| \geq 2$ such that $|f(A)| > |g(A)|$, $|f(B)| < |g(B)|$ and $|f(C)| = |g(C)|$. There is a strong form of Problem 8.

Problem 9.
Let $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ be integer-valued polynomials. Does there exist a sequence $\{A_i\}_{i=1}^{\infty}$ of finite sets of integers such that

$$\lim_{i \to \infty} \frac{|f(A_i)|}{|g(A_i)|} = \infty?$$

There is also the analogous modular problem. For every polynomial $f(x_1, x_2, \ldots, x_n)$ with integer coefficients and for every set $A \subseteq Z/mZ$, we define

$$f(A) = \{f(a_1, a_2, \ldots, a_n) : a_i \in A \quad \text{for} \quad i = 1, 2, \ldots, n\} \subseteq Z.$$

Problem 10.
Let $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ be polynomials with integer coefficients and let $m \geq 2$. Do there exist sets $A, B, C \subseteq Z/mZ$ with $|C| > 1$ such that $|f(A)| > |g(A)|$, $|f(B)| < |g(B)|$, and $|f(C)| = |g(C)|$.

Problem 11.
Let $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ be polynomials with integer coefficients. Let $M(f, g)$ denote

https://doi.org/10.17993/3cemp.2022.110250.186-196

the set of all integers $m \geq 2$ such that there exists a finite set A of congruence classes modulo m such that $|f(A)| > |g(A)|$. Compute $M(f, g)$.

Note that if there exists a finite set A of integers with $|f(A)| > |g(A)|$, then $M(f, g)$ contains all sufficiently large integers.

Finally we mention two famous open problems:

1. **Erdos:** if $a_i \in N$, such that $\sum_{i=1}^{\infty} \frac{1}{a_i} = \infty$, then $\{a_i\}$ contains arbitrarily long APs. Green-Tao is a special case since $\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty$, where p_i are all primes.
2. **The abc-conjecture:** This says roughly that if a lot of small primes divide a and b, then only a few large ones will divide their sum $a + b = c$. More precisely, we define the **radical** of $n \in N$ as $rad(n) =$ the product of the distinct prime factors of n. Eg: rad(16)=2, rad(17)=17, rad(18)=6.

Statement
For every $\epsilon > 0$, there exist only finitely many triples (a, b, c) of coprime positive integers with $a + b = c$ and $c > rad(abc)^{1+\epsilon}$.

This has lot of consequences for mathematics. In particular this also implies the truth of Fermat's Last Theorem.

Recently, Shinichi Mochizuki from Japan has claimed a proof of the abc-conjecture. The 500 odd -page proof is published by the RIMS(Research Institute of Mathematical Sciences), Kyoto, Japan. And the author is one of editors of the journal. However there is no consensus among mathematicians about the correctness of the proof. Only future will tell.

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ANALYSIS OF DIGITAL INFORMATION MANAGEMENT OF PRODUCT MARKET COMPETITION UNDER THE ENVIRONMENT OF AGRICULTURAL PRODUCT E-COMMERCE

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ABSTRACT

With the rapid development of China's regional agriculture, its consumption of energy, the squeezing and encroachment of the environment and the pollution of the ecological environment have been put on the agenda, which stems from the unreasonable management and regulation of the rapidly developing agricultural infrastructure. Aiming at the evaluation index cluster of agricultural management in Northeast China, the main structure of the digital information management platform of green ecological agriculture we built is divided into variable layer, middle layer and evaluation index layer, which is a superimposed and progressive layer structure design.The results show that compared with 2018, the use index of electronic agricultural products, per capita greening index, soil organic matter content index and per capita water content index increased by 30.34%, 6.14%, 25.34% and 30.26% respectively in 2019.The index of per capita desert land area decreased by 10.97%.The Sustainability Index experienced an unusual decrease in 2017-2018, with a drop of 0.08. Compared with 2018, the green ecological index from 2019 to 2021 increased by 5.94%, 8.58% and 12.87% respectively. This provides guidance for the structure and design of China's future agricultural development.

KEYWORDS

Agricultural ecology, green environmental protection, sustainable development, information management, data platform g

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Introduction

The development of green ecological agriculture can protect and improve the ecological environment, prevent pollution, maintain ecological balance [1, 2], improve the safety of agricultural products [3], take the road of sustainable ecological development [4], and closely integrate environmental construction with economic development. Combined [5, 6], it can improve the income of agricultural workers while developing agriculture. Green ecological agriculture, in simple terms, is to use the principles of ecology, ecological economics and systematic scientific methods to organically combine the achievements of modern science and technology with the essence of traditional agricultural technology [7, 8], and integrate agricultural production, rural economic development and ecological. It is a new comprehensive agricultural system with ecological rationality and a virtuous cycle of functions that integrates environmental governance and protection, resource cultivation and efficient utilization [9, 10]. There are three main models of green ecological agriculture, mainly including space-time structure type, food chain type and space-time food chain comprehensive type, as shown in Table 1. Ecological agriculture is an agro-ecological economic complex system. According to the principle of "whole, coordinated circulation and regeneration", the agricultural ecological system and the agricultural economic system are comprehensively integrated to realize the multi-level utilization of natural resources energy, so as to achieve the maximum ecological economy overall benefit [11, 12].At the same time, it can integrate agriculture, forestry, animal husbandry, sideline and fishery industries[13, 14] to form a comprehensive development model of large-scale agricultural production, processing and sales, adapting to the development of the socialist market economy[15]. However, with the rapid rise of China's e-commerce and the rapid development of informatization, the application of various digital high-tech information technology and the analysis and management of modern green ecological agriculture are the inevitable trends in the development of agricultural modernization.

Table 1 Main modes of green ecological agriculture

agricultural model	Features
space-time structure	According to the biological, ecological characteristics and a rationally formed ecosystem of mutually beneficial symbiotic relationships between organisms
food chain	A virtuous cycle agro-ecosystem designed according to the energy flow and material cycle laws of the agro-ecosystem
Integrated spatiotemporal food chain	The organic combination of space-time structure type and food chain type is a mode type with moderate input, high output, less waste, no pollution and high efficiency

Green ecological agriculture is an inevitable way to realize modern agriculture, and efficient and reasonable organization and management are the foundation and guarantee for the development of ecological agriculture. Scientific management concepts, tools and methods are the basic means to achieve green agriculture [16, 17]. Green ecological management reflects the choice of ecological agriculture development model and the innovation of green technology management. Select and manage agricultural production models from the perspective of agricultural product production and ecological economics, and research can best reflect ecological benefits and economy. Nowadays, in the field of analysis

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and management of green ecological agriculture, many experts and scholars have made a lot of discussions. For example, for agriculture and ecological management under uncertain conditions, Chen, J [18] proposed a reliability-based interval multi-objective crop area planning model. The integration, developed considering the economic and ecological benefits of the research system [19, 20], was developed to deal with interval and ambiguous uncertainties. It focuses on crop area optimization, and the interval objective function is to maximize system benefits, maximize watershed area, and maximize system benefits per unit area. Rural agro-ecosystems have an important impact on the development of China's economy, society and ecological environment at any time. Chen, F [21] took big data as the research background and based on the complex system theory to construct an indicator system for the ecological management system of rural agro-ecosystems. The fertilizer was used in the experiment, and the consumption, water pollution degree, pest and disease degree, carbon and nitrogen absorption and agricultural economic benefit of the rural agricultural ecosystem in a certain area were taken as the systematic indicators of the ecological management system. Using data mining technology in big data [22, 23], collecting and processing relevant data in the network, analyzing and understanding agricultural ecosystems through complex systems, and finally calculating and analyzing data of various indicators, their research shows that agro-ecological management Institutions have a positive effect on rural agro-ecosystems. In the development of green ecological agriculture, the optimal water distribution model is an effective tool to provide a reasonable water distribution scheme [24], Pan, Q. [25] proposed an interval multi-objective fuzzy interval credible constraint nonlinear programming model, combined with the estimation of ecological vegetation space water demand , to solve the problem of agricultural and ecological water allocation in irrigated areas under uncertain conditions. Excessive fertilization can cause land pollution [26, 27], which is not conducive to the development of ecological agriculture. Li, X [28] established a linear regression equation to predict the runoff in the study area, and then determine the pollution in the area. Zhu, Z [29] built a 5G IoT-based agricultural product circulation information system to realize real-time positioning, information sharing and security assurance of supply chain circulation. Liu, X [30]'s research shows that in agricultural product e-commerce, product quality, brand image, e-commerce platform and logistics distribution have a significant positive impact on customer satisfaction, and have an important impact on the sales of agricultural products. Based on the research of many scholars, we found that in e-commerce, the quality and safety of agricultural products are decisive factors for the sales of products. Through digital information management, we can achieve coordinated development and the environment, and form two virtuous circles in ecology and economy, the unity of the three major benefits of economy, ecology and society. By studying the data in the development of green ecological agriculture, and constructing a green ecological management system model by analyzing these data, data mining, etc., the economic benefits of ecological agriculture can be improved.

The research on green ecological agriculture management is of great significance to the development of ecological agriculture and the solution of various drawbacks and crises brought by modern agriculture. However, in the current e-commerce sales, the safety and quality of agricultural products cannot be presented to customers. Based on this, in our research, we build an information-based digital management platform, which includes developed languages, frameworks and database. In the digital information management platform, we track and monitor the agricultural product information of green ecological agriculture in Northeast China throughout the whole process, so as to ensure the safety and quality of the agricultural products during the sale of the agricultural products on the e-commerce platform. In addition, we also discussed the economic benefits of this digital information platform for green ecological agriculture.

Construction of information digital management platform

In order to better understand the situation of green ecological agriculture in Northeast China, this chapter mainly introduces the development languages, development frameworks and tools used in the electronic platform of agricultural products, and gives a brief introduction to them according to the situation of green ecological agriculture in Northeast China. The advantages and reasons for selection are analyzed one by one. These theories or tools include: languages, frameworks, and databases.

Java languages introduction

(1) There are not many Java language features, and there is no need to consider issues such as multiple inheritance, operator overloading, automatic forced conversion, etc.;

(2) Abstract the real green ecological agriculture by Java in Northeast China through classes, represent agricultural products through objects, and extract common attributes and behaviors between things through inheritance;

(3) Java can detect type errors in time in the process of compiling the electronic platform for agricultural products, and can automatically recycle garbage so that the memory of the electronic platform for agricultural products does not occupy much space;

(4) Java can perform language conversion in the virtual machines corresponding to different agricultural product electronic platforms, and parse and run on different agricultural product electronic platforms;

(5) Java can support the multi-threaded agricultural product electronic platform model to ensure the synchronization between the agricultural product electronic platform threads.

This system uses Java7 to develop the electronic platform of green ecological agricultural agricultural products in Northeast China. As of Java7, the information features of the electronic platform for agricultural products have been reflected, annotated, generic,

and concurrency.

Framework construction principle

The framework in the agricultural product electronic platform adopts a layered structure, which consists of five well-designed electronic framework sub-modules, as shown in Figure 1.

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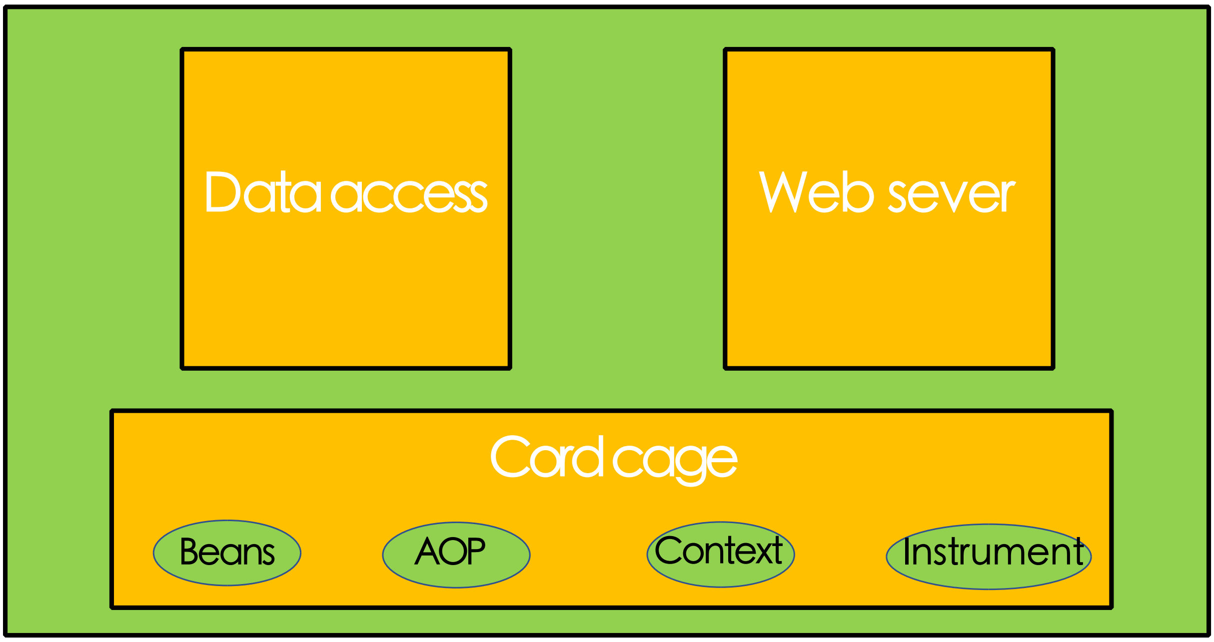

Figure 1: The Framework of the Agricultural Products Electronic Platform

Any module in the agricultural product electronic platform can be used independently, and can also be used in parallel with some modules of other agricultural product electronic platforms. The five electron spring framework submodules are as follows:

(1) Bean container. The Bean container is the basis for the electronic Spring framework to realize the IOC layered structure. By reading the XML file or by parsing the language of the agricultural product electronic platform, it generates the agricultural product electronic platform information of green ecological agriculture in Northeast China defined by Bean, and fills it into the core in a container;

(2) Spring AOP module.This module extracts the agricultural product information from the aspects in the business process of the agricultural product electronic platform, and encapsulates those behaviors that are not related to the business logic of the agricultural product electronic platform, but are required to be called by many functional modules in the agricultural product electronic platform. Duplicate codes in the platform reduce the coupling degree of the electronic platform of agricultural products;

(3) Spring DAO module.This module is not related to the specific situation of green ecological agriculture in Northeast China.Through this module system, abnormal semantics in the platform system can be identified;

(4) Spring Web module. Common development basic functions such as file uploading and downloading in the agricultural product electronic platform, binding request parameters to objects, etc. are included in this module.

(5) Spring MVC framework. This module is used to configure view parsing related to green ecological agricultural products in Northeast China, and to define the priority of processing.

database

The database supports the storage engine settings of various electronic platforms, and there are different data types of storage methods within the electronic platforms, so that the access speeds to the electronic platforms are different. In addition, the creation of monitoring electronic platform big data will only be used for query, and will not be added, deleted or modified. Since the addition of the database supports setting the storage engine at the table level of the electronic platform, combined with the characteristics of the green ecological agriculture in Northeast China, different storage engines can be selected for different electronic platforms in a more targeted manner to optimize their performance.

Model Validation

With the popularization of various smart mobile devices, the promotion of agricultural information and the promotion and sale of agricultural products can solve the problems of difficulty in obtaining rural information, low commercialization, and unsalable commodities. The functional test of this system is mainly based on black boxes. Testing is the main means. Therefore, iterative training is very necessary in the underlying data of the agricultural product electronic platform, and the model operation accuracy can be tested through iterative training. The details are as follows:

(1) Accuracy. Precision is the proportion of positive classes that resolve to samples identified as positive classes. The specific calculation process is as follows:

(1)

Among them, TP is a true example, and FP is a false positive example.

(2) Recall rate. Used to solve for the proportion of all positive class samples that are correctly identified as positive classes. The specific calculation process is as follows:

(2)

Among them, FN is a false negative example. In the field of agricultural product sales, Internet information technology should be fully utilized to help small farmers become the main body of agricultural product e-commerce business, break the restrictions on trading venues, reduce transaction costs, and reduce the number of agricultural products in circulation.

(3) Accuracy is a metric used to evaluate classification models. Simply put, it is the proportion of the total number of correct predictions by the model. The calculation process is as follows:

(3)

Among them, TN is a true negative example. Apply Internet information technology to market analysis, variety selection, intensive farming, and pest and disease analysis to create

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precision agricultural production, achieve fine management of the industrial chain, and transform traditional agriculture into smart agriculture.

We compared the different accuracy comparison models, took into account the background of the agricultural product electronic platform and other backgrounds, and adopted appropriate algorithms for evaluation, and finally considered the accuracy rate. In Figure 2, as the number of iterations increases, the training accuracy in the input agricultural product information is also increasing. When the number of iterations is 50, the accuracy of 97.33% is reached, and then the accuracy tends to stabilize; the number of iterations is 25. The second time, the test set accuracy reached 95.34%. As the number of iterations increases to 50, the accuracy rises to 97.52%, which shows that our agricultural product electronic platform has high prediction accuracy for the underlying data. The description of the data set parameters is shown in Table 1.

Table 1： The relationship between iteration accuracy and number of interactions

Number of interactions	Iteration accuracy
12.5	90.55
25	95.34
37.5	96.59
50	97.33
62.5	97.35
75	97.41
87.5	97.48
100	97.52

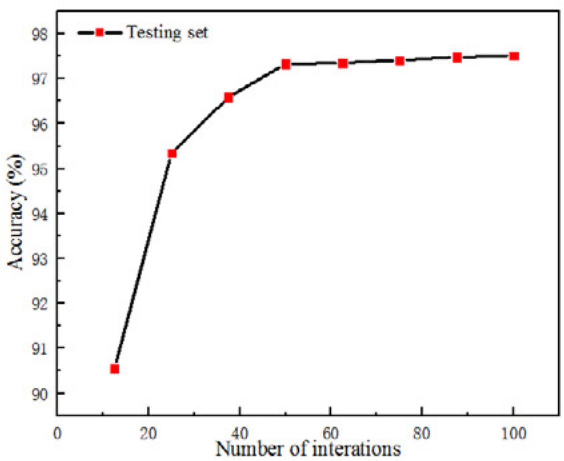

Figure 2: Iterative Accuracy Graph

Results and Analysis

For the development of green ecological agriculture in China, the rural revitalization strategy has greatly improved the level of regional agriculture. This improvement is related to economic benefits and the structure of agricultural allocation. However, the substantial development of agriculture is still a double-edged sword. With the rapid development of China's regional agriculture, its consumption of energy, the squeezing and encroachment of the environment and the pollution of the ecological environment have been put on the agenda. This problem largely stems from the unreasonable management and regulation of the rapidly developing agricultural infrastructure. Therefore, the demand and management of green ecological agriculture in China is a top priority.

First of all, the conventional evaluation indicators of regional sustainable utilization of agricultural resources can be used as the main evaluation indicators and guiding principles for the demand and management of green ecological agriculture in China, which provides a solid foundation for us to establish an effective management system. For the evaluation index cluster of agricultural management in Northeast China, we can regard the index cluster as a series of variables that are correlated and complementary, and have strong responsiveness to the sustainable utilization of agricultural natural resources and agricultural socio-economic resources of. The number of elements in the variable population is large, but they are all basically continuous distribution, so they can form an indicator vector or indicator matrix. A series of indicators formed by various digital information outputted by the spatial database. We have built an evaluation system for the demand and management of China's green ecological agriculture before, and introduced the construction principles and methods in the digital system. Among them, several index systems stored in the index library include the content of sustainable utilization of agricultural resources. In specific applications, they can be called directly through the user interface of scientific engineering, and then input into the evaluation model.

Specifically, the main structure of the digital information management platform of green ecological agriculture we built is divided into variable layer, middle layer and evaluation index layer, which is a superimposed and progressive layer structure design. The variable layer includes the utilization rate of electronic agricultural products, per capita green area, per capita desert land area, soil organic matter content and per capita water resources content. The hidden environmental variables in the middle layer are determined as natural population growth rate, desertification development rate, soil organic matter loss rate, water resource decay rate, and vegetation index. For the final evaluation index layer variables, we chose the sustainable development index and the green ecological index as the final comprehensive evaluation index.

Influence of variable layer parameters of digital information management platform

According to the collection of a large amount of relevant data in 2017, we have continuously revised and learned the forecasting module in the digital information management platform of green ecological agriculture, and used the digital information management platform to analyze various data of the variable layer during 2017-2021. Data collection and mining were carried out. This data collection and mining comes from multiple sources of information such as provincial agricultural bureaus, environmental bureaus and local regional monitoring points in the Northeast region. After processing the data, the platform retains data points that are useful for future evaluation metrics. The annual average data collected from 2017 to 2021 were normalized after screening to facilitate subsequent analysis and to build multiple regression curves. The results of the analysis are shown in Figure 3. It is observed that the use index of electronic agricultural products, the per capita greening index, the soil organic matter content index and the per capita water content index all show an upward trend with the years, while the per capita desert land area index shows a decreasing index. Among the related variables, one variable is regarded as the dependent variable, and one or more other variables are regarded as independent variables, and a statistical analysis method is used to establish a linear or nonlinear mathematical model quantitative relationship between multiple variables and use sample data for analysis. The overall trend of each variable parameter has a large change range from 2018 to 2019. It is observed that compared with 2018, the use index of electronic agricultural products, per capita greening index, soil organic matter content index and per capita water content index in 2019 are observed. Up 30.34%, 6.14%, 25.34% and 30.26% respectively. The index of per capita desert land area decreased by 10.97%. This shows that during the period of 2018-2019, the management and control of green ecological agriculture in Northeast China has achieved a more significant effect. In the following 2019-2021 years, the changes of per capita greening index, soil organic matter content index, per capita water content index and per capita desert land area index tended to be stable, which indicates that the management of green ecological agriculture in this region is in the realization of the underlying structure. After the transformation, the government began to carry out stable development, which is conducive to further evaluating the advantages and disadvantages brought about by the structural transformation and providing guidance for subsequent development.

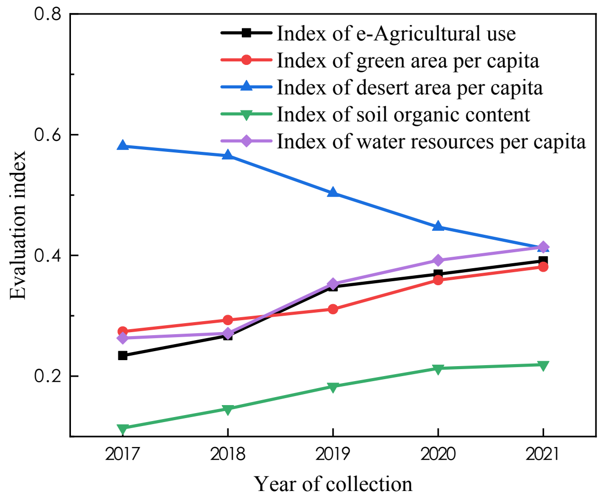

Figure 3 Changes of each index in the variable layer with years

Sustainable development assessment of green ecological agriculture

Then, we forecast the variable layer data for the period 2017-2021 after using the data collected in 2017 to revise the learning of the forecasting module within the digital information management platform of green ecological agriculture. Effective analysis data has been obtained. In this section, we use the prediction module in the digital information management platform of green ecological agriculture to analyze the output layer variables we care about. Among them, the change trend of the sustainable development index over the years is shown in Figure 4. An unusual decrease in the Sustainability Index was observed during 2017-2018, with a decrease of 0.08. This is inconsistent with the trend change results of the variable layer in Figure 3.

Therefore, we judged and analyzed the results according to the data changes in the middle layer. We found that the excessively large development area of farmland makes the corresponding soil and water resources environment polluted to a certain extent, which eventually leads to the reduction of the sustainable development index. And with the improved measures, in 2019, the observed sustainability index increased by 0.013, compared to the growth rate of 4.28% in 2018. This shows that the implementation of the adjustment measures of control is feasible. From 2019 to 2021, the growth of the sustainable development index also stabilized, at 0.317, 0.319 and 0.322, respectively.

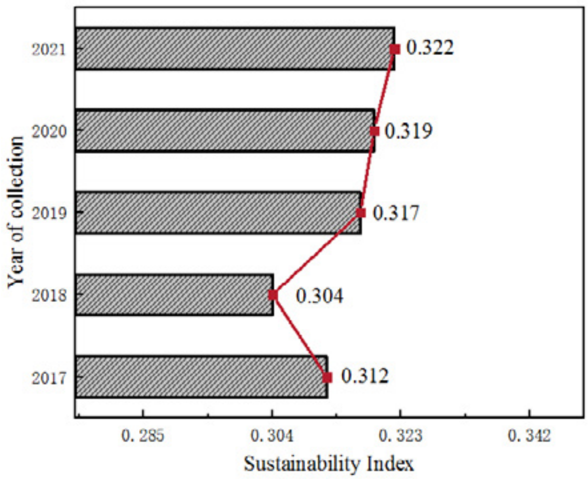

Figure 4 Changes in the sustainable development index over the years

Green ecological assessment of green ecological agriculture

Finally, we use the prediction module in the digital information management platform of green ecological agriculture to analyze the changes of the green ecological index with the development year. The results are shown in Figure 5. A smaller increase in the green ecological index was observed during 2017-2018, at only 2.36%. As can be seen from Figures 3 and 4, this period is a critical stage for structural and policy regulation. During the period from 2018 to 2019, the green ecological index has been significantly improved, which is due to the comprehensive results of the agricultural environment, the abundance of soil nutrients and water resources in Figure 3, which are conducive to green and sustainable development. It was observed that compared with 2018, the green ecological index from 2019 to 2021 increased by 5.94%, 8.58% and 12.87% respectively.

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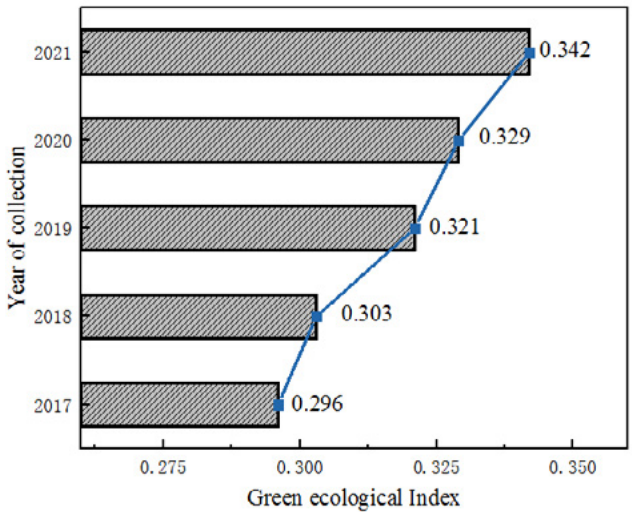

Figure 5 Changes of Green Ecological Index with Years

Discussion

With the rapid development of China's regional agriculture, its consumption of energy, the squeezing and encroachment of the environment and the pollution of the ecological environment have been put on the agenda. This problem largely stems from the unreasonable management and regulation of the rapidly developing agricultural infrastructure. Therefore, the demand and management of green ecological agriculture in China is a top priority. This provides a solid foundation for us to establish an effective management system. Aiming at the evaluation index cluster of agricultural management in Northeast China, the main structure of the digital information management platform of green ecological agriculture we built is divided into variable layer, middle layer and evaluation index layer, which is a superimposed and progressive layer structure design. We focus on the analysis of the variable layer and the evaluation index layer. The conclusions are as follows:

(1) The overall trend of each variable parameter has a large change range from 2018 to 2019. Compared with 2018, the use index of electronic agricultural products, per capita greening index, soil organic matter content index and per capita water content index increased respectively in 2019 30.34%, 6.14%, 25.34% and 30.26%. The index of per capita desert land area decreased by 10.97%. This shows that during the period of 2018-2019, the management and control of green ecological agriculture in Northeast China has achieved a more significant effect. In the following 2019-2021 years, the per capita greening index, soil organic matter content index, per capita water resource content index and per capita desert land area index tended to stabilize;

(2) The sustainable development index dropped abnormally during 2017-2018, with a drop of 0.08. This is inconsistent with the trend change results at the variable level. This is due to the excessive development of cultivated land, which pollutes the corresponding soil and water resources to a certain extent, which ultimately leads to a decrease in the sustainable development index. And with the improved measures, in 2019, the Sustainability Index rose by 0.013, compared to 4.28% in 2018. This shows that the implementation of the adjustment measures of control is feasible. From 2019 to 2021, the growth of the sustainable development index also stabilized, at 0.317, 0.319 and 0.322 respectively;

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(3) During the period of 2017-2018, the growth rate of the green ecological index was small, only 2.36%, because this period was a key stage of structural and policy regulation. During the period from 2018 to 2019, the green ecological index has been significantly improved, which is a comprehensive result of the improvement of the agricultural environment, soil nutrients and water resources, which is conducive to green and sustainable development. Compared with 2018, the green ecological index from 2019 to 2021 increased by 5.94%, 8.58% and 12.87% respectively.

In the process of ecological compensation, the government should coordinate and integrate ecological compensation funds, give unified leadership to ecological compensation activities, coordinate management and operation, and establish a supervision mechanism to make the process of ecological compensation open and transparent. Barriers, it is necessary to transform the ecological compensation mechanism of a single element into a comprehensive compensation mechanism centered on the entire region, and make overall planning and coordinated promotion. In the selection of compensation objects, this paper has not yet achieved accurate compensation, and only makes reasonable choices based on the local conditions of the compensation area. Strengthen mass participation, pay attention to the will of the masses before compensation, incorporate mass participation into the evaluation system of ecological compensation implementation effect, increase mass participation, and better play the positive role of public participation.

In the digital information management platform, we track and monitor the information of green and ecological agricultural agricultural products in Northeast of China throughout the process to ensure the safety and quality of agricultural products in the online sales process. business platform. In addition, we also discuss the economic benefits of this digital information platform for green ecological agriculture.

Data availability statement

The original contributions presented in the study are included in the article/ supplementary material, further in quiries can be directed to the corresponding author.

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Conflict of Interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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INDUSTRY 4.0: INTELLIGENT QUALITY CONTROL AND SURFACE DEFECT DETECTION

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ABSTRACT

Quality Control (QC) has recently emerged as a significant global trend among manufacturers, adopting intelligent manufacturing practices in view of Industry 4.0 requirements. Intelligent manufacturing is the process of enhancing production through the use of cutting-edge technologies, sensor integration, analytics, and the Internet of Things (IoT). The proposed paper mainly focuses on the study of the scope and the evolution of quality control techniques from conventional practices to intelligent approaches along with the state of art technologies in place. The challenges faced in building intelligent QC systems, in terms of security, system integration, Interoperability, and Human-robot collaboration, are highlighted. Surface defect detection has evolved as a critical QC application in modern manufacturing setups to ensure high-quality products with high market demand. Further, the recent trends and issues involved in surface defect detection using intelligent QC techniques are discussed. The methodology of implementing surface defect detection on cement wall surfaces using the Haar Cascade Classifier is discussed.

KEYWORDS

Quality Control, Industry 4.0, Internet of Things, Intelligent manufacturing, Interoperability, cutting-edge technologies, analytics, surface defect detection.

1. INTRODUCTION

Industrial societies are increasingly interested in intelligent manufacturing, especially with the advent of Industry 4.0, which calls for the majority of industrial tasks to be performed by robots with intelligence. It expressly indicates that the production systems will be fully connected and all production processes, including quality control and administration, can be made as intelligent as possible to run with the least amount of human involvement. Interoperability is a well-known necessity for the quick transformation of industry-specific processes. Hence it calls for integrating quality functions with other manufacturing operations to maintain intelligent collaboration so that quality-related knowledge may be shared with other manufacturing processes. On the other hand, the integration of manufacturing processes has ensured better performance.

Quality Control (QC) refers to a policy or set of practices created to satisfy a client's or customer's requirements or to fulfil a defined group of quality standards for a manufactured product or service [1]. It plays a significant role in maintaining and improving the quality of manufactured products. It involves testing the products to determine that they meet the necessary specifications. Testing is done to determine whether corrective measures are needed in finetuning the manufacturing processes to meet customer demands. QC ensures additional benefits such as reduced inspection and production costs, minimization of variations, and cost-effective use of resources. The process inspires employees to create high-quality goods leading to greater customer satisfaction [2]. Establishing customer-acceptable quality standards, finding defects or variations in the raw materials and manufacturing processes, ensuring smooth and uninterrupted production, assessing the degree of quality deviation in a product during the manufacturing process, thoroughly examining the contributing factors and thus achieving the objectives of quality control [3]. Some of the applications in that quality control are involved include preserving the quality of processes, products, and services, alerting for process abnormalities and fault detection, predicting the behaviour of machines, devices, and respective equipment in terms of the expected yield, machine maintenance and condition monitoring. Thus these practices ensure the effectiveness of the entire supply chain, starting from suppliers to customers [4]. The stepwise processes involved in quality control, starting from the inspection of manufactured products to meet the specified requirements up to the decision of acceptance/ rejection, are shown in Figure 1.

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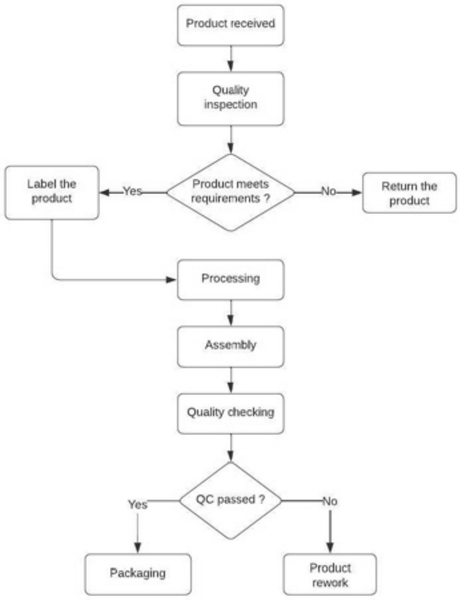

Figure 1. Quality control flowchart.

2. STATE-OF-THE-ART TECHNOLOGIES FOR QUALITY CONTROL

Production development is still guided by scientific management theory and has evolved from manually operated systems in Industry1.0 (I1.0) to intelligent systems in Industry4.0(I4.0). This strategy starts the management concept's scientific and technological growth. Control charts use graphical representations to determine whether the manufacturing processes or final products meet the intended specifications. Economic statistics defines econometrics as a field that helps to monitor and promote the economic factors influencing the quality of products. The Taguchi methods determine the ideal cost of quality throughout a product's life and emphasize product design and development in reducing defects in the manufactured products.

Quality costs involve a model that tracks the quality costs and is used to keep tabs on quality management's outcomes. The systems theory is used to model corporate systems through hierarchical breakdown and feedback. The primary organizational innovation paradigm for quality improvement in Japanese corporations was Quality circles, a collection of goals representing the best worldwide practices as a benchmarking model for building products and processes. A "flat" organizational structure with a specific position for quality management is the foundation of the lean system, an organizational, technical system then established in Japan. The material, process, and information flow from raw materials to goods delivered to customers are optimized and synchronized by the Quality Management supply chain model, which optimizes inventory and lowers the costs associated with product life. The concept of digital quality for intelligent systems has been improved by virtual quality using simulation models for the intelligent model quality of the integrated management system [5].

3. QUALITY CONTROL SYSTEMS WITH INTELLIGENCE

Industry 4.0, which calls for most industrial tasks to be performed by robots with intelligence, has made intelligent manufacturing popular with industrial societies. Setting up knowledge-intensive tasks to ensure quality and continual improvement is the fundamental prerequisite for developing intelligence in quality control. This calls for systematic performance monitoring and evaluation. Computer vision technology is mainly used to replace human eyes because many manufactured products' dimensions and surface characteristics dictate their quality. It is well known that automated machine vision systems can evaluate geometric and surface features to judge product quality and apply statistical analysis methods. Artificial intelligence can be used in industrial system design to

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ensure such capabilities. The manufacturers may require the use of lean manufacturing techniques and sophisticated machinery. To maintain proactive management of equipment, processes, services, and goods, they must increase predicted output while increasing the productivity and efficiency of their manufacturing hardware. Intelligent machinery should be able to process massive data collected from various sensors and use artificial intelligence and machine learning techniques. Utilizing cutting-edge manufacturing technologies is necessary to create intelligent systems that can be reconfigurable, interoperable, and reusable and cut down on potential wastes like scraps, overtime, and expenditures. AI-enhanced sensors, big data analytics-based decision support systems, and improved materials should be integrated throughout the industrial life cycle and serve as its drivers. A manufacturing system can predict and comprehend critical events and solve problems instantly before they result in any hazardous and dangerous situations or wastes, given the ability to process real-time data collected from the machines and conduct intelligent analysis over those through AI technologies. Thus it allows manufacturing systems to perform preventative maintenance and create fully functional manufacturing suits [4].

To provide the resources and capabilities required for satisfying standards and continuously enhancing the efficiency of the quality system, intelligent quality requires management commitment. The objective of the design of the manufacturing chain should be to maintain a constant and sustained level of quality for operations and services across the entire organization, with a high degree of integration. Another part of intelligent quality focuses on utilizing a knowledge-driven strategy rather than a conventional data-driven one. The system needs the appropriate operational and quality knowledge to establish the necessary level of intelligence. Methodologies and techniques are required to extract knowledge from the data that is currently accessible to produce self-behaviour with the desired quality function, as shown in Figure 2 [4].

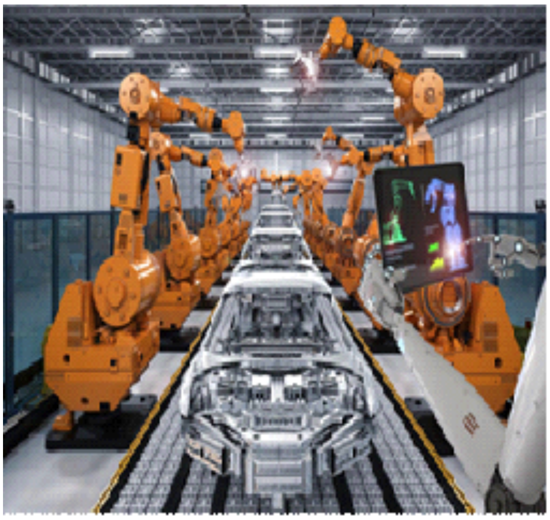

Figure 2. Robot manipulators for intelligent quality control.

Source: https://www.therobotreport.com/top-5-countries-using-industrial-robots-2018/.

4. CHALLENGES IN INTELLIGENT QUALITY CONTROL

- Security Issues in Smart Manufacturing

An intelligent manufacturing system uses an integrated network to share information between manufacturing or machining units and end users using the internet. A globally unique identity and end-to-end data encryption are required for internet-based information sharing to ensure data and information security throughout the system. Therefore, every network node must be secured against outside threats and data exploitation [6].

- System Integration

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Integrating new and existing technology equipment is a hurdle in developing an intelligent manufacturing system. A better communication system is also necessary for machine-to-machine communication and system interconnectivity. IPv6 connectivity is needed for the most modern production systems to enable more devices to be linked concurrently [6].

❖ Interoperability

The capacity of several systems to comprehend and utilize one another's features independently using proper synchronization of the communication protocols and standards is known as Interoperability.
The differences in transmission bandwidth, communication method, operational frequency, hardware capabilities, etc., limit the system's compatibility.

❖ Safety in Human-Robot Collaboration

Any instructions given in a human language should be converted into machine language by the multilingual intelligent manufacturing systems for performing the appropriate action. The instructions can be in verbal or text form of input from the operator.

❖ Multilingualism

Financial Analysis and return on investment (ROI) should be thoroughly examined for an existing manufacturing system before transitioning to other advanced technologies. The increased expenditure necessary compared to production losses during an upgrade and the time needed to recover the investment's return [6].

❖ Surface defect detection

The large and complicated curved-surface components present a challenge for inspection since it is challenging to assure shape accuracy using conventional inspection techniques, which is crucial to these components' functional performance [7]. The manual inspection methodology's accuracy for inspecting these components falls short of expectations. Thus it calls for the requirement of an inspection system that uses mobile manipulators to measure huge components with complicated curved surfaces in three dimensions accurately and automatically [8]. Surface defects are the lines or planes that divide a substance into sections, each having a distinct orientation but the same crystalline structure. Surface defects are often caused by surface finishing techniques like embossing, weather-related degradation, or environmental stress cracking [9]. Defects may also be produced when metals are used and treated for industrial reasons.

The aircraft and automotive industries reject any material with manufacturing defects since even a tiny defect in a finished product could result in a catastrophe. Steel rolling processes cannot do real-time inline surface flaw checking without automatic machine vision technology. Some spots may go unnoticed, costing production time and causing significant financial losses. Modern methods for finding surface defects include eddy current testing, electrical resistance testing, flux leakage testing, magnetic testing, thermographic testing, radiographic testing, resonant testing, ultrasonic testing, penetrant testing, and visual testing. A successful surface defect detection would ultimately identify all the surface defects on the product's surface, eliminate products with defective surfaces, and help repair the faulty surfaces. Automated visual inspection techniques for surface defect detection can help intelligent quality control automatically inspect each printed circuit board assembly and update measurements and observations without human interaction [10-13].

5. METHODOLOGY OF IMPLEMENTATION OF SURFACE DEFECT DETECTION

Here the process of implementation of intelligent Surface defect detection is presented further. Due to its high inference speed, the object detection model was trained using the Haar Cascade algorithm. This algorithm uses a set of 1000 images for training and uses Haar features that traverse through the entire image to detect a location of defects, namely, spalling, cracks, uneven surfaces, and holes. The samples of test images for defect detection are shown in figure 3.

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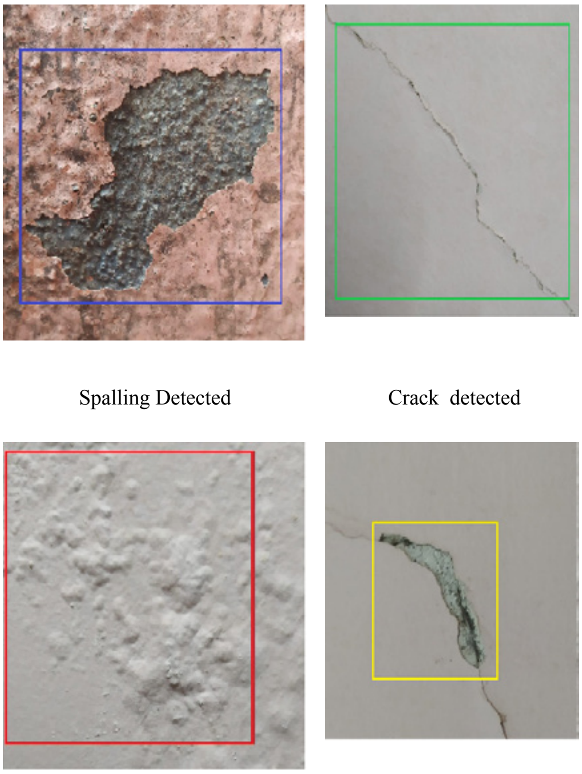
The experimentation results show that the Haar Cascade classifier is superior to other classifiers based on deep learning models in terms of higher inference speed. But the classifier suffers from some drawbacks, namely, low accuracy, false positives detection, and difficulty in training using custom datasets. The performance of the Haar Cascade classifier is evaluated in terms of inference time and accuracy of classification. As attaining accuracy in surface defect detection is crucial, autonomous quality inspection systems must adopt deep learning algorithms, namely, YOLO, Fast R-CNN, Faster R-CNN, etc., for better accuracy.



Spalling Detected Crack detected

Residual surface detected Hole detected

Figure 3. Surface defects detected.

6. CONCLUSIONS

The paper focuses on the study of quality control techniques, the evolution and processes involved and the state of art technologies in use. A review of challenges in intelligent quality control is explicitly done on surface defect detection. Further, the methodology and results of implementing surface defect detection using the Haar Cascade classifier on cement walls are presented. The performance of the Haar Cascade classifier is evaluated in terms of inference time and accuracy of classification. Further, it is proposed to analyze the surface defect detection performance using AI-based classifiers.

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A SECURED ARCHITECTURE FOR IOT-BASED HEALTHCARE SYSTEM

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ABSTRACT

Healthcare has gradually moved away from the model centered on traditional health centers due to the emergence of highly accurate sensors and Internet of Things (IoT) enabled medical equipment. Ambient intelligence takes whatever actions are required in response to a recognized event in order to enable continuous learning about patient data. The capabilities of IoT-assisted healthcare services might be improved by incorporating autonomous control and human-computer interface (HCI) technologies into ambient intelligence. Major unsolved issues include the privacy and security of information collected by medical IoT devices, both during transmission to and during cloud storage. This research explores different techniques, IoT factors, and features, with an emphasis on the data security concerns connected to data flow in medical IoT. In order to guarantee data security and privacy at all data levels, this study suggests a safe design for the IoT healthcare system.

KEYWORDS

HealthCare, Internet of Things, Secure IoT Networks, Privacy and Security HCI, Cloud Computing.

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1. INTRODUCTION

The Internet of Things principal objective is to connect every device on the earth. Today, IoT is mostly employed in healthcare to provide instant access to information. The IoT is a network of autonomous computing devices, mechanical and digital machines, animals, or humans. It connects everything to the Internet, encourages information exchange, organizes correspondence, and enables item positioning, tracking, administration, and monitoring. It provides information technology (IT) solutions, which employ computers to store, retrieve, transfer, and modify data without requiring human-to-human or human-to-computer interaction. According to its definition, the Internet of Items (IoT) is a "dynamic worldwide interconnected network technology with self-configuring capabilities based on standard and coherent communication protocols where virtual and physical things have identities, physical characteristics, and virtual personalities". Integrating a number of promising technologies will enable the IoT idea to be realized in the real world Jeong et al. [2016] Darshan et al. [2015] Thakre et al. [2022]. IoT may be combined with identification, sensing, and communication technologies. Security concerns, in addition to heterogeneity, scalability, connection, and many other problems, are significant hindrances to the growth of the Internet of Things and must be adequately addressed if IoT is to be successful. Among other security challenges like confidentiality, integrity, etc., authentication of devices participating in the IoT is one of the more significant ones and is the main theme of this article. IoT development has now been supported by a variety of technologies, including NOMA in wireless technology, MEMS, and the Internet Thakre et al. [2022] Kinhikar, et al. [2022]. A market worth more than 2.1 trillion dollars is predicted to emerge by 2025 owing to the low cost of sensor devices, resulting in more than 25 billion installed units by 2020. A market worth more than 2.1 trillion dollars is predicted to emerge by 2025 owing to the low cost of sensor devices, resulting in more than 25 billion installed units by 2020.

2. RELATED WORK

2.1 ARCHITECTURES

In the context of the IoT, the authors Almotiri et al. [2016] presented a mobile health (m-health) system (IoT). The use of mobile devices to obtain real-time health information from patients and store it on network servers connected to the Internet is known as m-Health.

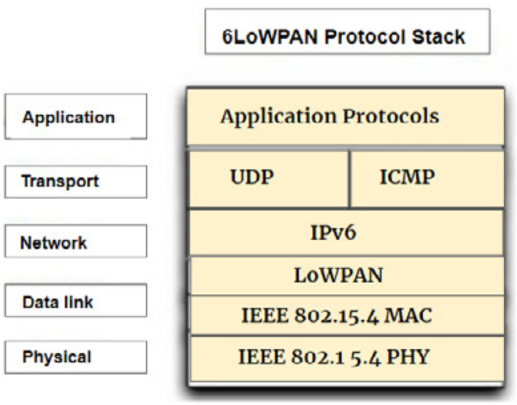

Fig 2.1: Protocol Stack.

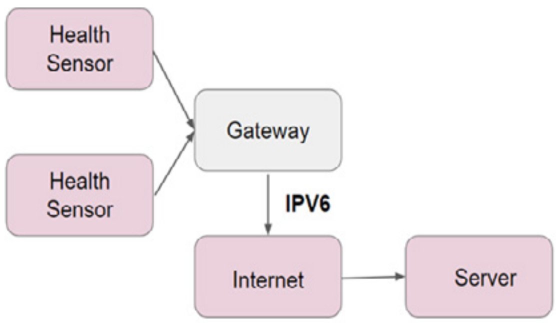

Fig 2.2: Related Architecture.

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In real-time, intelligence algorithms evaluate m-health data to find trends and raise different alarm levels based on the status of the observed patients. An m-health system's information technology architecture is a component of the IoT architecture, which is multi-layered and includes data collection, data storage, and data processing layers. In , T. N. Gia et al. [2014] the 6LoWPAN architecture, which is composed of low-power wireless area networks (LoWPANs)2, which are IPv6 stub networks for IoT networks, is described. 6LoWPAN has emerged as the favored option. Sensor nodes employ the 6LoWPAN protocol stack, represented in Figure 2, to transmit data to the network. The sensor data is encapsulated in a 6LoWPAN datagram and delivered over an IEEE 802.15.4 frame to the edge router or gateway. The packets are converted to IPV6 packets by the gateway and sent to the server for additional data processing through the standard IPV6 network. In Fig 2 depicts the end-to-end architectural features of the 6LOWPAN-based healthcare system.

2.2. SECURITY ENCURITY MODELS

Privacy protection has been widely researched and accepted as a bottleneck in smart medical healthcare. Goldwasser and Micali introduced a GM encryption system that proved semantic security under the assumption of quadratic residuosity (Shafi & Silvio, 1984).The ideal lattice-based encryption scheme proposed by Gentry [2009] was the first response to all encryption-related problems up to 2009 Cheon & Kim [2015]. A method to significantly boost totally homomorphic encryption's efficacy was put out by Chen, Ben, & Huang [2014]. In Cheon & Kim [2015] suggested sacrificing additional public keys in favor of lowering the exponentiation circuit's degree. In Ichibane et al. [2014] explained how to choose encryption parameters in several real-world settings.

3. PROPOSED ARCHITECTURE FOR IOT-BASED HEALTHCARE SYSTEM

The system under the proposed design uses IoT sensors to gather patient data, which is subsequently sent to the hospital's cloud storage. Five parts make up the architecture.

3.1. IOT DEVICES IDENTIFIED FOR MEASUREMENT ARE

Blood glucose sensor: An opto-physiological glucose sensor, which measures blood glucose levels using a photodiode and an accelerometer, can be utilized in a non-invasive IoT system for glucose monitoring.

Temperature monitoring sensor: A Raspberry Pi-based temperature monitoring system and a planned wireless network of sensors would alert a doctor if a patient's temperature rose over a certain level. The LM35 sensor is proposed as a device for sensing the body's core temperature.

Healthcare systems for the elderly: An approach for detecting falls among the elderly and informing concerned parties is described. The system detects falls using sensor values from the accelerometer and gyroscope. These should be simple for the monitoring systems to dismiss as false positives and not register as falls. The waist is the right location to implant the sensors to detect geriatric falls, according to an examination of diverse sensor implantation sites on the body using various types of sensors and algorithms for machine learning.

Electrocardiogram heart monitoring systems: A minimal-cost ECG system is recommended. The ECG sensor in this system is merely a data collection device. Multiple users' ECG values are collected with the ECG sensor and sent via Zigbee to a centralized server.

Heart Rate Monitoring: Instant heart rate monitoring is a feature of several popular models, including Fitbit and Garmin as reviewed in Kumar N. [2017]

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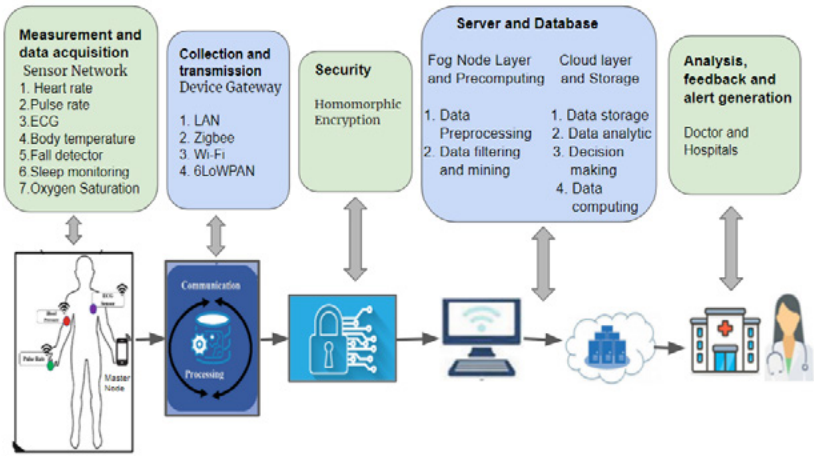

Fig 3.1: Proposed Architecture.

3.1.1. SENSORS IN IOT-BASED HEALTHCARE SYSTEMS

A minimal cost ECG system is recommended. The ECG sensor in this system is merely a data collection device. Multiple users' ECG values are collected with the ECG sensor and sent via Zigbee to a centralized server. The MMA7260QT accelerometer is specifically suggested for use in detecting body movement. Angular velocity can be measured along the x, y, and z axes using a device with three axes. A gyroscope can help alert medical workers when a person falls by detecting body tilt. A magnetometer aids in determining the relative direction by measuring the magnetic field. A magnetometer can detect a human fall when combined with an accelerometer, gyroscope, and other sensors, and is frequently seen in aged care equipment. Applications used: Raspberry Pi: According to, body temperature monitoring devices are created using the Raspberry Pi platform. A multispectral system that monitors body temperature, respiration rate, heart rate, and movement is developed using a Raspberry Pi as reviewed in Kumar N. [2017].

3.2. COLLECTION AND TRANSMISSION

Sensors are fastened to the patient's body to collect and transmit data to the master node in real time. The data from the master node is received by the mediator device and sent to the cloud. The collection and storage of data in the cloud is an ongoing process. The IoT sensors based on WiFi and Zigbee have the lowest latency, as shown in Table 3.2. Material is delivered to hospital servers using gateway nodes installed in the area for the protocol conversion from ZigBee to TCP/IP.

Table I. Comparison of available wireless communication technologies for smart healthcare.

Technology	Frequency	Types	Power usage	Range	Rate of Data
NFC	13.56 MHz	PAN	Very low	10 cm	100-400kbps
Bluetooth 4	2.4 GHz	PAN	Low	0.1km	1Mbps
Bluetooth 5	2.4 GHz	PAN	Very low	0.25 km	2Mbps
Z-wave Alliance	900 MHz	LAN	Very low	30m	9.6/40/100 kbps
Wi-Fi	2.4 GHz and 5GHz	LAN	Low-High	50 m	1Gbps
Zigbee	2.4GHz	LAN	Very low	10-100 m	250kbps

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Low-power sensor nodes can now be connected to the internet utilizing the Internet protocol, owing to communication protocols like 6LoWPAN. In order to function, the IPV6 protocol needs a substantial amount of computing power and bandwidth. They foresee "always-on" activity, which IoT devices do not. The study Ge et al. [2016] discovered that delivering these signals as JSON packets rather than XML packets significantly reduces response, processing, and interpretation times.

3.3. COMPUTING AND SECURITY

In fog computing, fog nodes (which can be access points, switchers, gateways, routers, etc.) are spread at the network's edge and move toward terminal facilities at a certain location. The fog computing layer is composed of the security, storage, and monitoring layers. Fog computing converts cloud data centers into uniformly spread platforms while aiming to sustain cloud services. As a result, there is a decrease in processing time for data from wireless medical sensors, which improves customer satisfaction and service level. Because storage nodes are low and also have limited power capacity, the public key encryption method cannot be employed for security. The research defines fog computing as the addition of cloud computing to the system's edge, which is a highly virtualized stage of the source pool that delivers computation, storage, and networking resources to local end users. The results indicate that fog computing may achieve more than 89% low latency and bandwidth efficiency. Because the fog nodule has a specified limited size, it cannot contain a large number of activities each second Yi et al. [2014], Kumar Y. et al. [2019]. Fog computing in health care was the focus of Isa et al. [2020]. In this study, a heart monitoring application was developed in which each patient was required to provide a 30-minute recording of their electrocardiogram signal to fog processing units for handling, analysis, and decision-making. The values are decreased so that less energy is utilized in both processors and networking equipment. When compared to the central cloud, the results show that energy may be saved by up to 69%.

To keep information confidential and private, it is necessary to encrypt it before it is delivered across a public channel and stored on fog nodes or cloud servers. After receiving information and instructions from the user and the cloud, they are responsible for filtering the raw data and uploading it to the cloud for long-term storage or additional analysis.

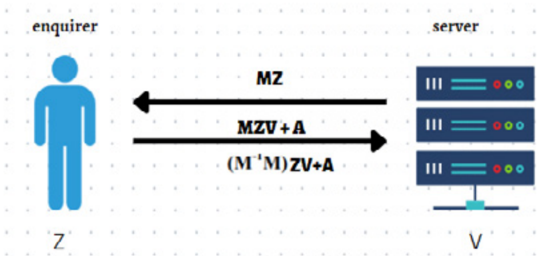

Fig 3.3: Homomorphic encryption.

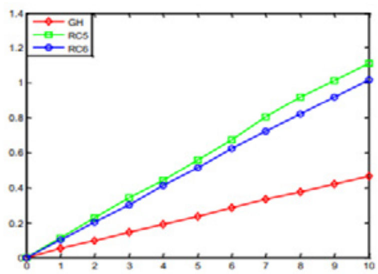

Fig 3.4: Comparison between different encryption models.

Steps for Homomorphic encryption

Step 1: The inquirer describes the data collected as a vector Z = (z1,z2,z3,.....,z8). After that, the secret eight-order invertible matrix M is generated. MZ is then computed and sent to the server.

Step 2: On the server, an intermediate vector V represented by V = (v1,v2,v3....v8) is stored. The greatest (denoted as Mx) and smallest (denoted as Mn) values of each physiological item's normal zone are likewise kept, and Li = Mx - Mn is discovered, with xi set to be 1/Li. In addition, we create matrix A, the elements that form the leading diagonal are equal to Mid, where Mid= -Mn/Li.

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Step 3: When the medical result MZ arrives at the server, it is multiplied by V to get MZV. MZV and matrix A are then returned to the enquirer by the server. The enquirer then obtains ZV by left multiplying it by M-1. Finally, Di is obtained by multiplying matrix A by ZV.

Step 4: If Di is in the range [0, 1], the data item remains normal. If Di is more than one, the data item appears too high; otherwise, it displays too low.

Even if the attackers are capable of stealing data passing through the communications platform, such as MZ and A, the data vector Z is left multiplied by M. If the proposed homomorphic encryption mechanism is used, hackers will be unable to access genuine data because they do not know M.

Fig (3.4) shows that the proposed encryption algorithm outperforms RC5 and RC6 in terms of efficiency. Several experiments were carried out to demonstrate its speed.

3.4. ANALYSIS OF DATA

Following the acquisition, the acquired data must be handled, filtered, and compressed to eliminate extraneous information. We can use the ontology technique, as proposed in the article Kumar V. [2015]. Clinic data for patients is specified as a source with a specific URL address in the recommended approach. To enable information transmission via Ontology data access, mapping with both previously collected and newly acquired records should be carried out after the acquisition of patient records. The doctor will create a treatment strategy for the patient after considering these variables, which will include the drugs and dosages that will be used. In this instance, the relationship is represented by the Protégé tool. Run a sparql query to get data or an owl/rdf file from the database.

4. CONCLUSION

This paper provides an in-depth examination of the most recent advancements in the IoT-based healthcare ecosystem, as well as a framework for IoT healthcare, a smart healthcare system, and its logical architecture. Sensors in a smart healthcare system collect medical information, data is collected via mobile and smart networks, data is transferred to cloud computing for analysis with advanced algorithms, and medical professionals can make treatment and diagnosis recommendations. The implementation of the IoT based healthcare system is split up into three features, with details on each presented. A homomorphic strategy based on a scrambling matrix was developed to address current problems of security in the IoT field. With the rapid development of IoT, we can anticipate that our medical healthcare system will have a wide range of applications.

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EMPIRICAL ANALYSIS OF MACHINE LEARNING-BASED ENERGY EFFICIENT CLOUD LOAD BALANCING ARCHITECTURES: A QUANTITATIVE PERSPECTIVE

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ABSTRACT

Design of energy efficient load balancing models in cloud environments requires in-depth analysis of the cloud architecture and nature of requests served by the cloud. Depending upon these parameters, machine learning models are designed which aim at assigning best possible resource combination to serve the given tasks. This assignment varies w.r.t. multiple task and cloud parameters; which include task time, virtual machine (VM) performance, task deadline, energy consumption, etc. In order to perform this task, a wide variety of algorithms are developed by researchers cloud designers. Each of these algorithms aim at optimizing certain load balancing related parameters; for instance, a Genetic Algorithm (GA) designed for optimization of VM utilization might not consider task deadline before task allocation to the VMs. While, algorithms aimed at performing deadline aware load balancing might not provide effective cloud-to-task-mapping before allocation of tasks. Thus, it becomes difficult for researchers to select the best possible algorithms for their cloud deployment. In order to reduce this ambiguity, the underlying text compares different energy efficient cloud load balancing algorithms; and evaluates their performance in terms of computational complexity, and relative energy efficiency. This performance evaluation is further extended via inter architecture comparison; in order to evaluate the most optimum load balancer implementation for a given energy efficient application. Thus, after referring this text, researchers and cloud system designers will be able to select optimum algorithmic implementations for their given deployment. This will assist in reducing cloud deployment delay, and improving application specific load balancer performance.

KEYWORDS

Cloud, Load, Balancing, Machine, Learning, Task, Deadline, Energy.

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1. INTRODUCTION

Due to the current CoVID-19 pandemic, most businesses are forced to adopt the work-from home (WFH) model. This model has increased dependency of users on cloud-based services, thereby requiring cloud service providers to optimize their load-balancing models. These models are broadly categorized into 2 types; which are hardware load balancing, and elastic load balancing. The former type consists of optimizing performance of hardware components like virtual machines, servers, memory utilization, task optimization, and central processing unit (CPU) optimization.

While, the later type consists of network load balancers, application load balancers and hybrid load balancers. Hierarchical categorization of these algorithms can be observed from figure 1, from where it can be observed that network load balancing depends on VM and Server load balancing; application load balancing depends on memory, task and CPU load balancing; while classic (or hybrid) load balancing depends on all the hardware-based load balancing models. VM load balancing models aim at optimizing virtual machine performance by assigning tasks in such a manner that most VM resources are utilized, thereby improving hardware utilization efficiency.

This model does not take into consideration deadline constraints, memory constraints, etc. while modelling the load balancer. In contrast, memory-based load balancer models only take into consideration memory utilization; and aim at optimizing task storage without considering resource or CPU load values. Server load balancing algorithms assist in optimization server utilization while performing load balancing, while CPU load balancers aim at optimizing CPU utilization while load balancing. Moreover, task load balancing models aim at executing tasks under a given deadline without considering CPU load, memory or virtual machine efficiency values.

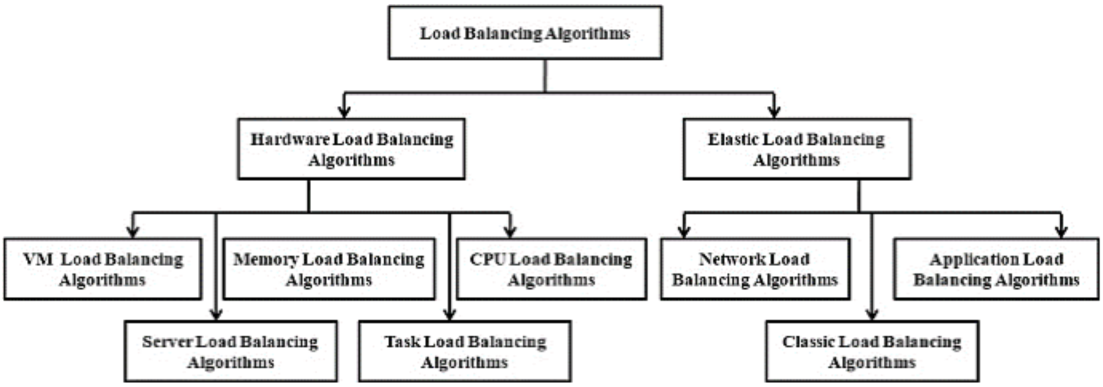

Figure1. Hierarchical categorization of load balancing models.

Elastic load balancers aim at optimizing a group of parameters during load balancing. For instance, network load balancers optimize virtual machine and server parameters; while application load balancers aim at optimization of memory, CPU load and task parameters. A combination of these parameters is optimized by classic load balancers, wherein depending upon the application; one or more cloud task parameters are optimized. An in-depth survey of these optimization models can be referred from the next section; wherein various machine learning models for load balancing are described. This is followed by statistical analysis of these models; their comparative evaluation. The evaluation assists in identification of best suited for models for any given application; which will assist researchers and system designers for high speed and high efficiency system design. Finally, this text concludes with some interesting observations about the reviewed algorithms; and recommends methods to improve them.

2. LITERATURE REVIEW

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A wide variety of energy efficient cloud load balancing models have been proposed by researchers and cloud designers. These algorithms aim at improving energy efficiency via application of different machine learning models including but not limited to bio-inspired models, swarm-optimization models, neural networks, etc. The work in (Panda, Moharana, Das, and Mishra, 2019)[1] introduces such an energy efficient model that utilizes virtual machine consolidation in cloud environments.

The model aims at minimizing energy consumption during virtual machine (VM) migration process via threshold-based sleep scheduling. Here, virtual machines with lower load levels are put to sleep for specified clock cycles, thereby assisting in energy reduction. Due to this sleep scheduling, VMs with high loads are easily identified. This identification assists in assigning tasks to the sleep mode VMs, thereby reducing the probability of load imbalance. Description of the model can be observed from figure 2, wherein load balancing and VM migration processes are described.

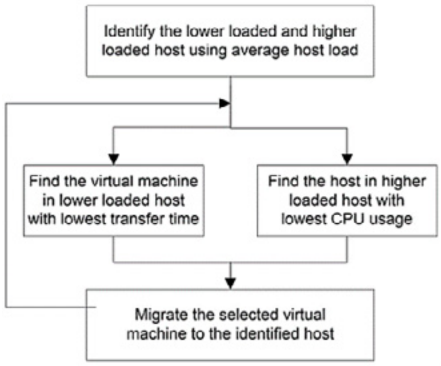

Figure2. Energy efficient VM migration and load balancing.

Figure 2. Energy efficient VM migration and load balancing (Panda et al., 2019)[1] Due to efficient migration of VMs, and proper load division, the model is able to reduce load imbalance levels by 15% energy consumption by 20% when compared with non-energy aware models. A similar model that uses dynamic energy efficient resource (DEER) allocation can be observed from (Rehman, Ahmad, Jehangiri, Ala'Anzy, Othman, Umar, and Ahmad, 2020)[2], wherein resource with minimum utilization is used for load balancing. This model also performs intelligent sleep scheduling, due to which an energy efficiency of 15% is achieved when compared with Dynamic Resource Allocation Strategy (DRAM) model. This model also reduces computational cost by 10% when compared with DRAM, thereby making is useful for real time load balancing applications. A load-based model that uses self-organizing maps (SOMs) can be observed from (Malshetty and Mathapati, 2019)[3], wherein cluster heads are created, and all load requests are handled by them.

The selected cluster heads are able to reduce computational load on centralized server, and select machines with minimum power consumption, thereby improving overall energy efficiency of the system. It is observed that the proposed model is able to reduce energy consumption by 18% and delay of processing by 10% when compared with Low-energy adaptive clustering hierarchy (LEACH) model. This approach can be compared with other models, a survey of these models can be observed from (Ala'anzy and Othman, 2019)[4], wherein effects load balancing & server consolidation are studied. The work compares algorithms like load-aware Global resource affinity management, advanced prediction-based minimization of load migration, multidimensional hierarchical VM migration, extended first fit decreasing algorithm, locusts inspired scheduling algorithm, etc.

It is observed that soft computing models outperform linear models in terms of energy efficiency & load balancing performance. An example of such a soft computing model can be observed from (Salem Alatawi and Abdullah Sharaf, 2020)[5], wherein Honey Bee optimization is combied with fuzzy logic for improved energy efficiency during load balancing. The model uses a fuzzy approach for host & VM selection, and then deploys a Honey Bee optimization model for load scheduling between these components. Due to incorporation of VM energy levels, and task length in the fitness function (described in equation 1), the model is able to schedule tasks with good energy efficiency.

$$Fitness = \frac{T_{length} * T_{deadline}}{E_{vm}} \ldots (1)$$

Where, T length T deadline are task length, and task deadline; while E vm is the per task execution energy of the VM. The model aims are reducing this fitness value in order to improve the energy efficiency, and execute tasks of the given length under the given deadline. The model is able to reduce energy consumption by up to 18% when compared with only fuzzy model, and up to 15% when compared with only the Honey Bee optimization model, thereby making it applicable for real time use. Execution of tasks with high energy efficiency must be accompanied with effective placement of VM services. This placement allows schedulers to select nearby VMs in order to execute tasks with high efficiency and low energy consumption. Example of such an architecture can be observed from (Alharbi, El-Gorashi, and Elmirghani, 2019)[6], wherein researchers have showcased the use of cloud-to-fog load balancing reduces energy consumption by 75% when compared with only-cloud load balancing architecture. This architecture can be observed from figure 3, wherein data from cloud is offloaded to fog nodes for efficient balancing.

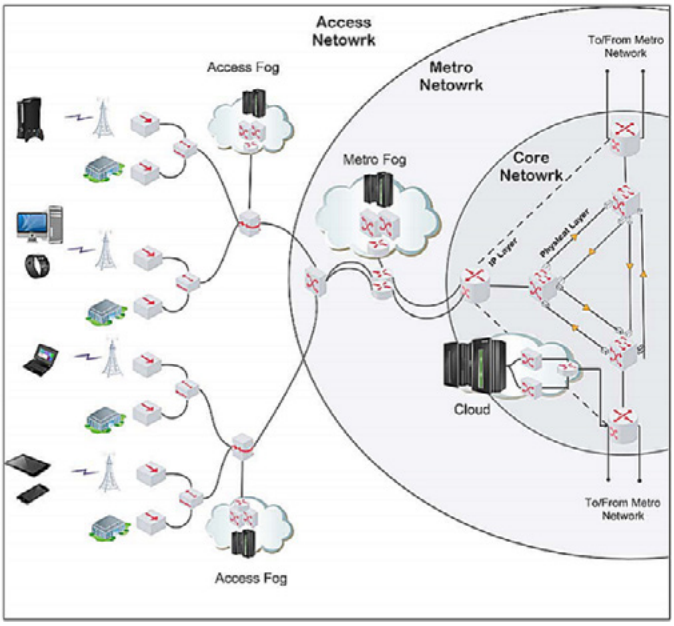

Figure3. Offloading data from single cloud to fog nodes for energy efficient load balancing [6].

Figure 3. Offloading data from single cloud to fog nodes for energy efficient load balancing [6] Due to offloading of tasks on fog nodes, the computational load is shared between different nodes, thereby assisting in faster task execution, and better energy efficiency. This efficiency can be further improved via addition of soft computing models. Such a model is defined in (Dong, Xu, Ding, Meng, and Zhao, 2019)[7], wherein glow worm swarm optimization (GSO) is used. The model combines clustering for extraction of large task resources, with sine cosine analysis (SCA) in order to modify the step size of GSO for energy adaptive scheduling. This modification in step size allows the GSO model to select load & energy optimized cloud resources for the given task.

The clustered input tasks are divided into edge group & cloud group; wherein each group has task elements based on CPU utilization and memory consumption. For instance, the edge group retains tasks with high CPU utilization but low memory consumption, while the cloud group retains tasks with high memory consumption. This task division is governed using the following equation,

$$L\_group = \ \vartheta * L\_cpu + \emptyset * L\_memory \ldots (2)$$

Where, L group is resource requirement for the given group, L cpu is resource requirement w.r.t CPU utilization, L memory is resource requirement w.r.t. memory utilization, while and are cloud and edge constants. The model is compared with First Fit Decreasing (FFD) & OTS models and it is observed that the GSO model has 25% better energy efficiency, 15% better throughput, and 18% better load balancing degree when compared with these algorithms. This efficiency can be further improved by performing computations on fog devices as observed from (Bhuvaneswari and Akila, 2019)[8], wherein comparison of different fog-based load balancing algorithms w.r.t. their energy performance is studied. Algorithms like ant colony optimization (ACO), max-min algorithm, active monitoring for load balancing (AMLB), and round robin (RR) are compared. It is observed that the ACO based soft computing model has 15% better efficiency when compared with RR, while AMLB when combined with ACO provides 25% better energy efficiency than individual algorithms. Another hybrid combination that uses ACO with support vector machines (SVM) can be observed from (Junaid, Sohail, Ahmed, Baz, Khan, and Alhakami, 2020)[9], wherein file type formatting (FTF) is used for load classification. Input requests are given to SVM model and depending upon file type at input, the model classifies load types into low power, medium power and high power. After this, the classified load is given to ACO, wherein VM to load mapping is performed using greedy heuristics. Flow of the model can be observed from figure 4, wherein training phases and testing phases can be seen with the final load balancing process. It is observed that the model performs SVM training on file formats, and provides the classified results to ACO. The ACO model internally maps suitable VMs to tasks for high energy efficiency. Due to this, the proposed SVMFTF model provides 8% better energy efficiency than random forest (RF), 6% than Na¨ıve Bayes, 9% than k-nearest neighbours (kNN), and 4% than convolutional neural network (CNN) models.

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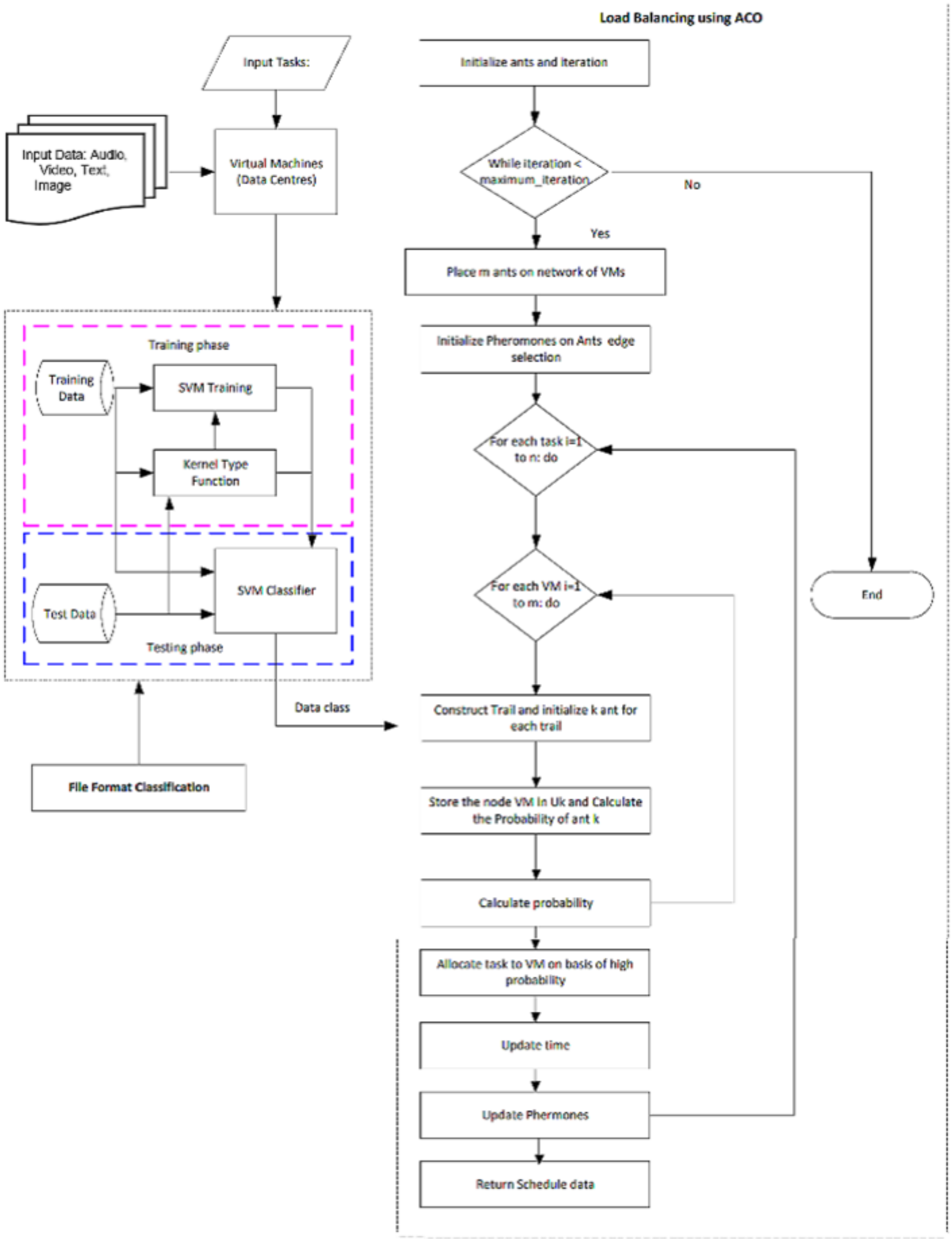

Figure4.SVMFTF model for energy efficient load balancing [9].

Figure 4. SVMFTF model for energy efficient load balancing [9] The model also showcases better throughput, and migration performance when compared with RF, Na¨ıve Bayes, kNN, CNN models. Specifically, the model provides 8% better throughput than RF, 3% better than Na¨ıve Bayes, 15% better than kNN, and 5% better than CNN; while it provides 10% better migration performance than RF, 4% better than Na¨ıve Bayes, 8% better than kNN, and 5% better than CNN models when compared on the same task set and cloud configurations . Comparison of this model can be done with other standard load balancing models as suggested in (Dey and Gunasekhar, 2019)[10], wherein models like first in first out (FIFO), fair scheduling, capacity scheduling, hybrid scheduling, longest approximate time to end scheduling, self-adaptive mapreduce scheduling, and context-aware scheduling for Hadoop are discussed. Each of these models are compared on parameters like Job Characteristics, Responsiveness, Resource Pool Configuration, Queue Characteristics, Parallelization of Tasks, Queue Responsiveness, Dynamic Priority, Locality Management, Remaining Burst Time, Task Priority, Context and Energy Efficiency. All these parameters are evaluated on the Planet Lab dataset, wherein it is observed that Robust Local Regression (RLR) methods like self-adaptive map-reduce scheduling, and context-aware scheduling outperform other methods by over 10% in terms of

energy efficiency. The same trend is observed for other parameters, due to the sophistication of the RLR methods, and in-depth analysis for the given tasks.

A similar study like [10] can be observed in (Kulshrestha and Patel, 2019)[11], wherein models like ALB, transport layer load balancer (TLLB), network layer load balancer (NLLB), VM provisioning on host, consolidation of VMs on host, and VM-level task scheduling are described. Out of these algorithm ALB outperforms other models in terms of energy efficiency by providing 8% better performance than TLLB, 5% better performance than VM provisioning, and 15% better performance than VM consolidation. An example the ALB scheme can be observed in (Zhang, Jia, Gu, and Guo, 2019)[12], wherein Matrix sparseness with normalizedWater-Filling (MSNWF) is described. This model is compared with Heterogeneous Network (HETNET), and OPT models, and it is observed that MSNWF provides 15% better energy efficiency than HETNET and 5% better energy efficiency than OPT. Thus, the MSNWF model can be used for high performance cloud load balancing applications, wherein along with efficiency of task scheduling, energy efficiency is also improved. This model performance can be further improved by integration of broker service policy for software as a service (SaaS) application. Modelling of such architectures requires high efficiency broker design, wherein any incoming task is first given to a broker for estimation of approximate processing site. This estimation allows the cloud VMs to pre-allocate resources for the task, thereby improving the task execution efficiency. In order to model such brokers, architectures like shortest job scheduling, Min-min, Max-min, Two-phase (OLB + LBMM), Modified active monitoring, Throttled Load Balancer, Genetic Algorithm, Honey Bee foraging algorithm, ACO, etc. are available (Jyoti, Shrimali, Tiwari, and Singh, 2020)[13]. It is observed that ACO and other soft computing models when utilizing fog and cloud computing, outperform other models in terms of energy efficiency. An example of such a model can be observed from (Lin, Peng, Bian, Xu, Chang, and Li, 2019)[14], wherein the soft computing models are deployed on cloud. The results of these models are VM-to-task mapping, which are executed either on the cloud infrastructure or offloaded to the fog device for better load balancing capabilities. The model for this architecture can be observed from figure 5, wherein offloading process is performed using different wireless standards.

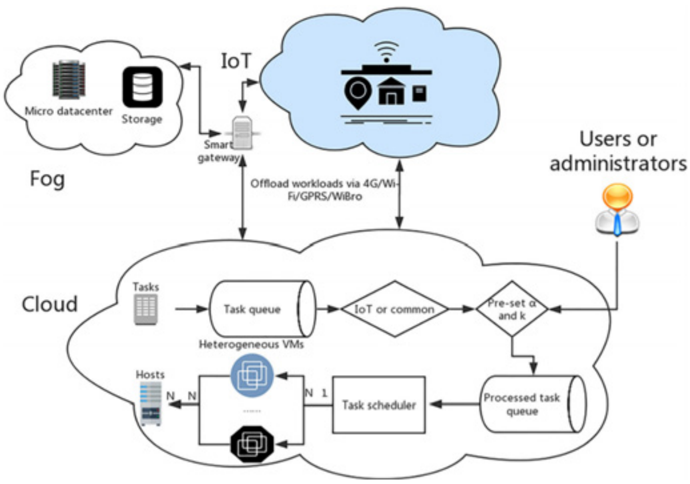

Figure5. Fog-cloud load balancing model with soft computing for efficient task mapping [14].

Figure 5. Fog-cloud load balancing model with soft computing for efficient task mapping [14] The model utilizes total amount of resource in host, total amount of resource in VM, estimated execution time of task, normalization of task average demand for resource, normal resource load of task, relative load of task for resource upon VM, task length, and task priority for scheduling. It is observed that the proposed soft computing model outperforms first come first serve by 15%, random assignment by 25%, trade-off by 8%, main resource load & time balancing by 16%, and main resource task balance by 9% in terms of energy efficiency. This is due to design of energy & task length aware fitness function design, which suggests that soft computing must be used for any kind of load balancing models. A similar model is proposed in (Mandal, Mondal, Banerjee, and Biswas, 2020)[15], wherein

service level agreement (SLA) is used for detection of task overload at different hosts. It uses a mapping ratio that consists of VM utilization and allocated resource characteristics in order to assign tasks to non-overloaded VMs. The value of this mapping ratio (Mapping r) can be observed from equation 3, wherein both the parameters are split into task & resource related characteristics.

$$Mapping_r = \frac{VM\_mips + VM\_RAM + VM\_BW}{T_{length} + T_{deadline}} \ldots (3)$$

where in, VM mips, VM RAM & VM BW are VM specific capacity, available RAM & bandwidth while Tlength & Tdeadline are task related length & deadline parameters. It is observed that the proposed model outperforms minimum migration time MMT by 45%, maximum correlation MaxCorr by 35%, minimum utilization (MU) by 46%, and random selection RS by 34%, thereby making it highly useful for real time cloud deployments. Context-aware load balancing models have better efficiency than energy-aware, or task-length aware models because these models adaptively modify their internal rules depending upon the context of given task and condition of the VMs. Such a model that utilizes context information for energy efficient load balancing is described in (Royaee, Mirvaziri, and Khatibi Bardsiri, 2021)16], wherein automata ant colony based multiple recursive routing protocol (AMRRPL) is used. The model solves issues like bottlenecking, efficient parameter selection, effect of upstream nodes, and congestion which are inherent with load balancing. The model uses destination oriented directed acyclic graph (DODAG) in order to perform load balancing via laying out all possible VM-to-Task combinations on an acyclic graph. Due to use of DODAG the model is able to achieve an energy efficiency of 8% when compared with ERPL (enhanced RPL), and 45% when compared with HECRPL (hybrid energy efficient RPL) and its configurations. This model can be applied to various applications including software defined network (SDN), content delivery network (CDN), cost-based distribution (CBD) networks, etc. for highly energy efficient load balancing. An example of this application for CDN can be observed in (Gupta, Goyal, and Gupta, 2015)[17], wherein a reliability aware load balancer model is applied.

The model uses a modified version of Genetic Algorithm (GA) for task scheduling, and is able to obtain 15% better energy efficiency when compared with queue length-based load balancing (QLBLB) model. Another low power model that uses first of maximum loss scheduling algorithm (FOML) is described in (Liang, Dong, Wang, and Zhang, 2020)[18], wherein relationship between energy utilization & average completion time is used. The model selects VM with maximum energy utilization and assigns it to a task that has average completion time (when compared to all tasks in queue). This task is then deleted, and a new average completion task is evaluated and assigned to the next maximum energy utilization VM. This process makes sure that all the high energy consuming VMs are assigned to moderate sized tasks, while other VMs are assigned to large & small sized tasks. Flow of this model can be observed from figure 6, wherein ETC (extended time of completion) and ACT (average completion time) matrices are evaluated for the given set of tasks.

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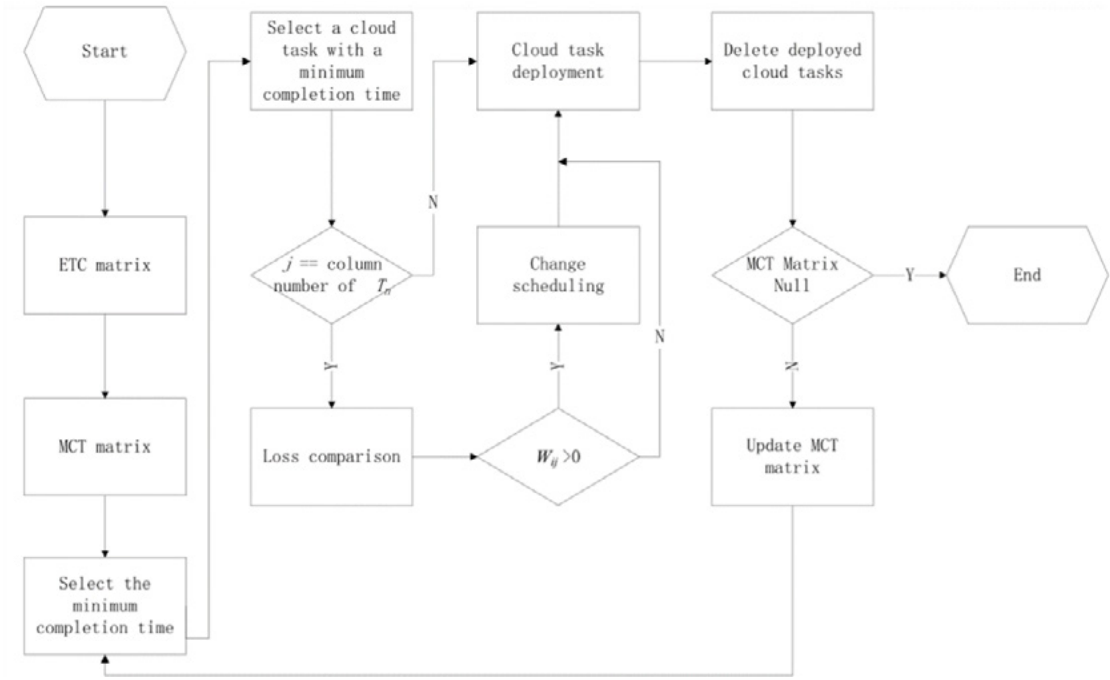

Figure6.Maximum energy utilization model for efficient task scheduling [18].

Figure 6. Maximum energy utilization model for efficient task scheduling [18] The FOML model is able to obtain 10% better energy utilization when compared with min-min model, 8% better utilization when compared with max-min model, 12% when compared with suffrage model, and 15% better than E-HEFT model, thereby making it highly effective for real time deployments.

Similar models are described in (Hadikhani, Eslaminejad, Yari, and Ashoor Mahani, 2020)[19], and (Sadeghi and Avokh, 2020)[20], wherein geographic information and Two-hop Routing Tree with Cuckoo search (CCTRT) models are defined. These models able to achieve 5% and 12% better energy efficiency when compared with E-HEFT model, thereby suggesting that the use of soft computing models is fundamental to design of energy efficient load balancing algorithms. The efficiency of these models can be further improved via use of predictive workload balancing, wherein the system is able to predict workloads depending upon task patterns, and pre allocate cloud & fog VMs for efficient execution. Architecture for such a system can be observed in (Jodayree, Abaza, and Tan, 2019)[21], wherein rule-based workload prediction is defined. It uses a combination of historical data analysis and random workload assignment in order to speed up workload balancing. Due to predictive analysis, the model is able perform host reduction and thereby reduce energy consumption by 10% when compared with random assignment algorithm.

This model can be further extended via use of deadline constrained task scheduling as suggested in (Ben Alla, Ben Alla, Touhafi, and Ezzati, 2019)[22], wherein a dynamic classifier is used to divide incoming tasks into priority queues, and each of these queues is processed using Fuzzy Logic and Particle Swarm Optimization model (FLPSO). The FLPSO model used in this approach is able to reduce energy consumption by 60% when compared with FCFS (first come first serve), 25% when compared with EDF (earliest deadline first), and 15% when compared with Differential Evolution (DE) with Multiple Criteria Decision Making (MCDM) algorithms. This comparison appears to be true for different VM and task combinations, thereby assisting in deploying the FLPSO model for a wide variety of cloud infrastructures.

This approach must be compared with other models like the ones mentioned in (Pourghebleh and Hayyolalam, 2020)[23] in order to evaluate its real time applicability and deployment capabilities.

Similar energy efficient models are discussed in (Rashid, Tripathi, Prakash, and Tripathi, 2019)[24], (Singh and Kumar, 2019a)[25] and (Kansal and Chana, 2018)[26] wherein load based energy efficiency, security aware energy efficiency, and migration aware energy efficiency models are

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described. Each of these models utilize soft computing techniques like ACO, PSO, GA, and GSO in order to achieve high energy efficiency. A resource aware load balancing model can be observed in (Ahmed, Aleem, Noman Khalid, Arshad Islam, and Azhar Iqbal, 2021)[27], wherein heterogenous clustering is used in order to perform resource-based task mapping. The model performs job to resource mapping depending upon resource availability, and resource aware load balancing for obtaining higher utilization ratio. It uses a predictive model for classification and forecasting job device suitability & job time estimation matrix as observed from figure 7, wherein the overall model is described. The model extracts features including front end clang (percentage of tasks remaining), kernel features ratio of task length to current machine configuration), and static features (initial performance of machines and number of tasks) from tasks and provides them to Resource aware load balancing and hierarchical clustering (RALBHC) model for improvement of resource utilization.

The model is able to achieve an energy efficiency of 25% when compared with Max-min algorithm, 15% when compared with Minimum Completion Time, 8% when compared with Resource-Aware Scheduling Algorithm (RASA), and 10% when compared with Task-Aware Scheduling Algorithm (TASA). Another energy efficient model that uses equal load distribution for fog-to-cloud & cloudto-fog migration (EDCW) is described in (Kaur and Aron, 2020)[28], wherein linear programming (LP) is used. The use of LP results into equal distribution of tasks between fog node and cloud node, thereby assisting in improved load balancer performance. The model is able to achieve 15% better energy efficiency when compared with Round Robin model, and 8% better efficiency when compared with throttled model, thereby making it useful for low energy load balancing applications.

Similar energy-efficient models are proposed in (Escobar, Ortega, D´ıaz, Gonz´alez, and Damas, 2019)[29], (?, ?)[30], (Taboada, Aalto, Lassila, and Liberal, 2017)[31],(Kumar, Singh, and Mohan, 2021)[32], and (Singh and Kumar, 2019b)[33], where in parallel evolutionary algorithms, context-based load balancing, energy-aware load balancing, resource-efficient load-balancing, and secure load balancing models are described. These models make use of different soft computing methods in order to perform task-based & resource-based load balancing. The underlying models are able to reduce energy consumption via optimization of the fitness function, wherein resource energy, task length, task deadline, and resource performance parameters are used. A quantitative analysis of these models is described in the next section, wherein the underlying models are compared in terms of relative energy efficiency values, thereby assisting cloud system designers to identify energy-efficient load-balancing models for their deployment.

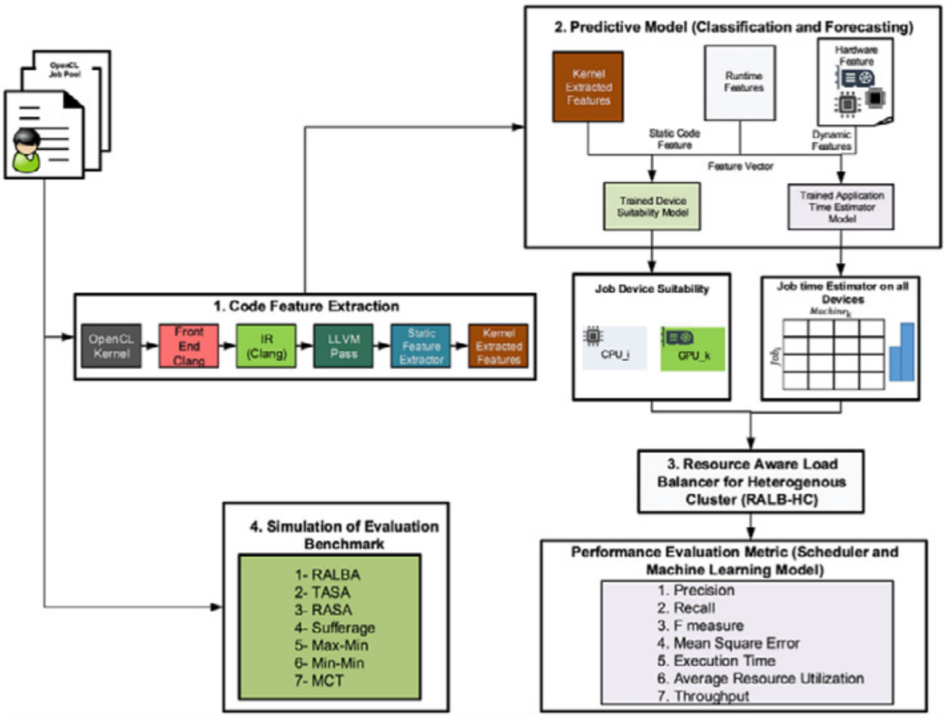

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Figure7.Maximum energy utilization model for efficient task scheduling [18].

3. QUANTITATIVE ANALYSIS

From the literature, it is observed that the energy efficiency of different cloud load balancing algorithms is estimated using their relative percentages. This limits the quantitative comparison capability of these algorithms. In order to resolve this drawback, this section evaluates absolute percentage energy efficiency when compared with the basic first come first serve (FCFS) algorithm.

This process will allow researchers to estimate energy performance of reviewed algorithms, along with their computational complexity. The computational complexity is divided into fuzzy ranges of low (L), medium (M), high (H) and very high (VH). Maintaining a balance between energy efficiency and computational complexity is a must while designing load balancing models for cloud. The quantitative results are tabulated in table 1, wherein the aforementioned parameters are compared across different algorithms.Based on this analysis it can be observed that the FLPSO (Ben Alla et al., 2019)[22], AMRRPL (Royaee et al., 2021)[16], MU (Mandal et al., 2020)[15], SLA based model (Mandal et al., 2020)[15], EDF(?, ?)[22], and GSO SCA(Dong et al., 2019)[7] model outperform other models in terms of relative energy efficiency.This performance evaluation can also be observed from the visualization in figure 8, wherein different algorithms and their accuracies are compared. It can also be observed that CNN and other deep learning models are not used for energy efficient load balancing, because training of these models for dynamic loads is resource intensive, thereby requires large amount of power.

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Model Used	Energy efficiency %	Computational Complexity
Threshold based sleep scheduling [1]	15	M
DEER [2]	12	M
DRAM [2]	8	M
Load based SOM [3]	22	H
Honey bee optimization [5]	25	H
Fuzzy logic [5]	15	M
Cloud to fog migration [6]	34	H
Single cloud [6]	18	M
GSO SCA [7]	33	M
FFD [7]	23	M
OTS [7]	15	L
ACO [8]	18	M
AMLB [8]	8	L
RR [8]	3	L
ACO FTF SVM [9]	12	H
RF [9]	5	M
Naïve Bayes [9]	8	M
kNN [9]	2	L
CNN [9]	0	VH
RLR [10]	15	H
NLLB [11]	10	M
MSNWF [12]	25	VH
HETNET [12]	15	H
Fog-cloud model [14]	15	H
Random assignment [14]	22	M
SLA based model [15]	48	H
MMT [15]	40	M
Max Corr. [15]	30	M
MU [15]	41	M
RS [15]	31	M
AMRRPL [16]	48	H
ERPL [16]	16	M
HECRPL [16]	5	M
FOML [18]	15	M
Min to Min [18]	6	L
E-HEFT [18]	23	M
CCTRT [20]	18	H
Rule based prediction [21]	15	M
FLPSO [22]	60	H
EDF [22]	48	M
DE [22]	26	M
MCDM [22]	31	M
RALBHC [27]	35	H
Max to Min [28]	18	M
RASA [28]	2	M
TASA [28]	25	M

Moreover, standard CNN models are also not available for this purpose, therefore it is a necessity that researchers should develop such models that aim towards energy efficiency. These models can then be extended via transfer learning or recurrent networks in order to incrementally tune their performance. Neural network models have very low energy consumption during evaluation, thus pre-training of models is further recommended.

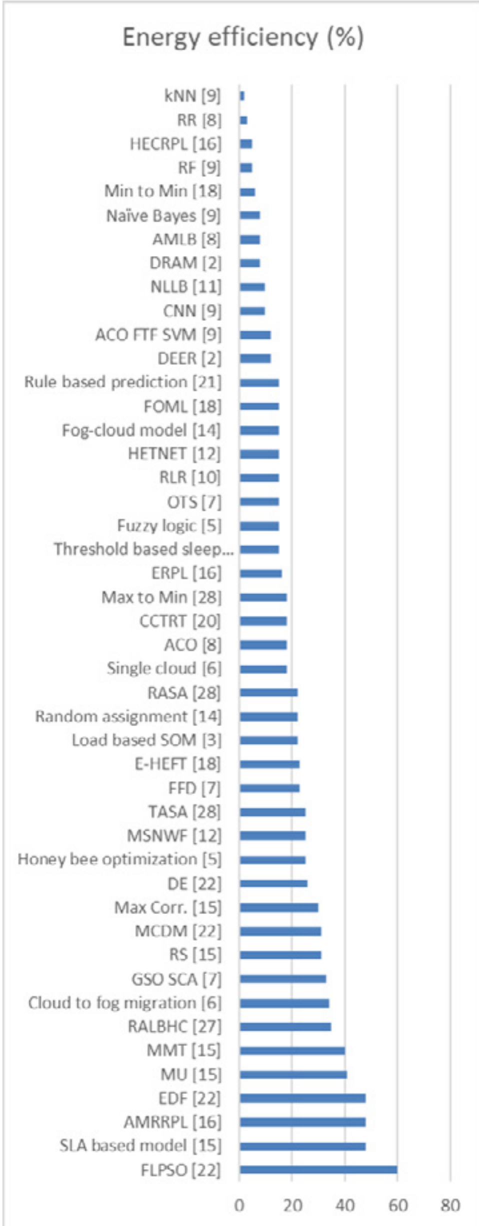

Figure8.Energy efficiency of different load balancing models.

4. CONCLUSION AND FUTURE SCOPE

The comparative quantitative analysis indicates that FLPSO (Ben Alla et al., 2019)[22], SLA based model (Mandal et al., 2020)[15], AMRRPL (Royaee et al., 2021)[16], EDF(Ben Alla et al., 2019)[22], MU (Mandal et al., 2020)[15], MMT [15], RALBHC (Ahmed et al., 2021)[27], Cloud to fog migration (Alharbi et al., 2019)[6], GSO SCA (Dong et al., 2019)[7], RS (Mandal et al., 2020)[15], MCDM (Ben Alla et al., 2019)[22], and Max Corr. (Mandal et al., 2020)[15] outperform linear models like kNN (Junaid et al., 2020)[9], HECRPL (Royaee et al., 2021)[16], CNN (Junaid et al., 2020)[9], and Rule based prediction (Jodayree et al., 2019)[21] in terms of energy efficiency.

Energy efficient models utilize soft computing techniques like PSO, ACO, GA, GSO, and Honey Bee Optimization in order to achieve this task via energy aware fitness function design. Deep learning models are not used for this purpose due to their energy intensive training process.

This limitation can be removed via using pre-trained CNN models that are optimized for energy efficient load balancing. Furthermore, existing models like SLA based model [15], AMRRPL (Royaee et al., 2021)16], EDF (Ben Alla et al., 2019)[22], MU (Mandal et al., 2020)[15], MMT [15], RALBHC (Ahmed et al., 2021)[27], etc. can be further improved via addition of soft computing for optimization

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of energy consumption. These additions will enhance system performance and help the models to be tuned with high energy efficiency.

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/18/

ROLE OF FRAME STRUCTURE IN THE DEVELOPMENT OF KRS FOR LEARNING DIALOGUES

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ABSTRACT

Dialogues are building blocks of tasks and non-tasks of communication, which happen between objects in the universe. Each dialogue is a source of linguistic knowledge within a natural language that explains and elaborates with frame structure in general. In this paper, it is noticed that various forms like (nouns, pronouns, yes-no questions and deletion) are essential part of each dialogue structure (DSS) in Chandan's work ਜੜ੍ਹਾਂ*/Roots. With the help of frames, knowledge representation system (KRS) is prepared for such dialogues in Punjabi. On the other hand, it is argued that highest numbers of nouns are total 45 in DS2 and only 1 deletion case finds in DS3. While DS1 and DS2 both have similar number of 2-2 cases of yes-no questions. The overall evaluation is successfully matched with proposed an algorithm based on frames.*

KEYWORDS

ਜੜ੍ਹਾਂ*/Roots, KRS, Frames, Nouns, Pronouns, Deletion.*

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1. INTRODUCTION

The universe consists of number of objects and each one has not a particular form but also certain characteristics that are source of information and knowledge. In this context, it is assumed that natural language is a very complex object and it has many layers and levels of knowledge representation. By introducing frames, it means that they are best ways to serve such knowledge. In general, frames report stereotyped situations and are essential part of frame system (Minsky, 1974). They are complete package of information for an object/a concept in spoken and written discourse. They look like an individual type; an abstract type and also were prototypes and exemplars (Steels, 1978).

On the other hand, when an object comes with huge numbers and classes then it is called generic frame (Brachman and Levesque, 2004). Indeed, it is an important to structure information and knowledge with the help of mental models, semantic networks, scripts, plans and frames (Crowley, 2012). Based on Minsky's frames and Fillmore's frames and Schank's scripts, a probablisitic model has already been designed to define corpus related matters and coherent has also been increased (Ferrarro and Durme, 2016). Also, it is noted down that frames oriented knowledge systems are functioning well in Google and Siri like platforms (Boroujerdi, 2018). At the textual level, it is also seen that knowledge is widely depended upon the context so that contextual study is as eqully important. When the context is explored then it is useful for terminology in the text. For this purpose, frames help to understand terminology within one word to group of words and group of words to another higher category in the text (Faber and Cabezas-Garcia, 2019). Likewise text, spoken conversation is a set of dialogues, which sometimes consist of group of four, five words and sometimes more. But today, the dialogues are going to be systemized with frames for special tasks whether it belongs to a doctor who tries to manage bad news with a patient and in this way, both they share same information (Blache and Houles, 2021).

In this paper, it is tried to analysis yes-no questions, noun-to-pronoun shift and deletion like few cases in Punjabi with frames and also present knowledge representation system (KRS). There are total six sections. First section discusses frames and KRS for Punjabi. Second section focuses on historical studies of frame structure and recent works. Third section indicates aims and objectives. Fourth section describes methodology (type of data sets and arrangement). Fifth section presents results and shows an algorithm. Last section draws conclusion and gives direction to future work. In brief, knowledge representation system (KRS) for Chandan's work ਜੜਾਂ/Roots (2006) is shown in Fig. 1.

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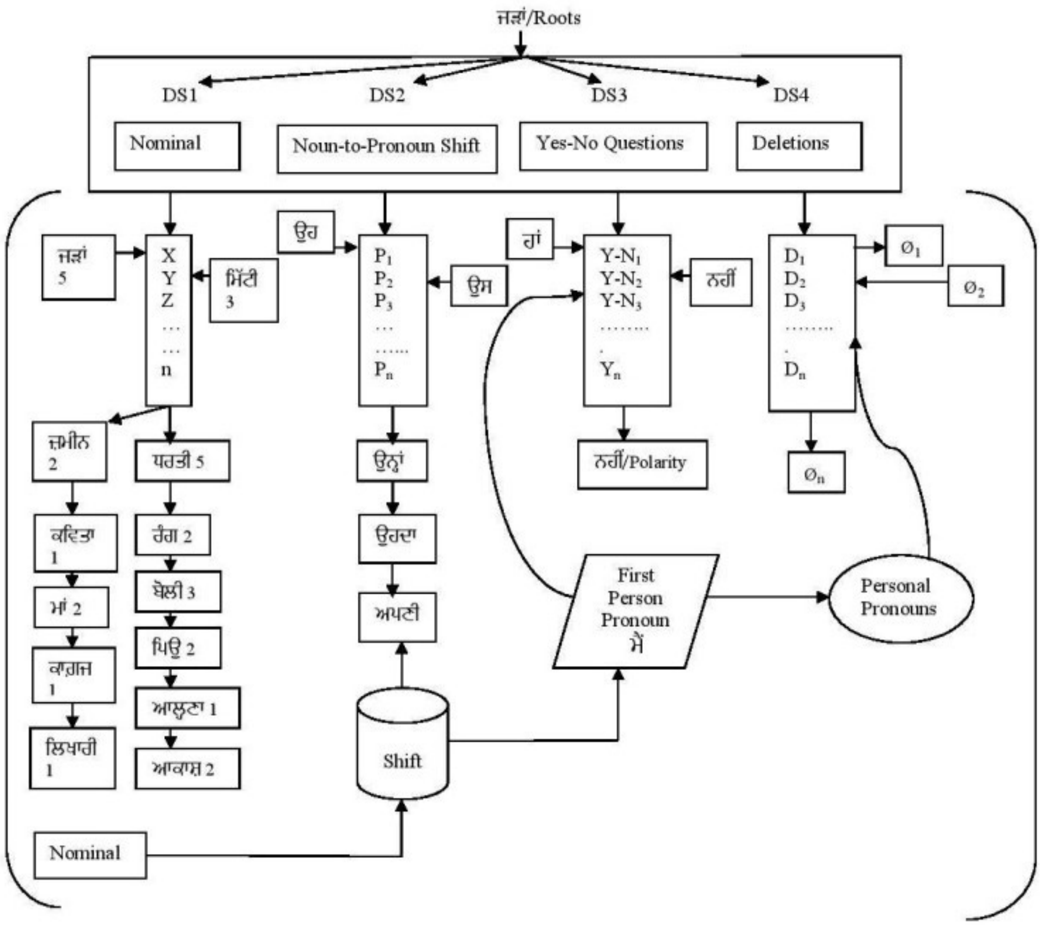

Fig. 1. KRS for dialogue structure.

Above fig. 1 shows that there are four dialogue structures where nominal contains ਜੜ੍ਹਾਂ/Roots, ਧਰਤੀ/earth, ਕਵਿਤਾ/poem, ਮਾਂ/mother, ਕਾਗਜ਼/paper, ਲਖਾਰੀ/writer, ਰੰਗ/colour etc in DS_1. Second DS_2 has ਉਹ/he, ਅਪਣਾ/his/her like third personal pronouns and reflexives respectively. Whereas, it seems that ਹਾਂ/yes and ਨਹੀਂ/no one word yes-no questions notice in DS_3. Similarly, personal pronouns such as ਉਹ/he, and ਅਪਣਾ/his/her like reflexives apppear in ommitted form in DS_3.

2. RELATED WORKS

Frames depict existed and flowing knowledge into dialogues and discourses (Thagard, 1984). Dialogues provide wonderful platform to discuss events, situations and tasks/non-tasks. Emotions within dialogues are captured by interface techniques (e.g. two tier mechanism) as suggested by (Ruttkay and Pelachaud, 2005). Spoken dialogue systems are performing well when they introduce with tasks/non-tasks (Jokinen and McTear, 2010).

On the other hand, it is said that the role of participants' impact on turns taking and maintains information flow within dialogue system (Thompson, 2013). Similarly, sets of phrases and small utterances of any dialogue can also be analyzed with frames (Khan, 2013). Frames are easily discharing knowledge through slots, values and so on. They are good source defining any particular doman in any corner of the world (Nazaruks and Osis, 2017). ASR and n-gram features are another way to track dialogue situations and they generalize dialogue contexts (Rudnicky et al., 2016). The use of ontology controls both users and robot to model dialogues (D'Haro, 2019).

Few factors like (the choice of word order, pause and frequency) also show personalities of characters' during dialogue processing (D'Haro, 2020). Based on dialogue or conversation between people or

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people with objects, it is necessary to capture frame knowledge to develop modern technology where the route direction could be simple to improve the manufacturing work (Simonova and Kapitonova, 2019). The frame knowledge in the form of frame semantics has also been adopted to see the relatedness between lexemes and to check them appropriately (Verdaguer, 2020).

It seems that frames also help to understand dialogues, particularly 'inner dialogues' where the special focus is given upon speaker's input and the mental state (Lopez-Soto, 2021). Regarding designing the set of conversation within a dialogue of any language, it is an essential to use linguistic knowledge in frames so that a dialogue can fully be represented in a systematic way (Chandrasegaran and Lioyd and Akdag Salah, 2022). Followed by linguistics, it is realized that frames with their semantic knowledge, they become effective tool to organize dialogues in English and German for the purpose of detecting disasters for the society's welfare. It has been argued that existing system "PAFIBERT" is again trained to improve the accuracy and so on (Skachkova and Kruijff-Korbayova, 2021).

Based on semantics, "framenet Brasil" has been introduced to improve contextual domain and generalizing sentences and texts computationally (Torrent et al., 2022). In this direction, "the research group of Düsseldorf" has presented the history of frames in relation to linguistics and cognitive science. It has been found that semantics and commmon sense knowledge are essential to develop linguistic frame model that covers word claases in natural language (Löbner, 2021). Frame knowledge has also been applied to discuss metaphors and it accurately mapping the metaphors in English (Stickles et al., 2014).

3. AIMS AND OBJECTIVES

To survey dialogue and knowledge representation system.

To find out nominal, noun to pronoun shift, yes-no question and deletion like cases into dialogue structures.

To analysis dialogue structures (DSs) through frames.

To present an algorithm based on KRS.

4. METHODOLOGY

It is declared that Chandan's work ਜੜ੍ਹਾਂ/Roots is selected to discuss few cases of nominal, shifting and etc. in dialogue structures (DSs). First, a single noun category is searched which is used to address, order and request against total no of nouns. Then pronouns, yes-no questions and deletion are selected one by one. In this procedure, it is suggested that each dialogue should be treated like a frame and it would be subject to an algorithm.

5. RESULTS

There are three dialogues like (ਜੜ੍ਹਾਂ/Roots, ਸਾਜ਼/Instrument, and ਮੋਰ/Peacock) have been extracted from Chandan's work ਜੜ੍ਹਾਂ/Roots. In ਜੜ੍ਹਾਂ/Roots (DS_1) total 26 nouns and 11 pronouns are found whereas noun-to-pronoun shift is commonly appeared. Only 02 cases find in yes-no questions. In ਸਾਜ਼/Instrument (DS_2), total 45 nouns and 05 pronouns are searched. Likewise ਜੜ੍ਹਾਂ/Roots (DS1), again 2 cases notice in yes-no questions. Finally, in ਮੋਰ/Peacock (DS_3), total 28 nouns and 02 pronouns (including 02 yes-no questions and 02 deletion cases) are found. Fig. 2. shows complete analysis for DS_1, DS_2 and DS_3.

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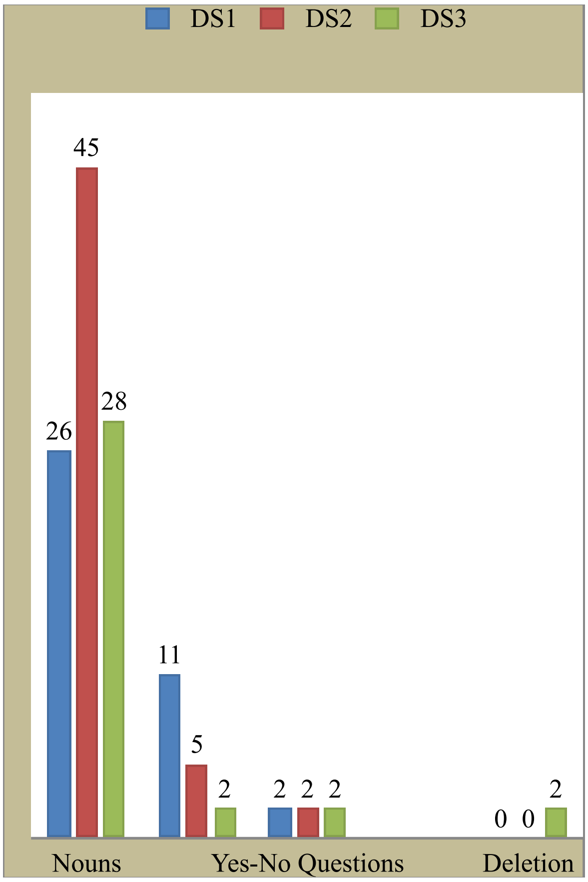

Fig. 2. Total no of four variables in dialogue structures.

According to Fig. 2 it is shown that deletion does not find in DS_1 and DS_2 except total 2 in DS_3. On the other hand, yes-no questions are total 3 find in DvS_1 and only 2-2 find in DS_2 and DS_3 respectively.

Here frame representation is significant to analysis above dialogue structures ($DS_{1,\ 2,\ 3}$) one by one.

5.1. NOUNS IN FRAME STRUCTURE

Nouns indicate towards person, place and thing in the universe. In general, common, proper and mass are kinds of nouns in natural language. Table 1 shows how frames are applied for nouns.

Table 1. Nouns in frames.

Slot	Value	Type
(Dhreja)		
Sex	Male	Human
Age	36 yrs	Biological
Home	Place	Stay
(God)		
Sex	Male	Spiritual
Age	Unimaginable	Non-Biological
Place	Everywhere	Stay
Religion	No	Belief and Trust
(Fire)		
Sex	Male	Deity
Element	Cooking	Life and Death bearer
(Well)		
Sex	Male	Non-Human
Item	Storage	Big/Small Size
Place	Somewhere	Village
(Peacock)		
Sex	Male	Non-Human
Age	20 yrs	Biological
Home	Rain forests	Stay
(Workers)		
Sex	Male/Female	Human
Age	40 yrs	Biological
Home	Place	Stay

Table 1 shows that **(Dhreja)**, **(God), (Fire), (Well), and (Workers)** are common nouns. Each example is a complete set of knowledge appearing with slot, value and type.

5.2. PRONOUNS IN FRAME STRUCTURE

Pronouns mostly stand against nouns in order to accomplish substitution tasks in natural language. Apart from personal pronouns, they are also called reflexives, reciprocals, zero and so on. However, only personal pronouns like (he), (it), (that) and (I) are found. The analysis for personal pronouns is mentioned in Table 2.

Table 2. Pronouns in frames.

Slot	Value	Type
(He)		
Sex	Male/Female	Human
Age	16 yrs	Biological
Home	Place	Stay
(It)		
Sex	Male/Female	Non-Human
Age	Not countable	Non-Biological
Use	Need based	Product (Pen, Book etc)
(That)		
Sex	Male/Female	Human/Non-Human
Age	32 yrs	Biological
Home	Place	Stay
(I)		
Sex	Male/Female	Human
Age	20 yrs	Biological
Home	Place	Stay

Table 2 shows that frames explain **(he), (it), (that) and** (I) like personal pronouns in a better way.

5.3 YES-NO QUESTIONS IN FRAME STRUCTURE

'Polar questions' and 'general questions' are selected for yes-no questions in linguistics. Each one receives one word answer (either affirmative or negative). See Table 3.

Table 3. Yes-no Questions in Frames.

Slot	Value	Type

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ਹਾਂ/Yes		
Sex	Male/Female	Human
Age	24 yrs	Biological
Query	Satisfaction	Acceptable/Rejection

Table 3 demonstrates that ਹਾਂ/Yes as affirmative answer comes under slot which is filled up by sex, age and query like variables. Value and type is another sort of information source for yes-no questions. Value which means that a person may belong to male category or not and it may be 24 yrs old. Under query, it is shown that satisfaction is kept against the slot and it falls down either accepted or rejected.

5.4 DELETION IN FRAME STRUCTURE

Deletion means to see omit items in spoken and written dialogues. For instance, ਉੜਦੇ ਮੋਰ/Flying Peacocks is deleted in the phrase 'ਉੜਦੇ ਮੋਰ' ਨੰ ੂ ਦੇਖੋ.... ਹੌਲੀ-ਹੌਲੀ ਦੇਖੋ but remember that this ____empty space is filled up with 'ਉੜਦੇ ਮੋਰ'only. See Table 4.

Table 4. Deletion in Frames.

Slot	Value	Type
ਉੜਦੇ ਮੋਰ (Flying Peacock)		
Sex	Male	Non- Human
Age	20 yrs	Biological
Home	Rain forest	Stay

Table 4 indicates that 'ਉੜਦੇ ਮੋਰ'/Flying Peacock is not appeared in the second line because it is in omitted mode.

5.5 PLANNING FOR AN ALGORITHM

It is pointed out that frames can generalize dialogue structures in a better way. Slot, value and type do simplification for mentioned each dialogue structure and they provide a complete package of information. In this regard, following algorithm is proposed for DS_1, DS_2, and DS_3.

Step 1

Check nouns in dialogues (1, 2 and 3)

Collect all possible nouns

Select each one for frames

Step 2

Check pronouns in dialogues (1, 2 and 3)

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Collect all possible pronouns

Select each one for frames

Step 3

Check yes-no questions in dialogues (1, 2 and 3)

Collect all possible polar questions

Select each one for frames

Step 4

Check deletion in dialogues (1, 2 and 3)

Collect all possible deletion/omitted cases

Select each one for frames

It is represented in **Fig. 3.**

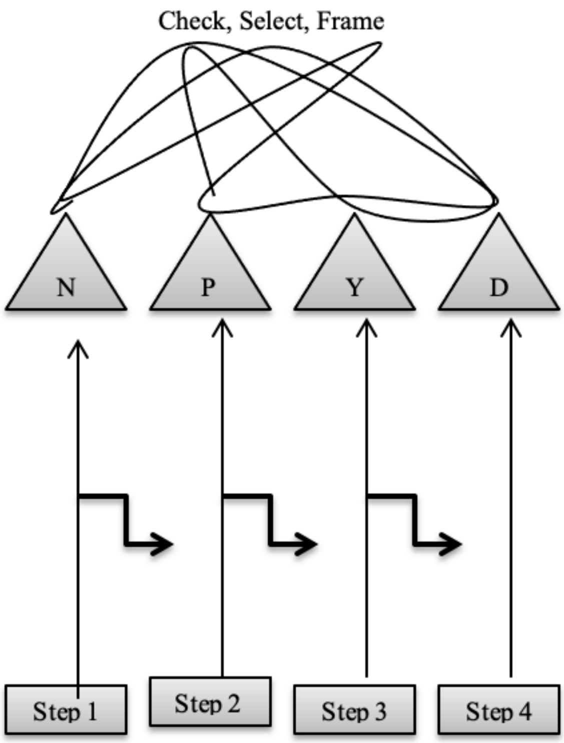

Fig 3. Steps of an Algorithm.

As per given an algorithm, it is suggested that there are total four steps where there is generally three conditions are applied. In step 1, capital N denotes nouns, capital P denotes pronoun in step 2. Similarly, it is seen that Y indicates (yes-no questions) and last D denotes deletion case in step 3 and 4 respectively. All four steps with corresponding N, P, Y, and D must follow the sequence of (check>select>frame) at the execution time.

In other words, it is simply mentioned that noun, pronoun, yes-no question and deletion kind of cases are seen in dialogue 1,2,3. All they pass through check, select and giving frame slot three criteria. It is pointed out that total number (as already given in fig 2) is successfully identified.

6. CONCLUSION AND FUTURE WORKS

Dialogues usually contain nouns, pronouns, anaphors and zero forms. Frames are used to explain them in the work of ਜੜ੍ਹਾਂ/Roots. The highest number of nouns is 45 that find in DS1 whereas a single deletion case is found in DS3. It is noticed that above mentioned algorithm fairly defines each DS and in future, it could be modified to incorporate other dialogues like 'ਚਿੱਟੇ ਹਾਸ਼ੀਏ ਵਾਲੀ ਤਸਵੀਰ', ਕਾਗ਼ਜ਼, ਲਾਲ ਝੰਡਾ, ਕੇਵਲ ਕੌਰਦੀ ਯਾਦ ਵਿਚ, ਤਿਰਕਾਲ ਸੰਧਯਾ, ਹਣ-ਖਿਣ, ਟੇਮਜ਼ ਕੰਢੇ, ਕੌਵਟ ਗਾਰਡਨ ਲੰਡਨ, ਭੋਗਾਵਸਥਾ, ਸਟੌਕਹੋਮ ਤੋਂ ਪਿਕਚਰ ਕਾਰਡ, ਸਾਂਭ ਕੇ ਰਖੀ ਚੀਜ਼ etc.

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/19/

MODELLING THE CRITICAL SUCCESS FACTORS FOR ADVANCED MANUFACTURING TECHNOLOGY IMPLEMENTATION IN SMALL AND MEDIUM SIZED ENTERPRISES

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ABSTRACT

In almost every part of the world, small and medium-sized businesses (SMEs) are seen as the backbone of economic expansion. Small and medium-sized enterprises (SMEs) typically have a simpler organisational structure than large corporations, which allows them to be more adaptable, provide instantaneous feedback, have shorter decision-making chains, and respond more quickly to customer needs. Even so, SMEs face enormous pressure to stay competitive in both domestic and international markets. Globalization, new technologies, and evolving consumer preferences are all contributing to a shift in the competitive landscape. These shifts are compelling small and medium-sized enterprises to adopt cutting-edge manufacturing techniques. The goal of this research is to identify the critical success factors (CSFs) that will help and guarantee that SMEs will be able to successfully implement AMTs (SMEs). Literature-based CSFs for AMT deployment are collected and fine-tuned using input from professionals in the field and scholars in the academy. The method of interpretive structural modelling (ISM) is applied to this CSF analysis. According to the ISM study, the three most important factors influencing the adoption of AMT are "Top management support and commitment," "entrepreneurial environment," and "financial availability." The desired goal of AMT implementation is found to be "performance improvement" and "sustainable AMT implementation." The identified CSFs and the structural relationship between them will help SMEs' top management create and prioritise business strategies that ease the implementation of AMT. The study's results point potential AMT financiers in the right direction by highlighting a handful of critical considerations that will improve the project's chances of success.

KEYWORDS

Small And Medium-Sized Enterprises (SMES), Advanced Manufacturing Technologies (AMTS), Success Factors, Interpretive Structural Modeling (ISM), MICMAC Analysis.

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1. INTRODUCTION

Today's businesses should be better prepared than ever to meet the challenges of a highly competitive market. Therefore, in today's world of ever-increasing competition, they must overcome the difficulty of discovering novel ways to boost their efficiency and effectiveness. Failure to rise to this challenge may prevent businesses from fortifying their position against rivals or expanding into new markets. Many factories are now using AMT to improve their competitiveness [1, 2]. It is possible that SMEs could benefit from AMT implementation if they had a better understanding of and ability to manage the drivers and barriers. They might be able to improve their efficiency as a result. There is substantial evidence in the literature that AMT implementation has helped organisations improve their operational and economic performance [3]. But in most cases, companies that have already implemented AMTs fail to see the expected benefits. This may be due to the fact that, in many cases, organisations overlook crucial aspects of AMT implementation that would improve its success. Critical success factors (CSFs) of AMT implementation refer to these actions. To succeed in a competitive environment, one must focus on a small number of critical success factors (CSFs). CSFs are the "things must go right" areas of a business that are essential to the achievement of the manager's goals. [4]. Thus, the CSFs approach is an effort to disentangle factors that are vital to project management's success [5]. When considered in the context of their significance at each stage of the implementation process, these CSFs take on a much richer meaning, helping to push the boundaries of process improvement. In this research, critical success factors (CSF) refer to anything considered important for the effective use of AMTs by Indian SMEs.

Small and medium-sized enterprises (SMEs) are the backbone of India's manufacturing sector (SMEs). Small and medium-sized enterprises (SMEs) in India are responsible for 43% of the country's industrial output and 40% of its exports. [7]. For India's economy to thrive and for new jobs and growth to be created over the long term, small and medium-sized enterprises (SMEs) must undergo a process of industrial modernization. These small and medium-sized enterprises (SMEs) face internal and external challenges as they adopt new technologies. [8].

Compared to large industries, which are more efficient at scale but slower to adapt to innovations, small and medium-sized businesses (SMEs) are more nimble when it comes to technology and niche markets. [9]. Since the decision to invest in AMT is so important, SMEs need to think through the entire implementation process before making a final decision.

Although the technical capabilities of AMTs are well established, a framework for effective implementation has not been agreed upon by practitioners or academics. The reason for this is that researchers have yet to identify all of the factors that either help or hurt when trying to implement AMT. Therefore, in order to hasten the spread of advanced manufacturing, it has been decided to study the factors that contribute to the success of implementing advanced manufacturing technology in small and medium-sized businesses. The connections between AMT factors and firm performance are of critical strategic importance. Potential investors who are thinking about investing in AMTs in the future can use the information gleaned from this study. Further, business leaders who take the time to comprehend these connections will have an easier time crafting efficient strategies for managing technology within the company.

2. RESEARCH METHODOLOGY

This study employed the following research methodology: I A comprehensive literature review of success factors for AMT implementation in SMEs. To learn more about how SMEs in India are using AMT, a questionnaire-based survey was conducted. To analyse the survey questionnaire data, the researchers used SPSS (20.0). There were two primary methods used to examine the data. The data was initially put to use for broad statistical purposes. Second, the proposed relationship between the business environment, competitiveness, and firm performance was tested using the standard Pearson correlation test. To model the intricate web of causality linking the most crucial factors influencing the adoption of AMT, the interpretive structural modelling (ISM) method is employed. The authors hope to determine which factors have the greatest impact on whether or not an AMT is adopted by using this method. With ISM, the chaos of such variables can be brought under control. v) Creating a framework for identifying the critical success factors for implementing AMT in SMEs. vi) The scope of each driver of the AMT implementation practise is critically examined using a Matrice d Impacts Croises - Multiplication Applique'and Classment (MICMAC) analysis. When conducting a MICMAC analysis, the significance of a variable is not determined by the strength of its direct relationships but rather by the number and types of indirect relationships it has. Understanding how different factors affect the whole system is revealed. The analysis's purpose is to categorise variables according to their driving and dependent powers.

3. ISM BASED MODELLING OF THE OF CRITICAL SUCCESS FACTORS OF AMT IMPLEMENTATION

This section included an Interpretive Structural Modeling of the important success criteria of AMT implementation in the context of Indian SMEs. The model can be employed to rank and understand the complex nature of hierarchy and explore the relationship existing among the critical success factors.

3.1. IDENTIFICATION OF CRITICAL SUCCESS FACTORS FOR AMT IMPLEMENTATION

Eighteen success factors are collected from literature survey and calibrated by industry experts and academicians are listed as follows:

Table 1: Success factors for AMT implementation.

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S. No.	CSF	Literature Relevanc
C1	Top management support and commitment	6,11,12,14,33,34,47,4
C2	Finance availability	11,12,14,33,43,47,48
C3	Entrepreneurial environment	6,35,36,37
C4	Absorptive capacity	38,39,40,41,42
C5	Organizational context	26,27,28,33,47,48
C6	Clear and long term AMT objectives	14,33,47,48
C7	Linking business and manufacturing strategy	3, 6,14,33,44,45,46,4
C8	Operations strategy	14,25,33,47,48
C9	Technology champion	12, 14,46,47,48
C10	Employee training and pilot project	12,14,45,48
C11	Sustainable AMT implementation	32,33
C12	Cross functional implementation team	6, 14,46,47,48
C13	Employee participation & empowerment	30,31,47,48
C14	High level system integration	3,12,14,33,47
C15	Technology know how	3,45,46,47,48
C16	Vendor development	2,6,14,11,29,33, 45,47,48
C17	Performance improvement	14, 33,47,48
C18	Customer involvement	11,12, 14,46,48

3.2. (SSIM) FOR CRITICAL SUCCESS FACTORS

Through the use of expert consultation and the identification of contextual relationships between the success factors included in the system, a structural self-interaction matrix for the critical success factors is developed. A 'leads to' type contextual relationship is selected to examine the interplay of the success factors. For instance, a "favourable company image" can result from "better quality." The interrelationships of the variables in their context are constructed in a similar fashion. The existence of a relation between any two variables I and j) and the direction of the relation are questioned, while taking into account the contextual relationship for each variable. There are four signs used to indicate the directional relationship between the I and j variables:

P: The stress caused by I will be reduced by j.

The two variables, I and j, will mutually alleviate one another, as shown in (A) and (X).

O: There is no connection between I and j.

The SSIM is built around the contextual relationships of the 18 variables found to be most important for the AMT implementation practises of Indian SMEs. The SSIM of critical success factors is shown in table 2.

Table 2: Structural self-interaction matrix of critical success factors.

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CSF	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
C1	X	V	X	V	V	V	V	V	V	V	V	V	V	V	V	V	V	V
C2	A	X	A	V	V	V	V	V	V	V	V	V	V	V	V	V	V	V
C3	X	V	X	V	V	V	V	V	V	V	V	V	V	V	V	V	V	V
C4	A	A	A	X	A	A	A	A	A	A	V	A	A	O	A	O	V	O
C5	A	A	A	V	X	A	A	A	V	V	V	V	V	V	A	O	V	O
C6	A	A	A	V	V	X	V	V	V	V	V	V	V	V	X	V	V	V
C7	A	A	A	V	V	A	X	X	V	V	V	V	V	V	A	V	V	V
C8	A	A	A	V	V	A	X	X	V	V	V	V	V	V	A	V	V	V
C9	A	A	A	V	A	A	A	A	X	V	V	X	V	V	A	O	V	O
C10	A	A	A	V	A	A	A	A	A	X	V	A	X	V	A	O	V	O
C11	A	A	A	A	A	A	A	A	A	A	X	A	A	A	A	A	V	A
C12	A	A	A	V	A	A	A	A	X	V	V	X	V	V	A	O	V	O
C13	A	A	A	V	A	A	A	A	A	X	V	A	X	V	A	O	V	O
C14	A	A	A	O	A	A	A	A	A	A	V	A	A	X	A	O	V	O
C15	A	A	A	V	V	X	V	V	V	V	V	V	V	V	X	V	V	V
C16	A	A	A	O	O	A	A	A	O	O	V	O	O	O	A	X	V	V
C17	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	X	A
C18	A	A	A	O	O	A	A	A	O	O	V	O	O	O	A	A	V	X

3.3. DEVELOPMENT OF REACHABILITY MATRIX FOR CRITICAL SUCCESS FACTORS

If you want to create a reachability matrix using SSIM, you'll need to have a firm grasp on transitivity and reachability. These are the two main tenets of the ISM then 'k' is also related to I this is what is meant by 'transitivity. The transitive property helps maintain internal coherence in one's ideas. The ISM methodology relies on the reachability concept. Element pairs with different identifications are compared with one another in terms of their interconnection. This data is represented as a binary matrix. If the ith factor aids in achieving the jth factor, then the cell I j) of the reachability matrix is assigned a 1, otherwise it is assigned a 0. (i, j). Moreover, some of the cells in the reachability matrix can be filled inductively thanks to the transitivity property [24]. Matrix entries I j) = 1 and (j, k) = 1 imply I k) = 1 because of the identity between the two variables. By exchanging 1s and 0s for Vs, As, Xs, and Os in the SSIM, we obtain a binary matrix we refer to as the initial reachability matrix. Rules for exchanging ones and zeros are as follows:if the (i, j) entry in the SSIM is V, then the (i, j) entry in the reachability matrix becomes 1 and the (j, i) entry becomes 0.

- if the (i, j) entry in the SSIM is A, then the (i, j) entry in the reachability matrix becomes 0 and the (j, i) entry becomes 1.
- if the (i, j) entry in the SSIM is X, then the (i, j) entry in the reachability matrix becomes 1 and the (j, i) entry becomes 1.
- if the (i, j) entry in the SSIM is O, then the (i, j) entry in the reachability matrix becomes 0 and the (j, i) entry also becomes 0.

Following these guidelines, the AMT drivers' initial reachability matrix is determined, and the final reachability matrix is obtained by incorporating the transitivities, this is shown in table 3. In this table, the driving power and dependence of each variable are also shown. The driving power of a particular variable is the total number of variables (including itself), which it may help to achieve while the dependence is the total number of variables, which may help to achieve it.

Table 3: Reachability matrix for critical success factors.

CSF	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	DP
C1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	18
C2	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	16
C3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	18
C4	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	3
C5	0	0	0	1	1	0	0	0	1	1	1	1	1	1	0	0	1	0	9
C6	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	15
C7	0	0	0	1	1	0	1	1	1	1	1	1	1	1	0	1	1	1	13
C8	0	0	0	1	1	0	1	1	1	1	1	1	1	1	0	1	1	1	13
C9	0	0	0	1	0	0	0	0	1	1	1	1	1	1	0	0	1	0	8
C10	0	0	0	1	0	0	0	0	0	1	1	0	1	1	0	0	1	0	6
C11	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	2
C12	0	0	0	1	0	0	0	0	1	1	1	1	1	1	0	0	1	0	8
C13	0	0	0	1	0	0	0	0	0	1	1	0	1	1	0	0	1	0	6
C14	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	3
C15	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	15
C16	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	1	4
C17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
C18	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	3
DPD	2	3	2	13	8	5	7	7	10	12	17	10	12	13	5	8	18	9	**163**

3.4. LEVEL PARTITIONS OF THE REACHABILTY MATRIX OF CRITICAL SUCCESS FACTORS

Using the final reachability matrix, we can establish the reachability and antecedent set of each factor. The antecedent set includes the element and any other elements that could be useful in achieving the goal, while the reachability set includes the element and any other elements that could be useful in reaching the goal. Next, we calculate the intersection of these sets across all variables. The root of the ISM is the element for which the reachability and intersection sets are identical. Nothing below the top-level element in the hierarchy could be achieved with the help of the top-level element. Separation from the other elements occurs after the top-level element has been identified. The same method is then used to uncover the following tier of elements. Incorporating these discovered levels into the diagraph and ultimate model is beneficial. Each critical success factor's position in the ISM-based hierarchical model was determined by first partitioning the reachability matrix into different levels. A total of 10 cycles were used to determine where each success factor stood in the system. Table 4 displays the first iteration's results, which show that the performance enhancement factor C17 is the most important variable in the underlying ISM model. In table 4, the results of iterations II through X are displayed, revealing the remaining success factors and their relative levels of rest. The ISM digraph and final model were constructed using the identified variable levels.

Table 4: Results of iteration I of the level partitions of the reachability matrix of critical success factors.

CSF	Reachability set	Antecedent set	Intersection	Level
C1	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,3	1,3	
C2	2,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,2,3	2	
C3	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,3	1,3	
C4	4,11,17	1,2,3,4,5,6,7,8,9,10,12,13,15	4	
C5	4,5,9,10,11,12,13,14,17	1,2,3,5,6,7,8,15	5	
C6	4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,2,3,6,15	6,15	

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CSF	Reachability set	Antecedent set	Intersection	Level
C7	4,5,7,8,9,10,11,12,13,14,16,17,18	1,2,3,6,7,8,15	7,8	
C8	4,5,7,8,9,10,11,12,13,14,16,17,18	1,2,3,6,7,8,15	7,8	
C9	4,9,10,11,12,13,14,17	1,2,3,5,6,7,8,9,12,15	9,12	
C10	4,10,11,13,14,17	1,2,3,5,6,7,8,9,10,12,13,15	10,13	
C11	11,17	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,18	11	
C12	4,9,10,11,12,13,14,17	1,2,3,5,6,7,8,9,12,15	9,12	
C13	4,10,11,13,14,17	1,2,3,5,6,7,8,9,10,12,13,15	10,13	
C14	11,14,17	1,2,3,5,6,7,8,9,10,12,13,14,15	14	
C15	4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,2,3,6,15	6,15	
C16	16,17,18,11	1,2,3,6,7,8,15,16	16	
C17	17	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	17	I
C18	17,18,11	1,2,3,6,7,8,15,16,18	18	

Table 5: Results of iteration II-X of the level partitions of the reachability matrix of critical success factors.

CSF	Reachability set	Antecedent set	Intersection	Level
C1	1,3	1,3	1,3	X
C2	2	1,2,3	2	IX
C3	1,3	1,3	1,3	X
C4	4	1,2,3,4,5,6,7,8,9,10,12,13,15	4	III
C5	5	1,2,3,5,6,7,8,15	5	VI
C6	6,15	1,2,3,6,15	6,15	VIII
C7	7,8	1,2,3,6,7,8,15	7,8	VII
C8	7,8	1,2,3,6,7,8,15	7,8	VII
C9	9,12	1,2,3,5,6,7,8,9,12,15	9,12	V
C10	10,13	1,2,3,5,6,7,8,9,10,12,13,15	10,13	IV
C11	11	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15	11	II
C12	9,12	1,2,3,5,6,7,8,9,12,15	9,12	V
C13	10,13	1,2,3,5,6,7,8,9,10,12,13,15	10,13	IV
C14	14	1,2,3,5,6,7,8,9,10,12,13,14,15	14	III
C15	6,15	1,2,3,6,15	6,15	VIII
C16	16	1,2,3,6,7,8,15,16	16	IV
C17	17	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	17	I
C18	18	1,2,3,6,7,8,15,16,18	18	III

3.5 FORMATION OF HIERARCHICAL MODEL

The structural model of the critical success factors is constructed using the level partition shown in table 4 and table 5, and the final digraph is developed by removing the transitivity as discussed in the ISM methodology. The digraph is finally transformed into the ISM as shown in Figure 2. 'Top management support and commitment' (C1), 'entrepreneurial environment' (C3) lead to 'finance availability' (C2). 'Finance availability' leads to 'clear and long term AMT objectives' (variable 6), 'technical know- how' (C15), which in turn leads to 'operations strategy' (C8), 'linking business and

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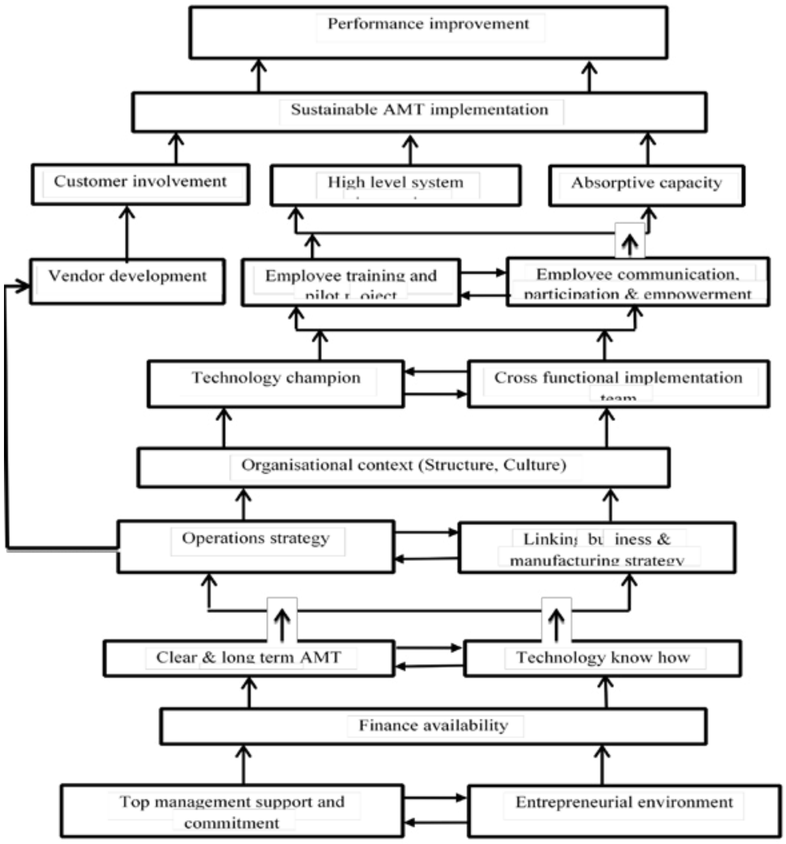
manufacturing strategy' (C7). Put together these variables leads to 'sustainable AMT implementation' (variable 11) which ultimately leads to 'performance improvement' (variable C17).



3.5 MICMAC ANALYSIS OF THE CRITICAL SUCCESS FACTORS

The critical success factors are categorized into four groups 'autonomous variables' (Cluster-I) 'dependent variable' (cluster-II), linkage variables (cluster-III) and independent variable (cluster-IV) through MICMAC analysis. The analysis requires construction of a driving power-dependence diagram (figure 3). Horizontal axis of this diagram represents the dependence potential while the vertical axis represents the driving potential of the critical success factors. Allocation of the critical success factors into different clusters of the driving power-dependence diagram is done based upon their driving and dependence potential values represented in table 3. For example, it is found from table 3, that 'top management support and commitment' (variable 1), entrepreneurial environment (variable 3) have a driving power of 18 and a dependence of 2; therefore, the factors are placed at a position corresponding to driver power of 18 and dependency of 2, in driving-dependence power diagram (figure 3). Similarly, factor 2 (finance availability), has driving power of 16 and dependence of 3 therefore, in figure 3, the factor is positioned at a place corresponding to driver power of 16 and dependency of 3. The factors 'sustainable AMT development', 'performance improvement', 'technology champion', 'cross functional implementation team' 'employee training', 'Employee communication, participation and empowerment' 'technology know-how, 'customer involvement', 'absorptive capacity' and 'high level system integration' are positioned in cluster-II, which indicates that these have strong dependence and weak driving power. Similarly, the variables 'top management support and commitment', 'entrepreneurial environment', 'finance availability' are placed in fourth cluster, which is an indication of their strong driving potential and weak dependence.

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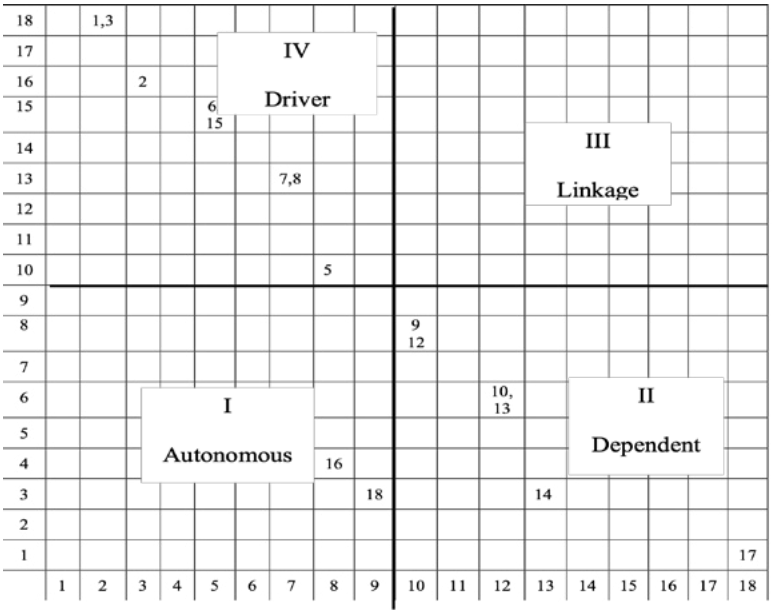

Figure 3: Driving power dependence diagram of the critical success factors.

4. DISCUSSION AND MANAGERIAL IMPLICATIONS

Implementation of AMT is a complex and difficult phenomena. Complexity of AMTs implementation is due to its dependency on several criteria. For effective AMT implementation all the relevant criteria have to be identified and the existing interrelationship between them has to be understood. This research has made an attempt to identify various success criteria for AMT implementation in Indian SMEs and used ISM approach to evaluate the critical success factors. The following managerial implications emerge from this study.

- The study explored a validated measure of 18 success factors of AMT implementation specific to Indian SMEs. Prior knowledge of these factors can be useful for the SMEs to consider a wide range of factors instead of focusing on few factors for successful AMT implementation.
- Interrelationships among the critical success factors were identified using a logical structure developed through ISM that can help managers to better prioritize their available resources while trying to bring desired changes in strategic adjustments that are necessary for improvements in AMT implementation practices.
- The driver power-dependence diagram (figure 3) indicates that 'customer involvement' and 'vendor development' are autonomous factors in this study. Autonomous variables generally appear as weak driver as well as weak dependent and are relatively disconnected from the system. These variables do not have much influence on the other variables of the system.
- Figure 3 (driver power-dependence diagram) shows factors placed in cluster II. The factors identified as dependent variables and have weak driving potential but strong dependence power.
- There is no factor positioned in the third cluster. The absence of linkage variables indicates that no identified critical success factor is unstable in nature.

Further, it can be observed from figure 3 that the variables that are positioned in fourth cluster are having strong driving power and weak dependence. These variables demand treatment of these factors as key drivers for an effective AMT implementation. Owners/managers of SMEs and practitioners should give priority while addressing these factors to achieve AMT implementation success.

The key findings of the present research are:

'Top management support and commitment', 'entrepreneurial environment' and 'finance availability'

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are positioned at the bottom of ISM based hierarchy are the critical success factors of the AMT implementation process. The factor 'performance improvement' occupies the highest hierarchical level and 'sustainable AMT implementation' is placed below it in the hierarchy. These factors represent the desired objective of successful AMT implementation. For obtaining these objectives the bottom level variables should be improved continuously.

5. CONCLUSION

AMT implementation has been viewed by SMEs as a significant step forward in their quest to stay competitive. However, it has been said that SMEs' inadequate resources and skills prevent them from using AMTs effectively, which is why the acceptance rate of AMT in Indian SMEs is uninspiring. It has been observed that most of the companies hesitate to adopt full integration whereas others adopt it partially. This could be due to the fact that the organizations that have adopted AMTs have shown mixed results. In this regard the knowledge of the potential critical success factors and their relative importance on effectiveness of AMT implementation practices, explored in this study could be beneficial for the organizations trying to implement AMTs in their plants. This research work has identified and prioritized the critical success factors that have to be considered for successful implementation of AMTs in Indian SMEs.

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OPTIMAL RESERVOIR OPERATION POLICY DETERMINATION FOR UNCERTAINTY CONDITIONS

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ABSTRACT

In recent years, optimising reservoir operations has emerged as a hot topic in the field of water resources management. Heuristic approaches to reservoir operation that incorporate rule curves and, to some extent, operator discretion have been the norm in the past. With so many stakeholders involved in water management, it can be difficult to strike a happy medium between everyone's needs and wants. The dissertation proposes a method for transforming traditional reservoir operation into optimal strategies, allowing users to take advantage of the rapid development of computational techniques.

This research creates and applies a Multi Objective Fuzzy Linear Programming (MOFLP) model to the monthly operating policies of the stage-I Jayakwadi reservoir located on the Godavari, the largest river in the Indian state of Maharashtra. In order to formulate the problem, we use two objective functions—maximizing irrigation releases and maximising power production releases. The constraints of the study are considered, including turbine release, irrigation demand, reservoir storage capacity, and continuity of reservoir storage. Utilizing linear membership functions, the objective functions are fuzzily defined. All other model parameters except for the goals are assumed to be hard and fast rules. Maximum Happiness Operating Policy (MOFLP) was used to determine the best course of action.

KEYWORDS

Jayakwadi Reservoir, Fuzzy Set Theory, Linear Programming, Irrigation.

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1. INTRODUCTION

The system engineering method for the water resources system employs a schematic analysis of the numerous alternatives available to policy and decision makers. Each option presents a complex problem with entangled effects, necessitating not only the consideration of a much larger number of alternatives but also the analysis of each option in light of its impacts at a number of locations. The system engineering method offers a flexible platform for constant assessment and re-planning in the face of unforeseen circumstances. When applied with an understanding of its limitations, this strategy has the potential to greatly enhance the management of water resource systems. The advent of digital computers has allowed for the efficient examination of problems for mathematical solutions and the management of vast amounts of data. Linear programming, dynamic programming, goal programming, integer programming, simulation techniques, etc. are all examples of system engineering approaches that find use in the water resources industry. However, there is no one solution that works for every conceivable problem. The technique selected is determined by factors such as the system's characteristics, the availability of data, the research's objectives, and the research's constraints.

There are many potential goals and objectives for the water resources system, so the planner must pick the best one. The nature of the system necessitates the use of deductive reasoning processes that can eliminate irrelevant options and reduce thousands of metrics to a more manageable set. The same is true for water resource development projects, in terms of both planning and management. Regulations or principles are applied to reservoir management based on the quantity and timing of water that must be stored and released. The following are some of the most widely accepted reservoir operation principles for flood control and conservational uses in the context of single purpose, multipurpose, and system reservoirs. These suggestions are meant to serve as broad, overarching guidelines. For the proper operation of a reservoir or network of reservoirs, unique regulation schedules must be developed after all relevant factors have been considered.

This research uses Multi Objective Fuzzy Linear Programming to create an optimal reservoir operation model for the stage-I Jayakwadi reservoir on the Godavari River in Maharashtra (MOFLP). This problem is framed with two goals in mind—irrigation release and hydropower generation—along with a number of constraints, and is then solved in an iterative fashion. Using linear membership functions, the objective function is fuzzy-valued. With the exception of the goals, it is assumed that all other model parameters are discrete. MOFLP is used to find a happy medium by maximising both the fuzzified objectives and the level of satisfactions. Potential outcomes for varying degrees of decision-maker satisfaction with objective measures are generated using the MOFLP model. Also, the optimal policies were determined for various incoming conditions using MOFLP.

The study's overarching objective was to demonstrate how system analysis methods can be used to optimise water resources management in service of measurable goals. Given the shift in policy and the growth in the agricultural, industrial, and domestic sectors, any water resource system, whether currently in place or soon to be implemented, should be able to meet the demand. Traditional methods dominate the system for managing food resources. However, system analysis and mathematical optimization techniques have been found to be helpful. Educating the public about the benefits of innovative approaches to water resource problems is, therefore, crucial.

The following are some of the goals of this research.

- Development of a MOFLP model featuring both loose and hard constraints and a fuzzy objective function.
- The optimization model is used to analyse the efficiency of the Jayakwadi reservoir at its initial stage.

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- Decision-makers are given a plethora of options thanks to lingo's (Language for Interactive General Optimization) use in the development of optimal operating policies.

The functioning of dam reservoirs is an important factor in water management research and planning. The research compared the effectiveness of three policies for improving reservoir performance: the Standard Operation Policy (SOP), the Hedging Rule (HR), and the Multi-Objective Optimization (MOO). The point of MOO was to boost dependability metrics while simultaneously reducing exposure. Coordination of the equilibrium between the interests of stakeholders in conventional ecological operations is difficult. It was proposed that multiple parties work together to manage a reservoir. The results show that the value of coordinated operation decreased by 0.184, 0.469, and 0.886 in a normal year, a dry year, and an exceptionally dry year, respectively. Soil and Water Assessment Tool (SWAT) and HEC-ResPRM were used to model and optimise the Nashe hydropower reservoir operation in the Blue Nile River Basin. Stream flow into the reservoir was determined using the SWAT model, which accounted for both short- and long-term effects of LULC changes [1-4].

A nested method is presented for the generation of reservoir scheduling models. Scheduled operations at the Three Gorges-Gezhouba (TG-GZB) cascade reservoirs serve as the basis for this system. A five-level framework for efficient scheduling has been developed using this method. It is unrealistic to expect DRSs to be redesigned to account for every conceivable negative scenario. An 11-step process is provided for dynamically modelling the available features of a DRS. The proposed framework was found to be useful for locating major influences on system performance [5,6].

The reservoir system can be optimised with the help of LP. A LP model was implemented by Palmer and Holmes [7] in the Seattle Water Department's expert system for managing drought. During a drought, Randall et al. [8] analysed how a multi-state water resource system functioned. Although most reservoir systems are non-linear, LP demands that they be made linear. This includes the constraints and the objective function. For the short-term, annual operation of an irrigation reservoir, Chaves and Kojiri[9] developed a deterministic LP model. Jangareddy and Nagesh Kumar [10] developed a chance constrained LP model to account for unpredictable cash inflows. Approximating solutions is possible via successive LP (SLP), just as approximating non-linear functions is possible via linear functions. Examples of SLP's application to multi-reservoir optimization problems are provided by Chang et al. [11]. In [12], Akter and Simonovic used LP to develop a system-wide operational and strategic plan for Adelaide's head works. Using LP, Shi et al.[13] detail a process for optimising power generation from the Highland Lakes on the Lower Colorado River in Texas over the course of a day. Consequently, LP can only be used for solving problems involving linear functions. In some cases, the optimization result may be worth less if simplified.

Short-term hydropower generation optimization research by Leta et al. [4] and Ghanbari et al. [14] demonstrated that the problem could be solved by rewriting it with only linear constraints on outflow release and storage content. An additional approach to the reservoir operation problem is the so-called Dynamic Programming method. Biswas et al. [15] developed a model of irrigation for the management of temporary reservoirs. The model consists of a crop water allocation model and an operating policy model developed with deterministic dynamic programming. Arunkumaret al. [16] also developed a DP model to solve the problem of water delivery from two reservoirs to an irrigation district at once. Predicted information is updated in the model, including evapotranspiration from crops, evaporation from reservoirs, and inflows. Nasseri et al. [17] were the first to introduce fuzzy linear programming as a variant of traditional LP. After looking at LP problems with fuzzy objectives and constraints and presenting an FLP-like LP problem, we see that the min operator is a useful aggregator for these functions. Ren et al[18] .'s proposal to use parametric programming to solve FLP has proven to be the most well-liked approach. Using their method, the optimal answer to the problem can be determined for a wide range of parameter values. RossT. J. [19] provided an illustration of how to use linear membership functions to solve fuzzy linear problems. In this research, we focused on the specific scenario of a fuzzy member with a linear membership function. They investigated problems where the right-side and technological coefficient are the only two uncertain variables. In their

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presentation of a fully fuzzy linear programming approach for multifunctional reservoir operating rules, Regulwar, Gaurav, and Kamodkar[20–23] outlined a number of advantages. This study investigates and applies the completely fuzzy linear system, which is a fuzzy linear system with fuzzy coefficients and fuzzy variables, to the reservoir operating problem, in order to determine the optimal release strategy for the Jayakwadi reservoir, which is located in the state of Maharashtra in India. And then they presented a paper on how to derive multipurpose single reservoir release policies using fuzzy constraints. Despite significant progress, reservoir operation research has been incredibly slow to translate into actual practise, as pointed out by Chaudhari and Anand [24]. Simonovic discussed the issues with reservoir operation models and the solutions to make them more appealing to operators.

Intuitionistic fuzzy set theory is a variant of fuzzy set theory that incorporates rejection and acceptance probabilities in such a way that their sum is less than one [25]. Solution proposals for intuitionistic fuzzy optimization [26] typically involve re-framing the optimization problem in light of the degree of rejection of restrictions and values of the impractical objectives. To rank agricultural best management practises, a case study of South Texas is used to illustrate the utility of a multi-criteria decision-making model based on Attanassous Intuitionistic Fuzzy Sets (A-IFS) methodology [26]. The intuitionistic fuzzy optimization method proposed in [20] is widely recognised as a powerful optimization tool by researchers around the world. This strategy aims to maximise acceptance while minimising rejection; the current strategy [28] additionally minimises hesitation when accepting new information. In most cases, the optimal irrigation planning model cannot be found by solving the crisp multiobjective problem because the objective functions, restrictions, and variables are highly uncertain, imprecise, and ambiguous in nature and depend on a large number of uncontrolled parameters. Since non-membership in the fuzzy set is a complement of membership in the set, the maximum of the membership function will always minimise non-membership. Since the degree of acceptance and rejection are defined simultaneously and are not additive, intuitionistic fuzzy sets tend to yield superior results [29]. A computational method for solving a multiobjective linear programming problem using an intuitive fuzzy optimization model is presented. To investigate how the model makes use of belonging/not-belonging status, a comparison of the effects of linear and nonlinear membership functions is provided [30]. A fuzzy multi-objective intuitionistic nonlinear programming model is developed for irrigation planning in both dry and wet conditions. The model's ability to accommodate uncertainty and resistance provides guidance to decision-makers in alleviating water scarcity [31]. Intuitive fuzzy multi-objective linear programming problem is provided using triangular fuzzy numbers and mixed constraints. Several linear and nonlinear membership functions are used to transform the original problem into a crisp linear/nonlinear programming problem, which can then be solved using the appropriate crisp programming approach [32]. Intuitionistic fuzzy optimization, an extended form of fuzzy optimization, considers user satisfaction, model rejection, and uncertainty as performance metrics [33]. Expert system, belief system, and information fusion model applications should consider both the truth membership supported by the evidence and the falsity membership opposed to the evidence [34], even though this is outside the scope of the fuzzy set and interval valued fuzzy set. However, intuitionistic fuzzy sets, a generalisation of fuzzy sets, account for both true and false membership. However, intuitionistic fuzzy sets are the only ones capable of dealing with incomplete information; contradictory or ambiguous data cannot be processed. Neutrosophic sets explicitly quantify truth membership, indeterminacy membership, and falsity membership, and these three types of membership are completely separate from one another [35-38]. Many single valued neutrosophic set (SVNS) operations have been established, and investigations into their basic properties continue [39–42]. A new multiobjective optimization framework is proposed for use in a neutrosophic context. The proposed approach [43–45] can be used to simultaneously deal with indeterminacy and falsehood.

2. FUZZY SET THEORY

First introduced by Regulwar and Kamodkar, fuzzy sets permit a looser membership criterion. For data that does not neatly fit into predetermined categories, fuzzy set theory provides a solution (i.e., fuzzy). Any method or theory that relies on "crisp" definitions, such as classical set theory,

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mathematics, and programming, can be "fuzzified" by replacing them with those of a fuzzy set with more nebulous boundaries. The extension of crisp theory and analysis to fuzzy techniques is powerful in solving real-world problems, which invariably involve some degree of imprecision and noise in the variables and parameters measured and processed for the application. Fuzzy logic uses language variables such as "high," "middle," and "low" to represent a range of numbers. Since fuzzy logic allows for overlap, these categories can be mixed. For instance, a flow of 10 units could be partially or fully classified as either "baseflow" or "interflow." Fuzzy set theory encompasses a wide range of related disciplines, including but not limited to fuzzy logic, fuzzy arithmetic, fuzzy mathematical programming, fuzzy topology, fuzzy graph theory, and fuzzy data analysis. There are two names for collections of clearly defined pieces: classical and crisp. In any given situation, there exists a set called the universal set that always and forever includes all of the elements of all other sets under consideration. The characteristic function of the set A is a formal way to say whether or not an element of A is in the set.

$$X_A(x) = \begin{cases} 1 & \text{If } f \quad x \in A \\ 0 & \text{If } f \quad x \notin A \end{cases}$$

Similar to this, a fuzzy set A of a set X can be described as a set of ordered pairs, each containing a first element from X and a second element from the interval [0, 1], with exactly one ordered pair present for each of X.

$\mu_A(x): X \to [0, 1]$

2.1 FUZZY RESERVOIR OPERATION MODEL

The specific stages that are taken for modelling reservoir operation with fuzzy logic are as follows:

Sharp inputs like inflow, reservoir storage, and release are converted into fuzzy variables during step a) Input Fuzzification; step b) Fuzzy Operator Creation Based on Expert Knowledge Base; step c) Fuzzy Operator Application to Create Single Number Representing Each Rule's Premise; step d) Rule Implications Definition; and step e) Defuzzication.

The first step in creating a fuzzy reservoir operation model is determining the degree of membership functions. Fuzzification yields a fuzzy degree of membership, wherein the inputs are members of all relevant fuzzy sets via the member, and the output is typically between 0 and 1. Regardless of the variable being used, the input is always a precise numerical value. The fuzzy rule set is formulated using the accumulated wisdom of professionals. If the storage is low and the inflow is moderate in period t, then the release is moderate. The rule basis should always be developed using the existing expert knowledge on the specific reservoir. Once the inputs have been fuzzified, it is possible to determine the extent to which each premise for each rule has been met. if the rationale behind a particular rule is recognised. When a premise of a given rule consists of more than one part, a fuzzy operator can be used to reduce the number of possible outcomes down to a single one. However many membership functions are fed into the fuzzy operator, the output is always a single trust value. Operators in fuzzy logic, like AND and OR, abide by the rules of traditional two-valued logic. Depending on the context, the AND operator can be interpreted as either the conjunction (min) of classical logic or the product (prod) of its two parameters. The probabilistic OR (prob or) approach is an alternate form of the OR method that is analogous to the disjunction operation in classical logic. The outcome of implication takes the shape of a fuzzy set. This is defuzzified for application. A fuzzy set is used as the input for the defuzzification process, and the output is one distinct integer. The "centroid" evaluation, which yields the centre of the area under the curve, is a typical defuzzification technique. The "bisection" defuzzification method is another option; it provides the bisection of the output fuzzy set's base.

Algorithm for MOFLP

The following algorithm (for maximisation problem) can be used to solve the MOFLP model.

Step 1:

Find the best (Z_1^+ and Z_2^+) values and worst (Z_1^- and Z_2^-) values corresponding to the set (decision variables) of solutions (X*) for each objective (Z_1 and Z_2) when you solve the model as a Linear Programming (LP) problem.

Step 2:

Define a linear membership function $\mu_k(x)$ for each objective as

$$\mu_{x1}(X)=\begin{cases}0 & Z_1 \le Z_1^- \\ (Z_1 - Z_1^-)/(Z_1^+ - Z_1^-) & Z_1^- \le Z_1 \le Z_1^+ \\ 1 & Z_1 \ge Z_1^+\end{cases}$$

$$\mu_{x2}(X)=\begin{cases}0 & Z_2 \le Z_2^- \\ (Z_2 - Z_2^-)/(Z_2^+ - Z_2^-) & Z_2^- \le Z_2 \le Z_2^+ \\ 1 & Z_2 \ge Z_2^+\end{cases}$$

Step 3:

An equivalent LP problem (crisp model) is then defined as

Maximize λ

Subject to

$$\lambda \le \frac{(Z_1 - Z_1^-)}{(Z_1^+ - Z_1^-)}$$ And

$$\lambda \le \frac{(Z_2 - Z_2^-)}{(Z_2^+ - Z_2^-)}$$

And all the original constraint sets and non negativity constraints for X and λ.

Step 4:

Solve the LP problem formulated in step 3. The solution is λ (i.e., maximum degree of overall satisfaction) which is achieved for the solution X*. The corresponding values of the objective functions are Z_1^* and Z_2^* obtained and this is the best compromise solution.

2.2 CASE STUDY

The Godavari River runs the length of the Deccan Plateau, from the Western Ghats to the Eastern Ghats. It all starts 80.46 km from the Arabian Sea, in the Nasik district of Maharashtra. Godavari, which rises to a height of 1066.81 m and flows south and east across Maharashtra and Andhra Pradesh, finally empties into the Bay of Bengal 96.56 km below Rajamuhendry. The Jayakwadi dam is located on the Godavari River in the Aurangabad district of the Indian state of Maharashtra. The catchment area of the reservoir is 21,750 km2 in size. There are currently 2171 Mm3 of usable storage and a total of 2909 Mm3 available. There is a total installed capacity of 12 MW for generating electricity (pumped storage plant). An area of 1,41,640 acres under command is irrigated. Kharif receives 22% of the total, Rabi 45%, two seasons 28%, hot weather 3%, and perennial crops 4.5% of the total. The entire power generation system has a capacity of 12 MW (pumped storage plant). The total irrigable area within the command zone is 1,41,640 acres. Kharif receives 22% of the total irrigation, Rabi 45%, two seasons 28%, hot weather 3%, and perennial crops 4.5%. Stage-1 of the Jayakwadi Project Report proposes the construction of a dam over the Godavari River in the Paithan

Tehsil of the Aurangabad district of Maharashtra, with a live storage capacity of 2170 cumec. The longest dimension of the dam is 9997 metres, and its greatest height is 37.73 metres (without the overflow). The dam has a discharge capacity of 18150 cumec, 27 radial gates measuring 12.50 by 7.9 metres, and an overflow section measuring 417 metres in length. A lined, left-bank canal, 208 km in length, receives water from the Paithan Dam and irrigates a 1,41,640 hectare (ICA) area in the districts of Aurangabad, Jalana, Parbhani, and Ahmednagar.. The index map is shown in figure 1.

Table 1 shows the maximum irrigation demand and 75% dependable inflow. 75% dependable monthly inflows are estimated using the Weibull plotting position formula.

Table 1: Maximum irrigation demands and 75% dependable inflow.

Sr no.	Months	Maximum irrigation demand Mm^3	75% dependable InflowMm^3
1	June	3.50	112.762
2	July	3.90	320.25
3	August	0.60	610.66
4	September	33.60	600.00
5	October	93.70	147.75
6	November	109.00	116.46
7	December	66.90	85.53
8	January	45.00	37.65
9	February	46.10	21.462
10	March	75.10	19.562
11	April	95.30	25.50
12	May	57.50	46.587

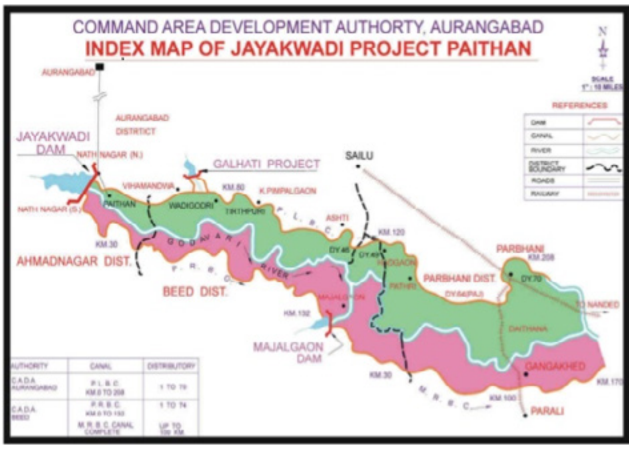

Figure 1. Index Map of Jayakwadi Project.

Formation of MOFLP model:

Application of MOFLP is demonstrate through the case study, Jayakwadi reservoir stage-1 in Maharashtra state, India. Problem is formulated with two objective function viz. Maximization of release for irrigation and maximization of release for hydropower production, with the following constraints and is solved in an iterative manner. All other model parameters other than the objectives are thought to be crisp in nature..

The study's two goals that were taken into account are:

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(1) Maximization of release for irrigation (i.e., RI) and
(2) Maximization of release for hydro power production (i.e., RP)

Max Z_1 = Max (TOTRI)

Max Z_2 = Max (TOTRP)

where TOTRP is the total release for hydropower generation over all time periods, and TOTRI is the total release for irrigation over all time periods (i.e., months). These objective functions can written as

$$\text{MAX } Z_1 = \sum_{t=1}^{t=12} RI_t$$

$$\text{MAX } Z_2 = \sum_{t=1}^{t=12} RP_t$$

Constraint

Turbine release constraint

Each month's release for the amount of hydropower the turbine will produce (RP) must be greater than or equal to both the firm power (FP) committed for that month as well as the turbine's capacity (TC).

$RP_t \leq TC$ $\forall t = 1, 2........., 12$
$RP_t \geq FP_t$ $\forall t = 1, 2........., 12$

Irrigation demand constraint

Release into the canal for irrigation (RI) need to be lower than or equal to the demand for irrigation (ID). Release must also be more than the minimum amount of irrigation necessary for all time periods in order to prevent crop wilting (30% of the irrigation demand in this instance is regarded as the minimum irrigation requirement).

$RI_t \leq ID_t$ $\forall t = 1, 2........., 12$
$RI_t \geq 0.3ID_t$ $\forall t = 1, 2........., 12$

Reservoir storage capacity constraint

For all time periods, the reservoir's live storage should be below or equal to its maximum capacity (SCAP).

$S_t \leq SC$ $\forall t = 1, 2........., 12$
$S_t \geq S_{min}$ $\forall t = 1, 2........., 12$

Reservoir storage continuity constraint

These restrictions apply to all time periods' turbine release (RP), irrigation release (RI), reservoir storage (S), inflow (I) into the reservoir, overflows (O), and evaporation losses (L).

$S_t + I_t – RI_t + 0.9RP_t – O_t – L_t – FCR – RWS = S_{t+1}$

By considering the evaporation losses as a function of storage (Loucks et al., 1981) and by assuming a linear relationship between reservoir water surface area and storage, continuity constraint can be written as follows.

$(1\text{-}a_t) S_t + I_t – RI_t - RP_t + 0.9RP_t – FCR – RWS –O_t – A_o e_t = S_{t+1}$

Where,

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$a_t = A_A e_t / 2$

A_A = Surface area of the reservoir per unit active storage volume.

A_o= Surface area of the reservoir corresponding to the dead storage volume.

e_t= Evaporation rate for month t in depth units.

RWS = Release for water supply.

FCR = Feeder canal releases.

3. PERFORMANCE ANALYSIS

For the purpose of reservoir management, a Multi Objective Fuzzy Linear Programming (MOFLP) model has been created. By focusing on one goal at a time, the optimal and worst-case values (Z+ and Z-) for both objectives (Z1 for release for irrigation and Z2 for release for power production) can be calculated. LINGO is used to maximise irrigation water release and power generation water release (Language for Interactive general optimization). When Z1 is maximised, Z2 is assumed to have its worst possible value, and vice versa. These values are specified in table 2.

Table 2: Objective function values (Best and Worst).

Objective function (Maximization)	Best value (Z^+)	Worst value (Z^-)
Release for irrigation (Z_1) Mm^3	630.20	392.0843
Release for Hydro-power Production (Z_2) Mm^3	408.00	336.00

When Z1 (release for irrigation) is maximised, hydropower generation (Z2) receives less power. Maximizing hydropower generation, denoted by Z2, gives higher priority to releasing water for power generation than releasing water for irrigation, denoted by Z1. The Jayakwadi scheme uses a reversible turbine for its pumped storage. The weirs that store the excess water from the hydropower generation and release it to the turbines downstream are only used during peak demand. The water is pumped upstream from the downstream weir and into the reservoir during off-peak hours (midnight, for example) using the same turbine.

After the objective function's upper and lower LINGO bounds are established, the second step is to fuzzify the objective functions by considering a suitable membership function. In this analysis, we focus on membership functions that are linear in nature.

$$\mu_{x1}(X) = \begin{cases} 0 & Z_1 \leq 392.0843 \\ \left(\dfrac{Z_1 - 392.0843}{630.20 - 392.0843} \right) & 392.0843 \leq Z_1 \leq 630.20 \\ 1 & Z_1 \geq 630.20 \end{cases}$$

$$\mu_{x2}(X) = \begin{cases} 0 & Z_2 \leq 336.00 \\ \left(\dfrac{Z_2 - 336.00}{408.00 - 336.00} \right) & 336.00 \leq Z_2 \leq 408.00 \\ 1 & Z_2 \geq 408.00 \end{cases}$$

The membership function for both the objectives Z_1 and Z_2 are shown in figures 2 and figure 3 respectively and can be stated as follows.

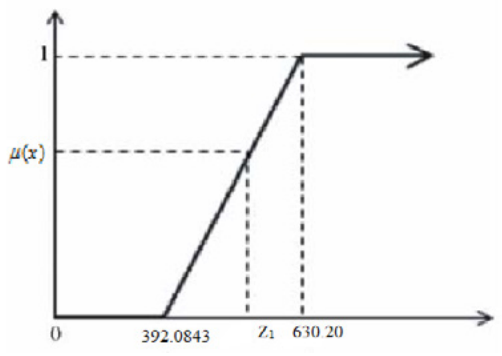

Fig.2 Membership function for Z_1.

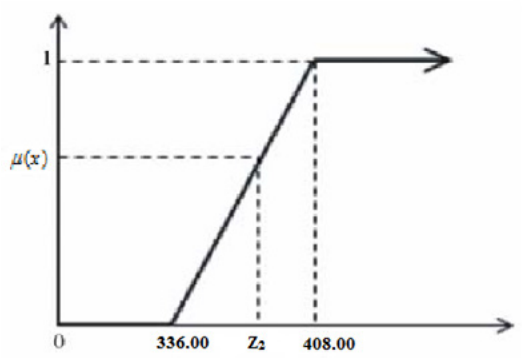

Fig.3 Membership function for Z_2.

The following updated LP problem is created as the third phase of the algorithm by combining the information mentioned before. Coefficients for constraints given below are obtained from the above two equations.

Maximizeλ

Subject to

$$\lambda \leq \frac{Z_1 - 392.0843}{630.20 - 392.0843}$$

$$\lambda \leq \frac{Z_2 - 336.00}{408.0 - 336.00}$$

And all the original constraints given in the model and $\lambda \geq 0$

The amount of satisfaction obtained by simultaneously optimising the fuzzified objectives Z1 and Z2 is represented by the symbol λ in this formulation. In the following stage, the LP model's solution is discovered.

The result obtained as follows.

λ (Maximum level of satisfaction = 1.00)

Z_1^* (Release of irrigation at the maximum level of satisfaction) = 630.20

Z_2^* (Release for Hydro power production corresponding to maximum level of satisfaction) = 408.00

The operating policy for maximization of release for irrigation is given in table 3 and maximization for power production is given in table 4. The operating policy corresponding to maximum level of satisfaction is given in table 5 and the results are shown in graph.

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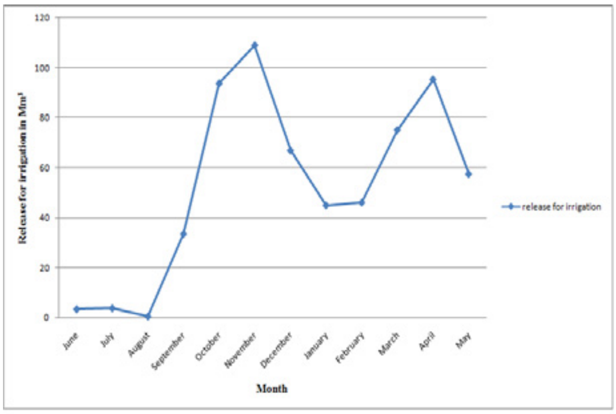

Figure 4. Release for irrigation (for maximization of Z_1).

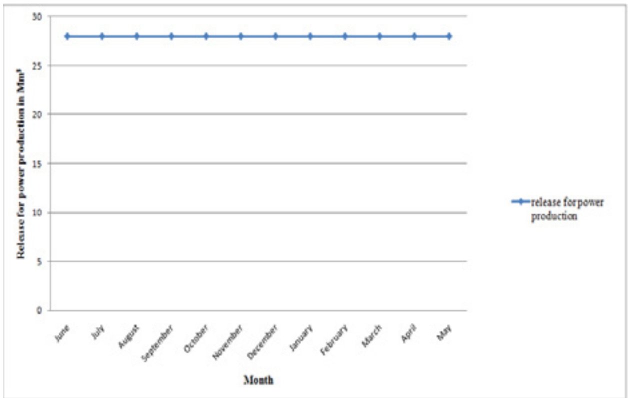

Figure 5. Release for power production (for maximization of Z_1).

Table 3: Operation policy for maximization of release for irrigation.

Month	Irrigation releases (RI) Mm³	Turbine releases (RP)Mm³	Head over turbine M	Storage Mm³	Overflow Mm³	Water supply releases Mm³	FCR Mm³
June	3.50	28.00	25.59	948.41	0.00	30.00	0.00
July	3.90	28.00	25.99	959.15	0.00	30.00	0.00
August	0.60	28.00	27.26	1187.225	0.00	30.00	0.00
Septembe r	33.60	28.00	28.94	1706.20	0.00	30.00	0.00
October	93.70	28.00	29.57	2170.599	0.00	30.00	50.00
November	109.00	28.00	29.15	2080.928	0.00	30.00	80.00
December	66.90	28.00	28.68	1924.644	0.00	30.00	70.00
January	45.00	28.00	28.17	1802.826	0.00	30.00	90.00
February	46.10	28.00	27.60	1626.343	0.00	30.00	60.00
March	75.10	28.00	27.03	1465.273	0.00	30.00	0.00
April	95.30	28.00	26.37	1288.857	0.00	30.00	0.00
May	57.50	28.00	25.80	1081.303	0.00	30.00	0.00
Total	**630.20**	**336.00**	**330.15**	**18241.758**	**0.00**	**360.00**	**350.00**

Table 4: Operation policy for maximization of release for Hydro power production.

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Month	Irrigation releases (RI) Mm³	Turbine releases (RP) Mm³	Head over turbine M	Storage Mm³	Overflow Mm³	Water supply releases Mm³	FCR Mm³
June	3.50	34.00	28.32	1766.84	0.00	30.00	0.00
July	3.90	34.00	28.66	1752.17	0.00	30.00	0.00
August	0.60	34.00	29.86	1960.379	0.00	30.00	0.00
September	33.60	34.00	31.48	2461.784	0.00	30.00	0.00
October	93.70	34.00	32.06	2909.00	0.00	30.00	50.00
November	109.00	34.00	31.60	2805.272	0.00	30.00	80.00
December	38.7843	34.00	31.13	2636.895	0.00	30.00	70.00
January	15.00	34.00	30.69	2533.645	0.00	30.00	90.00
February	16.00	34.00	30.17	2374.729	0.00	30.00	60.00
March	26.00	34.00	29.67	2231.045	0.00	30.00	0.00
April	32.00	34.00	29.10	2076.413	0.00	30.00	0.00
May	20.00	34.00	28.57	1895.989	0.00	30.00	0.00
Total	**392.0843**	**408.00**	**361.31**	**27404.161**	**0.00**	**360.00**	**350.00**

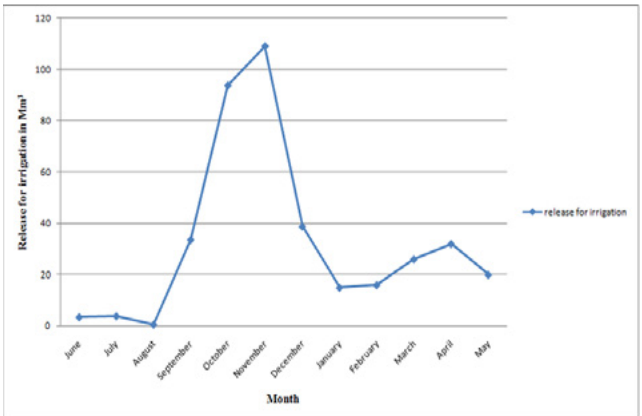

Figure 6. Release for irrigation (for maximization of Z_2).

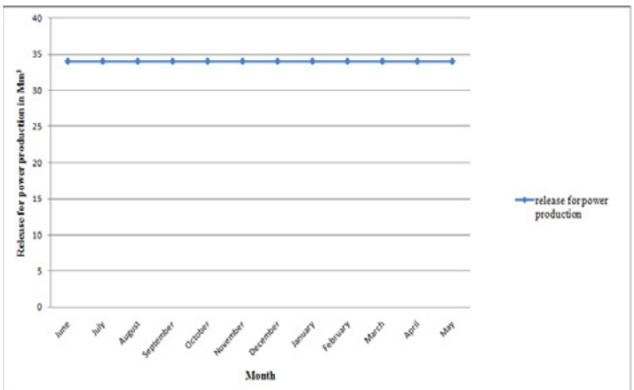

Figure 7. Release for power production (for maximization of Z_2).

Table 5: Operation policy for maximization of level of satisfaction i.e.λ = 1.00.

Month	Irrigation releases (RI) Mm³	Turbine releases (RP) Mm³	Head over turbine M	Storage Mm³	Overflow Mm³	Water supply releases Mm³	FCR Mm³

June	3.50	34.00	25.51	924.60	0.00	30.00	0.00
July	3.90	34.00	25.91	935.47	0.00	30.00	0.00
August	0.60	34.00	27.18	1163.530	0.00	30.00	0.00
September	33.60	34.00	28.85	1682.43	0.00	30.00	0.00
October	93.70	34.00	29.49	2146.759	0.00	30.00	50.00
November	109.00	34.00	29.07	2056.928	0.00	30.00	80.00
December	66.90	34.00	28.60	1900.43	0.00	30.00	70.00
January	45.00	34.00	28.10	1778.303	0.00	30.00	90.00
February	46.10	34.00	27.52	1601.624	0.00	30.00	60.00
March	75.10	34.00	26.94	1440.351	0.00	30.00	0.00
April	95.30	34.00	26.29	1264.189	0.00	30.00	0.00
May	57.50	34.00	25.71	1057.119	0.00	30.00	0.00
Total	**630.20**	**408.00**	**329.17**	**17951.733**	**0.00**	**360.00**	**350.00**

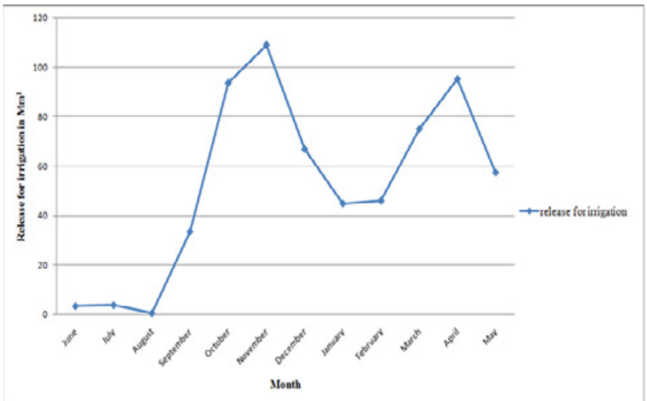

Figure 7. Optimal release for irrigation (for maximum satisfaction level).

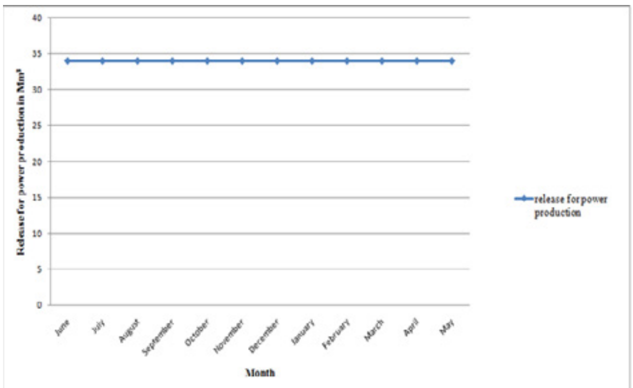

Figure 8. Optimal release for power production (for maximum satisfaction level).

Table 6: Maximized value of the objective function.

λ (Maximum level of Satisfaction)	**1.00**
Z_1^* (Irrigation release corresponding to the maximum level of satisfaction)	**630.20 Mm³**
Z_2^* (Release for Hydro power production corresponding to maximum level of satisfaction)	**408.00 Mm³**

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4. RESULT AND DISCUSSION

The fuzzy logic tool box available with the MATLAB package is used for developing the model(MATLAB).The inputs to the fuzzy system are inflows, storage, and time-of-year. The demand is assumed to be uniquely defined for a period, and hence the variable time-of-year(the period number) is taken as the equivalent input. The output is the release during the period. For the inputs and output operations the logical and implication operators are taken as (with conventional Fuzzy notation),

And Method	=	'Min';
Or Method	=	'Max';
Imp Method	=	'Min';
D e f u z z	=	'Centroi

Where the 'And' and 'Or' method corresponds to the conjuction(min) and disjunc- tion(max) operation of classical logic.

Step 1) fuzzy inference system tool:

The membership functions are used to determine the degree to which the inputs belong to each of the relevant fuzzy sets as the initial stage in creating a fuzzy inference system. Fuzzy Controller has five Inputs and one output. A fuzzy degree of Membership is the final outcome of the fuzzification process, and the input is always a crisp numerical value constrained to the universe of discourse of the input variable. The storage, inflow, RWS, ID, Evaporation and release were assigned the triangular membership functions. The salient membership function for the input inflow and output power are shown in figure 9.

Input For Data

Membership function values are traced to 'very low', 'low', 'med', 'high', 'very high' of storage, inflow, RWS, ID, Evaporation and release membership functions, respectively.

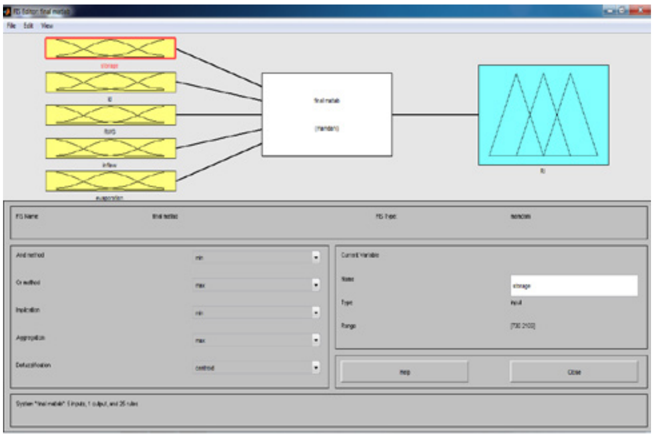

Figure 9. FIS editor.

Step 2) Membership function for input and output

The following describes the broad context in which the creation of a membership function takes place. The scenario includes a knowledge engineer, one or more subject-matter experts, and a particular knowledge domain of interest. The responsibility of a knowledge engineer is to draw out relevant knowledge from specialists and convey it in a necessary sort of operational form. The knowledge

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engineer tries to extract information in the first step using propositions that are presented in plain language. The knowledge engineer makes an effort to ascertain the definition of each language term used in these statements in the second stage. The techniques used to build a membership function, as determined by experts.

Types of triangular membership functions should be used as input. To display the various input fuzzy variable ranges, the five membership functions "Very Low", "Low", "Medium", "High," and "Very High," are employed.

Output: (RI)

The appropriate fuzzy rule for the period is activated once the reservoir storage and inflow levels (high, medium, etc.) have been determined. A fuzzy set for the release is produced via the fuzzy operator, implication, and aggregation. The Centroid of the fuzzy set is then utilized to produce a crisp release.

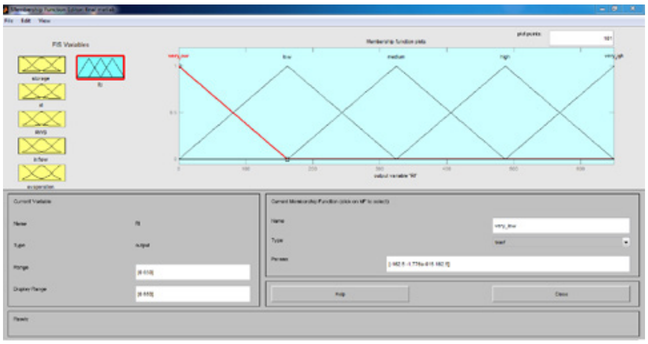

Figure 10. Membership Functions For Variables.

Step 3) Rule Viewer (adding and editing of rules):

The Rule Viewer is a show how the shape of certain membership functions influences the overall result. Rules shown in Rule Editor provide inference mechanism strategy and producing the control signal as output. Different numbers of rules that used in the system will give the different result, so the analysis for results will be conducted.

The operational rules were applied to generate a result for each rule before a combined operational rule were applied which then combines the results of the rules. These rules in figure 11 were applied to the inputs and the output of the Mamdani-type fuzzy inference system based controller. A new approach is therefore investigated through the use of fuzzy logic to serve as a base or platform to build an intelligent controller using a set of well-defined rules to guide its operational performance. By contrast, a fuzzy inference system employing "*if-then*" rules can model the qualitative aspects of human knowledge and reasoning processes without employing precise quantitative analyses. It is necessary to defuzzify the output fuzzy set in order to receive the output of the whole set of rules as a single integer.

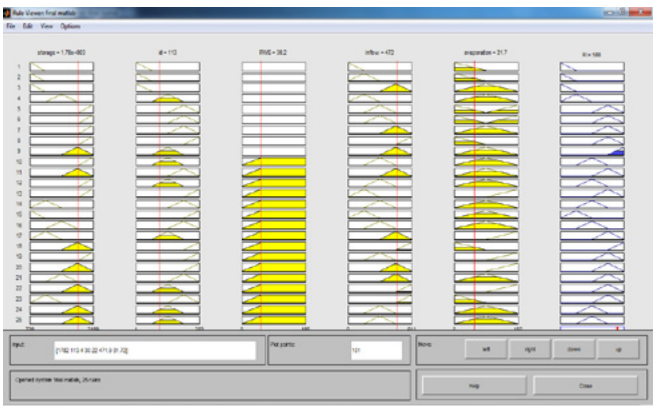

Figure 11. Generated Fuzzy Rules.

5. CONCLUSIONS

Application of system techniques to water management has gained momentum over the years. Many mathematical models have been developed and successfully applied for reservoir planning and operation studies. The use of these models has greatly aided in providing a good sight into the intricacies of the various aspects of problem in water management. The conclusions obtained from the present study of various are summarized. The Multi-Objective Fuzzy Linear Programming (MOFLP) model is created and employed to the reservoir operation problem to decide the optimal release policy for the Jayakwadi Reservoir stage-1, Maharashtra state, India. Optimal policies are determined for 75% dependable inflows using MOFLP. Depending on the decision-choice maker's of priorities for each target, these ideal policies may be put into practice for greater usage of the water resources. The two objectives i.e., release for irrigation and release for hydropower production are thought about in the study are maximization of irrigation release, and maximization of release for power production. First the model is solved for maximization of irrigation release. The maximized irrigation release obtained is 630.20 Mm^3 and corresponding release for power production 336.00 Mm^3. Then the model is run for maximization of release for hydropower production. The maximized release for hydropower production obtained is 408.00 Mm^3 and corresponding irrigation release is 392.0843 Mm^3.

The best and worst values of the two objective functions are decided. The objective functions are fuzzified over the best and worst values of each objective functions. The maximum satisfaction level for the fuzzified problem is obtained as 1.00. For this satisfaction level, maximized sum of release for irrigation is 630.20 Mm^3 and maximized sum of release for hydropower production is 408.00 Mm^3. Fuzzy rule based model considering singe objective is developed viz. release for irrigation. The model is based on the "if-then principle," where "if" represents a vector of ambiguous premises and "then" represents a vector of fuzzy consequences. Using mamdani method of FIS, the release for irrigation is 588 Mm^3.

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